Localized model reduction for nonlinear elliptic PDEs

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Objective
Parameterized model order reduction (pMOR) for PDEs

The goal of parameterized model order reduction (pMOR) is to reduce the **marginal** cost associated with the solution to parameterized problems.

pMOR is motivated by *real-time* and *many-query* problems design and optimization, UQ, control.
The goal of parameterized model order reduction (pMOR) is to reduce the *marginal* cost associated with the solution to parameterized problems.

pMOR is motivated by *real-time* and *many-query* problems in design and optimization, UQ, control.

**Pb:** find \( u_\mu \in \mathcal{X} : G_\mu(u_\mu, v) = 0 \ \forall v \in \mathcal{Y} \).

- \( G_\mu : \mathcal{X} \to \mathcal{Y}' \) variational (non)linear operator;
- \( \mu = [\mu_1, \ldots, \mu_P] \in \mathcal{P} \subset \mathbb{R}^P \) vector of parameters; material properties, geometric features,....
- \( \mathcal{M} := \{u_\mu : \mu \in \mathcal{P}\} \) solution manifold.
Monolithic projection-based pMOR: general recipe

**Pb:** find $u_\mu \in \mathcal{X} : \mathcal{G}_\mu (u_\mu, \nu) = 0 \ \forall \nu \in \mathcal{Y}, \ \mu \in \mathcal{P}$

**Approx:** $\hat{u}_\mu = Z \hat{\alpha}_\mu, \quad \hat{\alpha} : \mathcal{P} \rightarrow \mathbb{R}^n, \ Z : \mathbb{R}^n \rightarrow \mathcal{X}$

$n \ll N =$ dofs of the full-order model (FOM) $= FE,...$
Monolithic projection-based pMOR: general recipe

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**Offline (learning) stage:** (performed once)
- compute $u_{\mu^1}, \ldots, u_{\mu^{n_{\text{train}}}}$ using a FE (or FV...) solver;
- construct $Z = [\zeta_1, \ldots, \zeta_n]$ and define $Z = \text{span}\{\zeta_i\}_{i=1}^n$;
- define a reduced-order model (ROM) for $\hat{\alpha} : \mathcal{P} \to \mathbb{R}^n$.

**Online (prediction) stage:** (performed for new $\mu'$)
- estimate the solution coefficients $\hat{\alpha}_{\mu'} \in \mathbb{R}^n$.
- estimate $\|\hat{u}_{\mu'} - u_{\mu'}\|$.

$n \ll N = \text{dofs of the full-order model (FOM)} = \text{FE, ...}$
Limitation of monolithic approaches

Monolithic approaches rely on two assumptions:

- the solution is defined over a parameter-independent domain or over a family of diffeomorphic domains; no topology changes
- it is possible to solve the FOM for several parameters.

These conditions might not be fulfilled for several problems of interest.

**Example of topology change:**

![Diagram showing topology change](image-url)
Component-based model reduction

**Offline stage**
- Define a set of archetype components.
- Build local ROMs for each component.

**Online stage**
- Define the system as instantiation of components.
- Solve the global problem by gluing together the local ROMs.
Component-based model reduction: research questions

1. **Data compression**: how can we build “good” approximation spaces for each archetype component?

2. **Reduced formulation**: how can we glue together local ROMs to estimate the global solution?
   - non-overlapping conforming methods; scRBE
   - non-overlapping non-conforming methods; RBE, LRBMS
   - overlapping methods. zonal POD

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Aim of the talk

Define effective local approximation spaces for component-based MOR.

- Linear problems: optimal approximations, randomized methods.
- Randomized methods for nonlinear problems: theoretical and numerical considerations.

Bibliography: (for linear PDEs)
Babuska, Lipton, Multiscale Model. Simul., 2011.
Smetana, Patera, SISC, 2016.
Buhr, Smetana, SISC, 2018.
Working problem

Model pb: \[-\nabla \cdot (\kappa_\mu(u_\mu) \nabla u_\mu) = f_\mu \text{ in } \Omega, \quad u_\mu \big|_{\partial \Omega} = 0\]

\[\Omega = (0, 1)^2 = \bigcup_{i=1}^{N_{dd}} \Omega_i, \quad \mu = [\mu^{(1)}, \ldots, \mu^{(N_{dd})}, i^*],\]

\[\kappa_\mu(u) \big|_{\Omega_i} = \frac{36}{\mu_{2}^{(i)}} \left( \frac{u(1-u)}{u^3 + \frac{12}{\mu_{2}^{(i)}}(1-u)^3} \right)^2 + \mu_1^{(i)}, \quad i = 1, 2, \ldots\]

\[\mu^{(i)} \in \mathcal{P} = [0.1, 0.2] \times [30, 40], \quad i^* \in \{1, \ldots, N_{dd}\}.\]

