Localized model reduction for nonlinear elliptic PDEs

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Acknowledgements: A Iollo, G Sambataro

Objective

Parameterized model order reduction (pMOR) for PDEs

The goal of parameterized model order reduction (pMOR) is to reduce the **marginal** cost associated with the solution to parameterized problems.

pMOR is motivated by *real-time* and *many-query* problems
design and optimization, UQ, control.

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design and optimization, UQ, control.

Pb: find $u_\mu \in \mathcal{X} : \ \mathcal{G}_\mu(u_\mu, v) = 0 \ \ \forall \, v \in \mathcal{Y}$.

- $\mathcal{G}_\mu : \mathcal{X} \to \mathcal{Y}'$ variational (non)linear operator;
- $\mu = [\mu_1, \ldots, \mu_P] \in \mathcal{P} \subset \mathbb{R}^P$ vector of parameters;
  material properties, geometric features,....
- $\mathcal{M} := \{u_\mu \ : \ \mu \in \mathcal{P}\}$ solution manifold.

Monolithic projection-based pMOR: general recipe

Pb: find $u_\mu \in \mathcal{X} : \; \mathcal{G}_\mu(u_\mu, v) = 0 \;\; \forall\, v \in \mathcal{Y}, \; \mu \in \mathcal{P}$

Approx: $\hat{u}_\mu = Z\,\widehat{\alpha}_\mu,$ $\qquad \widehat{\alpha} : \mathcal{P} \to \mathbb{R}^n$, $Z : \mathbb{R}^n \to \mathcal{X}$

$n \ll N =$ dofs of the full-order model $\Big(\underbrace{\mathrm{FOM}}_{=\mathrm{FE},\dots}\Big)$

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Offline (learning) stage: (performed once)

compute $u_{\mu^1}, \ldots, u_{\mu^{n_{\mathrm{train}}}}$ using a FE (or FV...) solver;

construct $Z = [\zeta_1, \ldots, \zeta_n]$ and define $\mathcal{Z} = \mathrm{span}\{\zeta_i\}_{i=1}^n$;

define a reduced-order model (ROM) for $\widehat{\alpha} : \mathcal{P} \to \mathbb{R}^n$.

Online (prediction) stage: (performed for new μ')

estimate the solution coefficients $\widehat{\alpha}_{\mu'} \in \mathbb{R}^n$.

estimate $\| \hat{u}_{\mu'} - u_{\mu'} \|$.

$n \ll N =$ dofs of the full-order model $\left(\underbrace{\mathrm{FOM}}_{=\mathrm{FE}, \ldots} \right)$

Monolithic approaches rely on two assumptions:

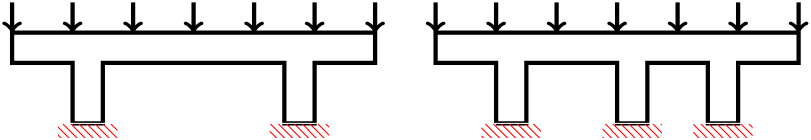
- the solution is defined over a parameter-independent domain or over a family of diffeomorphic domains;

  no topology changes

- it is possible to solve the FOM for several parameters.

These conditions might not be fulfilled for several problems of interest.

Example of topology change:

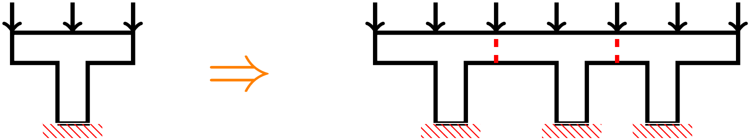


Offline stage

- Define a set of archetype components.
- Build local ROMs for each component.

Online stage

- Define the system as instantiation of components.
- Solve the global problem by gluing together the local ROMs.



Component-based model reduction: research questions

1. **Data compression:** how can we build "good" approximation spaces for each archetype component?
2. **Reduced formulation:** how can we glue together local ROMs to estimate the global solution?