- Idealized model for subsurface flows; challenging problem for monolithic MOR — \(2N_{dd} + 1\) parameters.
Working problem

Model pb: \(-\nabla \cdot (\kappa_\mu(u_\mu) \nabla u_\mu) = f_\mu \text{ in } \Omega, \ u_\mu \big|_{\partial \Omega} = 0\)

\(\Omega = (0, 1)^2 = \bigcup_{i=1}^{N_{dd}} \Omega_i, \ \mu = [\mu^{(1)}, \ldots, \mu^{(N_{dd})}, i^*]\),

\[f_\mu(x) = 100e^{-100\frac{\|x-x_{c,i^*}\|^2}{2}} \mathbb{1}_{\Omega_i^*}(x), \ x_{c,i^*} := \frac{1}{\Omega_i^*} \int_{\Omega_i^*} x \, dx.\]

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- Idealized model for subsurface flows; challenging problem for monolithic MOR — \(2N_{dd} + 1\) parameters.
Working problem

Model pb: 
\[-\nabla \cdot (\kappa_\mu(u_\mu) \nabla u_\mu) = f_\mu \text{ in } \Omega, \quad u_\mu \big|_{\partial \Omega} = 0\]
Goal: assess long-term behavior of radioactive waste districts.

Mathematical model: THM equations — nonlinear unsteady parabolic equations with internal variables.

Localized training

- Oversampling
- Linear problems
- Randomization
- Numerical results
Localized training

- Oversampling
  - Linear problems
  - Randomization
  - Numerical results
Archetype components

Three archetype components: corner (co), edge (ed) and internal (int).

- \( \hat{\Omega} = (0, h)^2 \) reference domain;
- \( \Phi_i : \hat{\Omega} \rightarrow \Omega_i \) \( i \)-th deformation map;
- \( I \in \{co, ed, int\}^{N_{dd}} \) instantiated components’ labels.
Oversampling (I): basic idea

**Task:** find low-dimensional spaces $Z^\text{co}, Z^\text{ed}, Z^\text{int}$ s.t.

$$\min_{\zeta \in Z_i^\text{co}} \| u_\mu |_{\Omega_i} \circ \Phi_i - \zeta \|_{H^1(\hat{\Omega})} \ll 1 \text{ for } i = 1, \ldots, N_{dd}$$

**Idea:**

- Define the patch $U \supset \hat{\Omega}$ with input boundary $\Gamma_{\text{in}} \subset \partial U$.
- Define the transfer operator $T$ s.t., given $g \in G \subset H^{1/2}(\Gamma_{\text{in}})$ and $\mu \in \mathcal{P}$, $T_\mu(g) = w|_{\hat{\Omega}}$ with

$$-\nabla \cdot (\kappa_\mu(w) \nabla w) = f_\mu \text{ in } U, \quad w|_{\Gamma_{\text{in}}} = g, \quad w|_{\partial U \setminus \Gamma_{\text{in}}} = 0.$$  

- Determine a low-dimensional approximation space for the manifold $\hat{\mathcal{M}} = \{ T_\mu(g) : g \in G, \mu \in \mathcal{P} \}$.

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Oversampling (II): role of $U, G$

Input boundary $\Gamma_{\text{in}}$ should be well-separated from $\hat{\Omega}$ to ensure decay of high-frequency modes, but $|U| \ll |\Omega|$ to ensure computational speed-ups.

Example: corner

Set $G$ should be representative of the behavior of $u_\mu|_{\Gamma_{\text{in}}}$:

$\Rightarrow$ in absence of prior information, $G$ is high-dimensional;

**Question:** why should the localized manifold $\hat{\mathcal{M}} = \{ T_\mu(g) : g \in G, \mu \in \mathcal{P} \}$ be reducible?
Motivating example: semi-infinite wave-guides (I)

Consider the problem for $g(x_2) = \sum_{n=1}^{\infty} c_n \cos(n\pi x_2)$

\[
\begin{cases}
-\Delta u_g - \mu^2 u_g = 0 \text{ in } U \\
\partial_{x_2} u_g = 0 \text{ on } \mathbb{R}_+ \times \{0, 1\} \\
u_g = g \text{ on } \{0\} \times (0, 1)
\end{cases}
\]

Define $\hat{\Omega} = (L, \infty) \times (0, 1)$, $C, \bar{\mu} \in \mathbb{R}_+$ and the extracted manifold $M_L = \{ u_g|_{\hat{\Omega}} : \|g\|_{H^{1/2}} \leq C, \mu = \bar{\mu} \}$.

$G = \{ g : \|g\|_{H^{1/2}} \leq C \}$ is infinite-dimensional and is irreducible.
Motivating example: semi-infinite wave-guides (II)

\[ u_g = \sum_{n=1}^{N_{pr}} c_n e^{-i\alpha_n x_1} \cos(n\pi x_2) + \sum_{n=N_{pr}+1}^{\infty} c_n e^{-\alpha_n x_1} \cos(n\pi x_2), \]

= (I) \hspace{1cm} = (II)

with \( N_{pr} = \left\lfloor \frac{\mu}{\pi} \right\rfloor \), \( \alpha_n = |n^2\pi^2 - \mu^2| \). (I) = propagating modes; (II) = evanescent modes.

The differential operator acts as a \textbf{low-pass filter}.

- Effect of evanescent modes is negligible far from \( \Gamma_{\text{in}} \).
- Manifold \( \mathcal{M}_L \) can be well-approximated by an \( n \)-dimensional space if \( e^{-\alpha_{n+1} L} \ll 1 \).

Filtering properties depend on the value of \( \bar{\mu} \).
Localized training

- Oversampling
- Linear problems
- Randomization
- Numerical results
Define the transfer operator $T : U \times \mathcal{P} \rightarrow \mathcal{Y}$. For second-order elliptic PDEs, $G \subset U \subset H^{1/2}(\Gamma_{in})$, $\mathcal{Y} \subset H^1(\Omega)$; $(U, \| \cdot \|_U)$, $(\mathcal{Y}, \| \cdot \|_{\mathcal{Y}})$ Hilbert spaces. We assume that $\mathcal{P} = \{ \bar{\mu} \}$ and that $T$ is compact.
Transfer operator

Define the transfer operator $T : U \times P \rightarrow Y$. For second-order elliptic PDEs, $G \subset U \subset H^{1/2}(\Gamma_{in})$, $Y \subset H^1(\hat{\Omega})$; $(U, \| \cdot \|_U), (Y, \| \cdot \|_Y)$ Hilbert spaces. We assume that $P = \{\bar{\mu}\}$ and that $T$ is compact.

**Proposition:** Sufficient conditions for compactness of $T$:

- $\text{dist}(\hat{\Omega}, \Gamma_{in}) > 0$;
- $\| T(g) \|_{H^1(U')} \leq C(U, U') \| T(g) \|_{L^2(U)}$ for $U' \subset U$.

Caccioppoli’s ineq.

Caccioppoli’s inequality is satisfied by a broad class of elliptic problems (Helmholtz, advection-diffusion, Stokes).

Introduce the **transfer eigenproblem**:

\[(\varphi_i, \lambda_i) \in \mathcal{U} \times \mathbb{R}_+ \quad (T(\varphi_i), \ T(g))_Y = \lambda_i(\varphi_i, g)_U \quad \forall \ g \in \mathcal{U}\]

with \( i = 1, 2, \ldots, \ \lambda_1 \geq \lambda_2 \geq \ldots \)

Define the **transfer eigenspace** \( \mathcal{Z}_{te}^n = \text{span}\{ T(\varphi_i) \}_{i=1}^n \).
Transfer eigenproblem (TE): properties (I)

Introduce the **transfer eigenproblem**:

\[(\varphi_i, \lambda_i) \in \mathcal{U} \times \mathbb{R}_+ \ (T(\varphi_i), T(g))_\mathcal{Y} = \lambda_i(\varphi_i, g)_\mathcal{U} \ \forall \ g \in \mathcal{U}\]

with \( i = 1, 2, \ldots, \ \lambda_1 \geq \lambda_2 \geq \ldots \)

Define the **transfer eigenspace** \( \mathcal{Z}_{n}^{te} = \text{span}\{ T(\varphi_i) \}_{i=1}^{n} \).

If \( \mathcal{U} = \text{span}\{ \phi_k \}_{k=1}^{N_{in}} \), then

\[
\varphi_i = \sum_{k=1}^{N_{in}} (\varphi_i)_k \phi_k, \quad A \varphi_i = \lambda_i B \varphi_i
\]

with \( A_{k,k'} = (T(\phi_k), T(\phi_k'))_\mathcal{Y}, \ B_{k,k'} = (\phi_k, \phi_k')_\mathcal{U} \).