- non-overlapping conforming methods; scRBE
- non-overlapping non-conforming methods; RBE, LRBMS
- overlapping methods. zonal POD

RBE: Maday, Rønquist, 2002. scRBE: Huynh, Knezevic, Patera, 2013. LRBMS: Kaulmann et al., 2012; Ohlberger, Schindler, 2015. zonal POD: Bergmann et al., 2018.

Aim of the talk

Define effective local approximation spaces for component-based MOR.

- Linear problems: optimal approximations, randomized methods.
- Randomized methods for nonlinear problems: theoretical and numerical considerations.

Bibliography: (for linear PDEs)

Babuska, Lipton, Multiscale Model. Simul., 2011.

Smetana, Patera, SISC, 2016.

Buhr, Smetana, SISC, 2018.

Working problem

Model pb: $-\nabla \cdot (\kappa_\mu(u_\mu)\nabla u_\mu) = f_\mu \text{ in } \Omega, \; u_\mu\big|_{\partial\Omega} = 0$

$\Omega = (0,1)^2 = \bigcup_{i=1}^{N_{\mathrm{dd}}} \Omega_i$, $\mu = \left[\mu^{(1)}, \ldots, \mu^{(N_{\mathrm{dd}})}, i^\star\right]$,

$$\kappa_\mu(u)\big|_{\Omega_i} = \frac{36}{\mu_2^{(i)}}\left(\frac{u(1-u)}{u^3 + \frac{12}{\mu_2^{(i)}}(1-u)^3}\right)^2 + \mu_1^{(i)}, \; i = 1, 2, \ldots$$

$\mu^{(i)} \in \mathcal{P} = [0.1, 0.2] \times [30, 40], i^\star \in \{1, \ldots, N_{\mathrm{dd}}\}$.

- Idealized model for subsurface flows; challenging problem for monolithic MOR — $2N_{\mathrm{dd}} + 1$ parameters.

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$\Omega = (0,1)^2 = \bigcup_{i=1}^{N_{\mathrm{dd}}} \Omega_i, \; \mu = \left[\mu^{(1)}, \ldots, \mu^{(N_{\mathrm{dd}})}, i^\star\right],$

$$f_\mu(x) = 100 e^{\left(-100 \frac{\|x - x_{\mathrm{c,i^\star}}\|_2^2}{2}\right)} \mathbb{1}_{\Omega_i^\star}(x), \quad x_{\mathrm{c,i^\star}} := \frac{1}{\Omega_i^\star} \int_{\Omega_i^\star} x \, dx.$$

$\mu^{(i)} \in \mathcal{P} = [0.1, 0.2] \times [30, 40], i^\star \in \{1, \ldots, N_{\mathrm{dd}}\}.$

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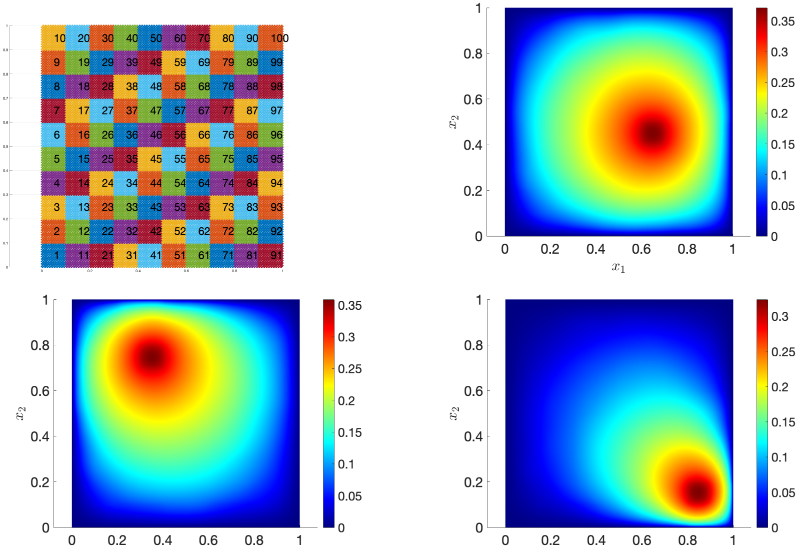
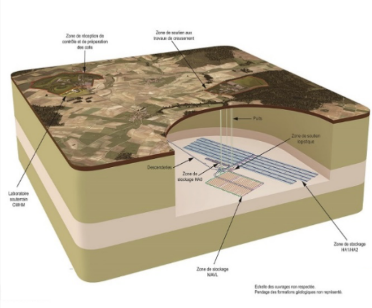
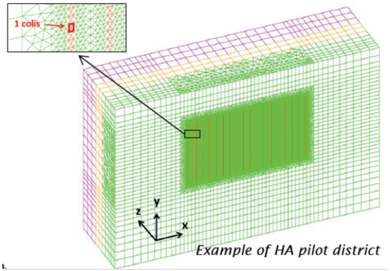