- Solution to TE requires computation of \( \{ T(\phi_k) \}_k \).
- If \( \{ \phi_k \}_k \) is orthonormal, then \( \mathcal{Z}_{n}^{te} \) is the POD space associated with the snapshots \( \{ T(\phi_k) \}_k \).
Optimality: (Pinkus, 1985)

\[ Z_{te}^n \in \arg \inf_{W \subset Y, \dim(W) = n} \sup_{g \in \mathcal{U}} \frac{\| \Pi_Y W \perp T(g) \|_Y}{\| g \|_U} \]

- The transfer eigenspace is optimal in the sense of Kolmogorov.

Error analysis: (Taddei, Patera, 2018)

\[ \frac{\| \Pi_Y (Z_{te}^n) \perp T(g) \|_Y}{\| g \|_U} \leq \| T \|_{\mathcal{L}(\mathcal{U}^\perp,Y)} \frac{\| \Pi_{\mathcal{U}^\perp} g \|_U}{\| g \|_U} + \sqrt{\lambda_{n+1}} \]

for any \( g \in H^{1/2}(\Gamma_{in}) \).

- If \( g \notin \mathcal{U} \), performance of \( Z_{te}^n \) depends on the product \( \| T \|_{\mathcal{L}(\mathcal{U}^\perp,Y)} \| \Pi_{\mathcal{U}^\perp} g \|_U \).
Extension to the parametric case

**TE + POD:** given \( \mathcal{U} = \text{span}\{\phi_i\}_{i=1}^{N_{\text{in}}}, \{\mu^k\}_{k=1}^{n_{\text{train}}}, n \in \mathbb{N} \)

For \( k = 1, \ldots, n_{\text{train}} \)

Compute \( \mathcal{Z}_{n,k}^{\text{te}} := \text{span}\{T_{\mu^k}\phi_i^k\}_{i=1}^n \) s.t. \( (\phi_i^k, \phi_j^k)\mathcal{U} = \delta_{i,j} \).

EndFor

Return \( \mathcal{Z}_n := \text{POD}\left(\{T_{\mu^k}\phi_i^k\}_{i,k,n}\right). \)

\( i = 1, \ldots, n, k = 1, \ldots, n_{\text{train}} \)

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Taddei, Patera, SISC, 2018.
Extension to the parametric case

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Compute \( \mathcal{Z}_{n,k}^{\text{te}} := \text{span}\{T_{\mu^k}\phi_i^k\}_{i=1}^{n} \) s.t. \( (\phi_i^k, \phi_j^k)_{\mathcal{U}} = \delta_{i,j} \).

EndFor

Return \( \mathcal{Z}_n := \text{POD}\left(\{T_{\mu^k}\phi_i^k\}_{i,k,n}\right) \).

\[ i = 1, \ldots, n, k = 1, \ldots, n_{\text{train}} \]

- Equivalent to hierarchical approximate POD. 
  Himpe, Leibner, Rave, 2018.

- Alternative approach: TE + strong Greedy in parameter. 
  Smetana, Patera 2016.

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Taddei, Patera, SISC, 2018.
Localized training

- Oversampling
- Linear problems
- Randomization
- Numerical results
Motivation

TE + POD requires $N_{in} \cdot n_{train}$ PDE solves. **Possible fix:** Krylov subspace methods for the TE.

Randomized methods offer an alternative strategy to tackle the problem.

**Mahoney 2016:** *randomness is an algorithmic resource creating efficient, unbiased approximations of nonrandom operations.*

**Key features of randomized algorithms:**
- Incremental (iterative) approach.
- Probabilistic error estimation.

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**Randomized SVD:** Mahoney, Martinsson, Tropp...  **Randomized model reduction:** Buhr & Smetana; Smetana, Zahm, Patera.
Adaptive randomized algorithm

Inputs: \( \text{maxit}, r_{\text{train}}, n \in \mathbb{N}, \text{pdfs } p_{\mu}, p_{G} \) (sampling);

Output: \( \mathcal{Z}_n = \text{span}\{\zeta_i\}_{i=1}^n \) reduced space. \( (\zeta_i, \zeta_j) = \delta_{i,j} \)

Set \( \mathcal{Z}_n = \emptyset, \lambda_1 = \ldots = \lambda_n = 0. \)

For \( i = 1, \ldots, \text{maxit} \)

- Generate \( \mu^{(k)} \overset{\text{iid}}{\sim} p_{\mu}, \ g^{(k)} \overset{\text{iid}}{\sim} p_{G}, \ k = 1, \ldots, r_{\text{train}}. \)
- Compute \( u^{k,i} = T_{\mu^{(k)}}g^{(k)} \) for \( k = 1, \ldots, r_{\text{train}}. \)
- Compute \( \hat{E} = \frac{1}{r_{\text{train}}} \sum_{k=1}^{r_{\text{train}}} \| \Pi_{\mathcal{Z}_n^\perp} u^{k,i} \|_Y / \| u^{k,i} \|_Y. \)

if \( \hat{E} \leq \text{tol} \), BREAK

else \( [\mathcal{Z}_n, \{ \lambda_j \}_j] = \text{POD} \left( \{ \zeta_j \sqrt{\lambda_j} \}_{j=1}^n \cup \{ u^{k,i} \}_{k=1}^{r_{\text{train}}} , n \right) \)

EndFor
Adaptive randomized algorithm: comments

Interpretation: incremental randomized POD in parameter and boundary data. Yu, Chakravorty, 2015

- Snapshots $\{u^{k,i}\}_{k,i}$ are used for testing and then to update the reduced space.

- $[Z_n, \{\lambda_j\}_j] = \text{POD} \left( \{\zeta_j \sqrt{\lambda_j}\}_{j=1}^n \cup \{u^{k,i}\}_{k=1}^{r_{\text{train}}} , n \right)$
  - Snapshots from previous iterations are not stored.
  - Factors $\{\sqrt{\lambda_j}\}_j$ serve to properly fuse information from different iterations. Himpe, Leibner, Rave 2018

- $\hat{E}$ is a MC estimate of $\mathbb{E}_{\mu \sim p_\mu, g \in p_G} \left[ \frac{\| \Pi Z_n^\perp T_\mu(g) \|_Y}{\| T_\mu(g) \|_Y} \right]$
  For linear problems, see also Buhr, Smetana, 2018.

Choice of the boundary distribution

**Idea:** since we expect solution to be smooth, we control the smoothness of the samples.

Consider $g_{\text{co}}(s; c^{\text{re}}, c^{\text{im}}) = \sum_{k=0}^{N_f-1} \frac{c_k^{\text{re}} + ic_k^{\text{im}}}{\sqrt{1 + (2\pi k)^{2\alpha}}} e^{2\pi k s i}$,

with $c_k^{\text{re}}, c_k^{\text{im}} \sim \mathcal{N}(0, 1)$. Then, if $\alpha \in \mathbb{N}$

$$\|g_{\text{co}}(\cdot; c^{\text{re}}, c^{\text{im}})\|_{L^2(0,1)}^2 + \|g_{\text{co}}^{(\alpha)}(\cdot; c^{\text{re}}, c^{\text{im}})\|_{L^2(0,1)}^2 \sim \chi^2(2N_f)$$
Choice of the boundary distribution

**Idea:** since we expect solution to be smooth, we control the smoothness of the samples.

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with \( c_k^{re}, c_k^{im} \overset{iid}{\sim} \mathcal{N}(0, 1) \). Then, if \( \alpha \in \mathbb{N} \)

\[ \|g_{co}(\cdot; \mathbf{c}^{re}, \mathbf{c}^{im})\|_{L^2(0,1)}^2 + \|g_{co}^{(\alpha)}(\cdot; \mathbf{c}^{re}, \mathbf{c}^{im})\|_{L^2(0,1)}^2 \sim \chi^2(2N_f) \]

**Proposal:** \( g(\cdot; \mathbf{c}^{re}, \mathbf{c}^{im}) = \text{Real} \left[ g_{co}(\cdot; \mathbf{c}^{re}, \mathbf{c}^{im}) \right] \).

For edge and corner components, we need to enforce Dirichlet boundary conditions.

details omitted
Localized training

- Oversampling
- Linear problems
- Randomization
- Numerical results
**Linear advection-diffusion-reaction equation**

**Pb:** given $\mu \in \mathcal{P}$, $g \in H^{1/2}(\Gamma_{in})$, $\kappa(x) = 1 + \|x\|^2_2$, $U = [0, 0.3]^2$, $\mathcal{P} = [0.2, 1] \times [-1, 1]^2 \times [0, 1]$, find $u_\mu$ s.t.

\[
\begin{align*}
\nabla \cdot \left( \mu_1 \kappa^{-1} \nabla u_\mu + [\mu_2, \mu_3] u_\mu \right) + \mu_4 u_\mu &= 0 \quad \text{in } U \\
\left. u_\mu \right|_{\partial U = \Gamma_{in}} &= g,
\end{align*}
\]