Target application (PhD thesis of G Sambataro)

Goal: assess long-term behavior of radioactive waste districts. Projet CIGEO

Mathematical model: THM equations — nonlinear unsteady parabolic equations with internal variables.





Monolithic pMOR: Iollo, Sambataro, Taddei, Arxiv, 2021.

Localized training

- Oversampling
- Linear problems
- Randomization
- Numerical results

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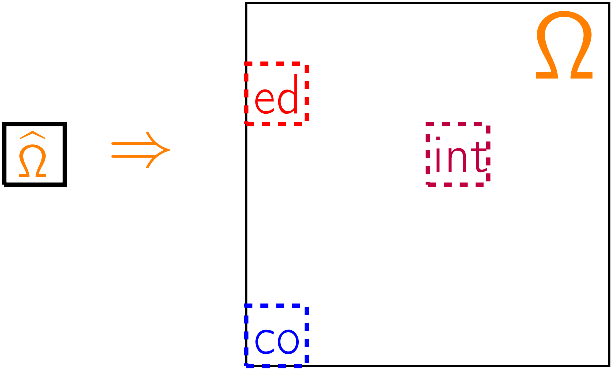
Three archetype components: corner ($\mathtt{co}$), edge ($\mathtt{ed}$) and internal ($\mathtt{int}$).

- $\widehat{\Omega} = (0, h)^2$ reference domain;
- $\Phi_i : \widehat{\Omega} \to \Omega_i$ i-th deformation map;
- $l \in \{\mathtt{co}, \mathtt{ed}, \mathtt{int}\}^{N_{\mathrm{dd}}}$ instantiated components' labels.



Oversampling (I): basic idea

Task: find low-dimensional spaces $\mathcal{Z}^{\mathtt{co}}, \mathcal{Z}^{\mathtt{ed}}, \mathcal{Z}^{\mathtt{int}}$ s.t.

$$\min_{\zeta \in \mathcal{Z}^{l_i}} \| u_\mu|_{\Omega_i} \circ \Phi_i - \zeta \|_{H^1(\widehat{\Omega})} \ll 1 \text{ for } i = 1, \dots, N_{\mathrm{dd}}$$

Idea:

- Define the patch $U \supset \widehat{\Omega}$ with input boundary $\Gamma_{\mathrm{in}} \subset \partial U$.
- Define the transfer operator T s.t., given $g \in G \subset H^{1/2}(\Gamma_{\mathrm{in}})$ and $\mu \in \mathcal{P}$, $T_\mu(g) = w|_{\widehat{\Omega}}$ with

$$-\nabla \cdot (\kappa_\mu(w) \nabla w) = f_\mu \ \text{ in } U, \quad w|_{\Gamma_{\mathrm{in}}} = g, \quad w|_{\partial U \setminus \Gamma_{\mathrm{in}}} = 0.$$

- Determine a low-dimensional approximation space for the manifold $\widehat{\mathcal{M}} = \{ T_\mu(g) : \ g \in G, \ \mu \in \mathcal{P} \}$.

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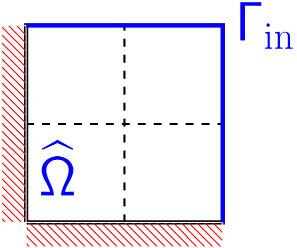
scRBE: Eftang, Patera 2013; Smetana, Patera, 2016. multiscale FE: Henning, Peterseim, 2013.

Oversampling (II): role of U, G

Input boundary $\Gamma_{\rm in}$ should be well-separated from $\widehat{\Omega}$ to ensure decay of high-frequency modes, **but** $|U| \ll |\Omega|$ to ensure computational speed-ups.

Example: corner



Set G should be representative of the behavior of $u_\mu|_{\Gamma_{\rm in}}$:

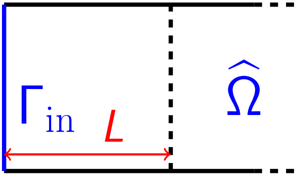
$\Rightarrow$ in absence of prior information, G is high-dimensional;

Question: why should the localized manifold $\widehat{\mathcal{M}} = \{T_\mu(g) : \; g \in G, \; \mu \in \mathcal{P}\}$ be reducible?

Motivating example: semi-infinite wave-guides (I)

Consider the problem for $g(x_2) = \sum_{n=1}^{\infty} c_n \cos(n\pi x_2)$

$$\begin{cases} -\Delta u_g - \mu^2 u_g = 0 \text{ in } U \\ \partial_{x_2} u_g = 0 \text{ on } \mathbb{R}_+ \times \{0,1\} \\ u_g = g \text{ on } \{0\} \times (0,1) \end{cases}$$



Define $\widehat{\Omega} = (L, \infty) \times (0,1)$, $C, \bar{\mu} \in \mathbb{R}_+$ and the extracted manifold $\mathcal{M}_L = \{u_g\big|_{\widehat{\Omega}} : \|g\|_{H^{1/2}} \leq C,\ \mu = \bar{\mu}\}$.

$G = \{g : \|g\|_{H^{1/2}} \leq C\}$ is infinite-dimensional and is irreducible.

Motivating example: semi-infinite wave-guides (II)

$$u_g = \underbrace{\sum_{n=1}^{N_{\mathrm{pr}}} c_n e^{-i\alpha_n x_1} \cos(n\pi x_2)}_{=(\mathrm{I})} + \underbrace{\sum_{n=N_{\mathrm{pr}}+1}^{\infty} c_n e^{-\alpha_n x_1} \cos(n\pi x_2)}_{=(\mathrm{II})},$$

with $N_{\mathrm{pr}} = \left\lfloor \frac{\mu}{\pi} \right\rfloor$, $\alpha_n = |n^2\pi^2 - \mu^2|$. (I) = propagating modes; (II) = evanescent modes.

The differential operator acts as a **low-pass filter**.

- Effect of evanescent modes is negligible far from Γ_{in}.
- Manifold $\mathcal{M}_L$ can be well-approximated by an n-dimensional space if $e^{-\alpha_{n+1} L} \ll 1$.

Filtering properties depend on the value of $\bar{\mu}$.

Localized training

- Oversampling
- Linear problems
- Randomization
- Numerical results

Define the transfer operator $T : \mathcal{U} \times \mathcal{P} \to \mathcal{Y}$. For second-order elliptic PDEs, $G \subset \mathcal{U} \subset H^{1/2}(\Gamma_{\mathrm{in}})$, $\mathcal{Y} \subset H^1(\widehat{\Omega})$; $(\mathcal{U}, \|\cdot\|_{\mathcal{U}})$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ Hilbert spaces.

We assume that $\mathcal{P} = \{\bar{\mu}\}$ and that T is **compact**.

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We assume that $\mathcal{P} = \{\bar{\mu}\}$ and that T is **compact**.

Proposition: Sufficient conditions for compactness of T:

- $\mathrm{dist}\left(\widehat{\Omega}, \Gamma_{\mathrm{in}}\right) > 0$;
- $\| T(g) \|_{H^1(U')} \leq C(U, U') \| T(g) \|_{L^2(U)}$ for $U' \Subset U$.

Caccioppoli's ineq.