We consider the extracted domain $\hat{\Omega} = [0.1, 0.2]^2$. 

\[
U = [0, 0.3]^2, \quad \mathcal{P} = [0.2, 1] \times [-1, 1]^2 \times [0, 1],
\]

\[
\mu_1, \mu_2, \mu_3, \mu_4 \text{ iid } \sim \text{Normal}(0, 1)^4.
\]
Pb: given $\mu \in \mathcal{P}$, $g \in H^{1/2}(\Gamma_{in})$, $\kappa(x) = 1 + \|x\|_2^2$, $U = [0, 0.3]^2$, $\mathcal{P} = [0.2, 1] \times [-1, 1]^2 \times [0, 1]$, find $u_\mu$ s.t. 

$$
\begin{cases}
-\nabla \cdot (\mu_1 \kappa^{-1} \nabla u_\mu + [\mu_2, \mu_3] u_\mu) + \mu_4 u_\mu = 0 \text{ in } U \\
u_\mu \big|_{\partial U = \Gamma_{in}} = g,
\end{cases}
$$

We consider the extracted domain $\hat{\Omega} = [0.1, 0.2]^2$.

**Description of the test**

- P3 FE with $N_{in} = 360$ dofs on $\Gamma_{in}$.

- TE+POD based on $n_{train} = 100$. 
Linear advection-diffusion-reaction equation

**Pb:** given $\mu \in \mathcal{P}$, $g \in H^{1/2}(\Gamma_{\text{in}})$, $\kappa(x) = 1 + \|x\|^2_2$, $U = [0, 0.3]^2$, $\mathcal{P} = [0.2, 1] \times [-1, 1]^2 \times [0, 1]$, find $u_\mu$ s.t.

\[
\begin{cases}
-\nabla \cdot (\mu_1 \kappa^{-1} \nabla u_\mu + [\mu_2, \mu_3] u_\mu) + \mu_4 u_\mu = 0 \text{ in } U \\
u_\mu \big|_{\partial U = \Gamma_{\text{in}}} = g,
\end{cases}
\]

We consider the extracted domain $\hat{\Omega} = [0.1, 0.2]^2$.

**Description of the test**

- Maximum relative projection error $E_{\text{max,rel}}$ based on $n_{\text{test}} = 100$ samples \(\{T_{\mu(k)} g^{(k)}\}_k\) with $g^{(k)} \overset{iid}{\sim} p_G^{\text{smooth}}(\alpha = 1)$, $\mu^{(k)} \overset{iid}{\sim} \text{Uniform}(\mathcal{P})$.

- Randomized training with $r_{\text{train}} = 10$, $g^{(k)} \overset{iid}{\sim} p_G^{\text{smooth}}$, $\mu^{(k)} \overset{iid}{\sim} \text{Uniform}(\mathcal{P})$. 
Choice of the boundary distribution: visualization (int)

Samples of the datum on $\Gamma_{\text{in}}$ for $\alpha = 1$. 

![Graph showing samples of the datum on $\Gamma_{\text{in}}$ for $\alpha = 1$.]
Samples of the datum on $\Gamma_{\text{in}}$ for $\alpha = 2$. 

![Diagram showing samples of the datum on $\Gamma_{\text{in}}$ for $\alpha = 2$.]
Samples of the datum on $\Gamma_{in}$ for $\alpha = 3$. 
Out-of-sample performance

Gaussian sampling: \( g \sim p_G^{gauss} \iff g(x) = \sum_{i \in I_{dir}} c_i \phi_i^{fe}(x) \)

- Randomized training performs similarly to TE+POD after 5 iterations.
- Smooth sampling is (slightly) more effective than Gaussian sampling at early iterations.
Effectivity of the error indicator

We compare estimates \( \mathbb{E}_{\mu \sim p_\mu, g \in p_G} \left[ \frac{\| \prod_{n} Z_n^\perp T_\mu(g) \|_Y}{\| T_\mu(g) \|_Y} \right] \)

computed during the iterations of the randomized algorithm with the estimates on the \( n_{\text{test}} = 100 \) datapoints.

Randomized training based on smooth sampling \( (\alpha = 1) \).

Effectivity \( \eta = \frac{\hat{E}_{r_{\text{train}} = 10}}{\hat{E}_{n_{\text{test}} = 100}} \).

Results over 100 independent runs.
Randomized sampling for nonlinear PDEs

Numerical results
Towards the extension to nonlinear PDEs

**Objective:** extend the localized training procedure to nonlinear PDEs.

- Error estimation and oversampling can be extended *as is*.
- How should we choose $p_G$?