Caccioppoli's inequality is satisfied by a broad class of elliptic problems (Helmholtz, advection-diffusion, Stokes).

Babuska, Lipton, 2011. Taddei, Patera, 2018.

Transfer eigenproblem (TE)

Introduce the **transfer eigenproblem**:

$$(\varphi_i, \lambda_i) \in \mathcal{U} \times \mathbb{R}_+ \;\; (T(\varphi_i),\; T(g))_{\mathcal{Y}} = \lambda_i (\varphi_i, g)_{\mathcal{U}} \;\; \forall\, g \in \mathcal{U}$$

with $i = 1, 2, \ldots, \;\; \lambda_1 \geq \lambda_2 \geq \ldots$

Define the **transfer eigenspace** $\mathcal{Z}_n^{\mathrm{te}} = \mathrm{span}\{T(\varphi_i)\}_{i=1}^n$.

Transfer eigenproblem (TE): properties (I)

Introduce the **transfer eigenproblem**:

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with $i = 1, 2, \ldots, \quad \lambda_1 \geq \lambda_2 \geq \ldots$

Define the **transfer eigenspace** $\mathcal{Z}_n^{\mathrm{te}} = \mathrm{span}\{T(\varphi_i)\}_{i=1}^n$.

If $\mathcal{U} = \mathrm{span}\{\phi_k\}_{k=1}^{N_{\mathrm{in}}}$, then $\varphi_i = \sum_{k=1}^{N_{\mathrm{in}}} (\boldsymbol{\varphi}_i)_k \phi_k$, $\mathbf{A}\boldsymbol{\varphi}_i = \lambda_i \mathbf{B}\boldsymbol{\varphi}_i$

with $\mathbf{A}_{k,k'} = (T(\phi_k), T(\phi_{k'}))_{\mathcal{Y}}$, $\mathbf{B}_{k,k'} = (\phi_k, \phi_{k'})_{\mathcal{U}}$.

- Solution to TE requires computation of $\{T(\phi_k)\}_k$.
- If $\{\phi_k\}_k$ is orthonormal, then $\mathcal{Z}_n^{\mathrm{te}}$ is the POD space associated with the snapshots $\{T(\phi_k)\}_k$.

Transfer eigenproblem: properties (II)

Optimality: (Pinkus, 1985)

$$\mathcal{Z}_n^{\mathrm{te}} \in \arg \inf_{\mathcal{W}\subset\mathcal{Y},\, \dim(\mathcal{W})=n} \sup_{g\in\mathcal{U}} \frac{\|\Pi^{\mathcal{Y}}_{\mathcal{W}^\perp} T(g)\|_{\mathcal{Y}}}{\|g\|_{\mathcal{U}}}$$

- The transfer eigenspace is optimal in the sense of Kolmogorov.

Error analysis: (Taddei, Patera, 2018)

$$\frac{\|\Pi^{\mathcal{Y}}_{(\mathcal{Z}_n^{\mathrm{te}})^\perp} T(g)\|_{\mathcal{Y}}}{\|g\|_{\mathcal{U}}} \leq \|T\|_{\mathcal{L}(\mathcal{U}^\perp;\mathcal{Y})} \frac{\|\Pi^{\mathcal{U}}_{\mathcal{U}^\perp} g\|_{\mathcal{U}}}{\|g\|_{\mathcal{U}}} + \sqrt{\lambda_{n+1}}$$

for any $g \in H^{1/2}(\Gamma_{\mathrm{in}})$.

- If $g \notin \mathcal{U}$, performance of $\mathcal{Z}_n^{\mathrm{te}}$ depends on the product $\|T\|_{\mathcal{L}(\mathcal{U}^\perp;\mathcal{Y})} \|\Pi^{\mathcal{U}}_{\mathcal{U}^\perp} g\|_{\mathcal{U}}$.

TE + POD: given $\mathcal{U} = \mathrm{span}\{\phi_i\}_{i=1}^{N_{\mathrm{in}}}$, $\{\mu^k\}_{k=1}^{n_{\mathrm{train}}}$, $n \in \mathbb{N}$

`For` $k = 1, \ldots, n_{\mathrm{train}}$

Compute $\mathcal{Z}_{n,k}^{\mathrm{te}} := \mathrm{span}\{T_{\mu^k}\varphi_i^k\}_{i=1}^{n}$ s.t. $(\varphi_i^k, \varphi_j^k)_{\mathcal{U}} = \delta_{i,j}$.

`EndFor`

Return $\mathcal{Z}_n := \mathrm{POD}\left(\{T_{\mu^k}\varphi_i^k\}_{i,k}, n\right)$.
$i = 1, \ldots, n, k = 1, \ldots, n_{\mathrm{train}}$

Taddei, Patera, SISC, 2018.

Extension to the parametric case

TE + POD: given $\mathcal{U} = \mathrm{span}\{\phi_i\}_{i=1}^{N_{\mathrm{in}}}$, $\{\mu^k\}_{k=1}^{n_{\mathrm{train}}}$, $n \in \mathbb{N}$

`For` $k = 1, \ldots, n_{\mathrm{train}}$

Compute $\mathcal{Z}_{n,k}^{\mathrm{te}} := \mathrm{span}\{T_{\mu^k}\varphi_i^k\}_{i=1}^n$ s.t. $(\varphi_i^k, \varphi_j^k)_{\mathcal{U}} = \delta_{i,j}$.

`EndFor`

Return $\mathcal{Z}_n := \mathrm{POD}\left(\{T_{\mu^k}\varphi_i^k\}_{i,k}, n\right)$.

$i = 1, \ldots, n, k = 1, \ldots, n_{\mathrm{train}}$

- Equivalent to hierarchical approximate POD.

  Himpe, Leibner, Rave, 2018.

- Alternative approach: TE + strong Greedy in parameter.

  Smetana, Patera 2016.

Taddei, Patera, SISC, 2018.

Localized training

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TE + POD requires $N_{\mathrm{in}} \cdot n_{\mathrm{train}}$ PDE solves.
Possible fix: Krylov subspace methods for the TE.

Randomized methods offer an alternative strategy to tackle the problem.

Mahoney 2016: *randomness is an algorithmic resource creating efficient, unbiased approximations of nonrandom operations.*

Key features of randomized algorithms:

- Incremental (iterative) approach.
- Probabilistic error estimation.

Randomized SVD: Mahoney, Martinsson, Tropp... Randomized model reduction: Buhr & Smetana; Smetana, Zahm, Patera.

Adaptive randomized algorithm

Inputs: $\texttt{maxit}, r_{\mathrm{train}}, n \in \mathbb{N}$, pdfs p_μ, p_G (sampling);

Output: $\mathcal{Z}_n = \mathrm{span}\{\zeta_i\}_{i=1}^n$ reduced space. $(\zeta_i, \zeta_j) = \delta_{i,j}$

Set $\mathcal{Z}_n = \emptyset$, $\lambda_1 = \ldots = \lambda_n = 0$.

$\texttt{For}$ $i = 1, \ldots, \texttt{maxit}$

Generate $\mu^{(k)} \overset{\mathrm{iid}}{\sim} p_\mu$, $g^{(k)} \overset{\mathrm{iid}}{\sim} p_G$, $k = 1, \ldots, r_{\mathrm{train}}$.

Compute $u^{k,i} = T_{\mu^{(k)}} g^{(k)}$ for $k = 1, \ldots, r_{\mathrm{train}}$.