**Theoretical questions**

- Compactness of the solution manifold\(^1\).
  - Caccioppoli’s ineq.

- Constructive optimal approximations\(^2\)

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\(^1\)Compactness of the transfer operator is key for approximability of the nonlinear set of functions, and ultimately justifies oversampling.

\(^2\)Generalization of transfer eigenproblems to nonlinear PDEs.
Choice of $p_G$ for the model problem

**Model pb:** $- \nabla \cdot (\kappa_\mu (u_\mu) \nabla u_\mu) = f_\mu$ in $\Omega$, $u_\mu \big|_{\partial \Omega} = 0$

$$\kappa_\mu (u) \big|_{\Omega_i} = \frac{36}{\mu_2^{(i)}} \left( \frac{u(1 - u)}{u^3 + \frac{12}{\mu_2^{(i)}} (1 - u)^3} \right)^2 + \mu_1^{(i)}, \ i = 1, 2, \ldots$$

$$f_\mu (x) = 100e \left( -100 \frac{\|x - x_{c,i*}\|_2^2}{2} \right) \mathbb{1}_{\Omega_i^*} (x), \ x_{c,i*} := \frac{1}{\Omega_i^*} \int_{\Omega_i^*} x \, dx.$$  

**Obs (I):** $u_\mu \in [0, 1] \Rightarrow$ we assume that $g(x) \in [0, \bar{u}_{\text{max}}]$  

**Obs (II):** $u_\mu \big|_{\Gamma_{\text{in}}}$ is expected to be smooth (at least $H^{1/2}$).  

$\Rightarrow$ smooth sampling.
We consider the following sampling algorithm:

1. Draw $c^{\text{re}}, c^{\text{im}} \in \mathbb{R}^N_f$ s.t. $c_k^{\text{re}}, c_k^{\text{im}} \sim \mathcal{N}(0, 1)$.

2. Draw $X_1, X_2 \sim \text{Uniform}(0, \bar{u}_{\text{max}})$, set $a = \min\{X_1, X_2\}$, $b = \max\{X_1, X_2\}$.

3. Set $g' = \text{Real} \left[ g_{\text{co}}(\cdot; c^{\text{re}}, c^{\text{im}}) \right]$.

4. Set $g = a + \frac{b - a}{\max g' - \min g'} (g' - \min g')$.

Memo: $g_{\text{co}}(s; c^{\text{re}}, c^{\text{im}}) = \sum_{k=0}^{N_f-1} \frac{c_k^{\text{re}} + ic_k^{\text{im}}}{\sqrt{1 + (2\pi k)^{2\alpha}}} e^{2\pi k s i}$.
Choice of $p_G$: internal component (visualization)

- $p_G$ encodes the fact that $\text{Im}[g] \subset [0, \bar{u}_{\text{max}}]$.
- $\alpha$ encodes prior knowledge on the regularity of $g$.

Samples of the datum on $\Gamma_{\text{in}}$ for $\alpha = 1$ ($\bar{u}_{\text{max}} = 0.5$).
Choice of $p_G$: internal component (visualization)

- $p_G$ encodes the fact that $\text{lm}[g] \subset [0, \bar{u}_{\text{max}}]$.
- $\alpha$ encodes prior knowledge on the regularity of $g$.

Samples of the datum on $\Gamma_{\text{in}}$ for $\alpha = 2$ ($\bar{u}_{\text{max}} = 0.5$).
Choice of $p_G$: internal component (visualization)

- $p_G$ encodes the fact that $\text{Im}[g] \subset [0, \bar{u}_{\text{max}}]$.
- $\alpha$ encodes prior knowledge on the regularity of $g$.

Samples of the datum on $\Gamma_{\text{in}}$ for $\alpha = 3$ ($\bar{u}_{\text{max}} = 0.5$).
Randomized sampling for nonlinear PDEs

- Numerical results
We perform localized training based on $n_{\text{train}} = 10^2$ simulations for each component to build $Z^{\text{co}}, Z^{\text{int}}, Z^{\text{ed}}$.

**Oversampling domains:** each subregion is characterized by a value of $\mu \in \mathcal{P}$; source term is activated with probability $p = 0.5$. 
Description of the test: global solves

We compute $n_{\text{test}}$ global solutions for $N_{\text{dd}}$ subdomains and we extract solution in each element.

$$n_{\text{test}} = 15, \quad N_{\text{dd}} = 100, \quad \Omega = (0, 1)^2$$

We use the datasets of local solutions to measure performance of localized training.

$$n_{\text{int}}^{\text{test}} = 960, \quad n_{\text{co}}^{\text{test}} = 60, \quad n_{\text{ed}}^{\text{test}} = 480$$
Error measures

- Average $H^1$ relative projection error at component level.
- Average $L^2$ relative global projection error.

We compare performance of
- smooth sampling for various $\alpha$;
- Gaussian sampling with clipping $\max(\min(g, \bar{u}_{\text{max}}), 0)$;
- benchmark: POD space based on test set “opt” \(^3\).

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\(^3\)Not computable for practical applications due to the impossibility to perform global solves.
Results: local errors for non-overlapping domains

- Randomized sampling provides accurate spaces for moderate $n$.
- Gaussian sampling is inaccurate for edge and corner components.
- Performance is nearly independent of $\alpha$. 

![Graph showing internal component and $E_{\text{avg}}$ vs. $n$ with different values of $\alpha$.]
Results: local errors for non-overlapping domains

- Randomized sampling provides accurate spaces for moderate $n$.
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Results: local errors for non-overlapping domains

- Randomized sampling provides accurate spaces for moderate $n$.
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![Graph showing corner component error vs. $n$ for different values of $\alpha$.](image)
Results: global errors

Relative error is below $10^{-4}$ for $n = 30$.

Total number of degrees of freedom: ROM $n \cdot 100$, FOM 90601.
Conclusions
Component-based model reduction offers the opportunity to address large-scale parameterized problems. 

**Tasks:** localized training and domain decomposition.

- **topology changes, high-dimensional parameterization**
Component-based model reduction offers the opportunity to address large-scale parameterized problems. 

- topology changes, high-dimensional parameterization

**Tasks:** localized training and domain decomposition.

In this talk, we discussed localized training for steady PDEs:

- transfer operator and optimal approximations;
- randomized training based on random samples of BCs;
- probabilistic error estimation.
Ongoing work and perspectives

Extension to nonlinear problems requires major advances to state-of-the-art methods.

Localized training

- Problem-aware sampling distribution of BCs.  
- Online basis enrichment.  
  Schlinder, Ohlberger, 2015

Domain decomposition

- Partition-of-unity overlapping methods.  
  Melenk, Babuska, 1996
- Hyper-reduction.
Thank you for your attention!

For more information, visit the website:

math.u-bordeaux.fr/~ttaddei/.
Backup slides

- More plots
Backup slides

- More plots
Choice of the boundary distribution: visualization (corner)

\[ g(s; c^{\text{re}}, c^{\text{im}}) = \text{Real} \left[ g_{\text{co}}(0.7 \cdot s; c^{\text{re}}, c^{\text{im}}) \right] s(1 - s). \]

- \( g_{\text{co}} \) is not periodic in \((0, 1)\).
- \( g_{\text{co}}(s) = 0 \) for \( s \in \{0, 1\} \).

Samples of the datum on \( \Gamma_{\text{in}} \) for \( \alpha = 1 \).
Choice of the boundary distribution: visualization (corner)

\[ g(s; c^{\text{re}}, c^{\text{im}}) = \text{Real} \left[ g_{\text{co}}(0.7 \cdot s; c^{\text{re}}, c^{\text{im}}) \right] s(1 - s). \]

- \( g_{\text{co}} \) is not periodic in \((0, 1)\).
- \( g_{\text{co}}(s) = 0 \) for \( s \in \{0, 1\} \).

Samples of the datum on \( \Gamma_{\text{in}} \) for \( \alpha = 2 \).
Choice of the boundary distribution: visualization (corner)

\[ g(s; c^\text{re}, c^\text{im}) = \text{Real} \left[ g_{\text{co}}(0.7 \cdot s; c^\text{re}, c^\text{im}) \right] s(1 - s). \]

- \( g_{\text{co}} \) is not periodic in \((0, 1)\).
- \( g_{\text{co}}(s) = 0 \) for \( s \in \{0, 1\} \).

Samples of the datum on \( \Gamma_{\text{in}} \) for \( \alpha = 3 \).
Is the Newton solver affected by $p_G$?

For $\bar{u}_{\text{max}} = 0.5$, Newton solver with line search converges for all training points.

For $\bar{u}_{\text{max}} = 0.75$, Newton solver with line search fails to converge $1\%$ of the times for smooth sampling $\alpha = 0.5$ and $30\%$ of the times for Gaussian sampling.

Results for the edge component:

$\bar{u}_{\text{max}} = 0.5$

$\bar{u}_{\text{max}} = 0.75$