Compute $\widehat{E} = \dfrac{1}{r_{\mathrm{train}}} \sum_{k=1}^{r_{\mathrm{train}}} \|\Pi_{\mathcal{Z}_n^\perp} u^{k,i}\|_{\mathcal{Y}} / \|u^{k,i}\|_{\mathcal{Y}}$.

if $\widehat{E} \leq tol$, $\texttt{BREAK}$

else $[\mathcal{Z}_n, \{\lambda_j\}_j] = \mathrm{POD}\left(\left\{\zeta_j \sqrt{\lambda_j}\right\}_{j=1}^n \cup \left\{u^{k,i}\right\}_{k=1}^{r_{\mathrm{train}}}, n\right)$

$\texttt{EndFor}$

Adaptive randomized algorithm: comments

Interpretation: incremental randomized POD in parameter and boundary data. Yu, Chakravorty, 2015

- Snapshots $\{u^{k,i}\}_{k,i}$ are used for testing and then to update the reduced space.
- $[\mathcal{Z}_n, \{\lambda_j\}_j] = \mathrm{POD}\left(\{\zeta_j \sqrt{\lambda_j}\}_{j=1}^{n} \cup \{u^{k,i}\}_{k=1}^{r_{\mathrm{train}}}, n\right)$
  - Snapshots from previous iterations are not stored.
  - Factors $\{\sqrt{\lambda_j}\}_j$ serve to properly fuse information from different iterations. Himpe, Leibner, Rave 2018
- $\widehat{E}$ is a MC estimate of $\mathbb{E}_{\mu\sim p_\mu, g\in p_G}\left[\frac{\|\Pi_{\mathcal{Z}_n^\perp} T_\mu(g)\|_{\mathcal{Y}}}{\|T_\mu(g)\|_{\mathcal{Y}}}\right]$

For linear problems, see also Buhr, Smetana, 2018.

Randomized Greedy: Cohen et al, M2AN, 2020.

Idea: since we expect solution to be smooth, we control the smoothness of the samples.

Consider $g_{\mathrm{co}}(s; \mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}}) = \sum_{k=0}^{N_{\mathrm{f}}-1} \frac{c_k^{\mathrm{re}} + \mathtt{i}\, c_k^{\mathrm{im}}}{\sqrt{1 + (2\pi k)^{2\alpha}}} e^{2\pi k s \mathtt{i}}$,

with $c_k^{\mathrm{re}}, c_k^{\mathrm{im}} \overset{\mathrm{iid}}{\sim} \mathcal{N}(0,1)$. Then, if $\alpha \in \mathbb{N}$

$\|g_{\mathrm{co}}(\cdot; \mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}})\|^2_{L^2(0,1)} + \|g_{\mathrm{co}}^{(\alpha)}(\cdot; \mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}})\|^2_{L^2(0,1)} \sim \chi^2(2N_{\mathrm{f}})$

Choice of the boundary distribution

Idea: since we expect solution to be smooth, we control the smoothness of the samples.

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with $c_k^{\mathrm{re}}, c_k^{\mathrm{im}} \overset{\mathrm{iid}}{\sim} \mathcal{N}(0,1)$. Then, if $\alpha \in \mathbb{N}$

$$\|g_{\mathrm{co}}(\cdot; \mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}})\|_{L^2(0,1)}^2 + \|g_{\mathrm{co}}^{(\alpha)}(\cdot; \mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}})\|_{L^2(0,1)}^2 \sim \chi^2(2N_{\mathrm{f}})$$

Proposal: $g(\cdot; \mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}}) = \mathtt{Real}\left[g_{\mathrm{co}}(\cdot; \mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}})\right].$

For edge and corner components, we need to enforce Dirichlet boundary conditions. details omitted

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Linear advection-diffusion-reaction equation

Pb: given $\mu \in \mathcal{P}$, $g \in H^{1/2}(\Gamma_{\text{in}})$, $\kappa(x) = 1 + \|x\|_2^2$, $U = [0, 0.3]^2$, $\mathcal{P} = [0.2, 1] \times [-1, 1]^2 \times [0, 1]$, find u_μ s.t.

$$\begin{cases} -\nabla \cdot \left(\mu_1 \kappa^{-1} \nabla u_\mu + [\mu_2, \mu_3] u_\mu\right) + \mu_4 u_\mu = 0 \text{ in } U \\ u_\mu\big|_{\partial U = \Gamma_{\text{in}}} = g, \end{cases}$$

We consider the extracted domain $\widehat{\Omega} = [0.1, 0.2]^2$.

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We consider the extracted domain $\widehat{\Omega} = [0.1, 0.2]^2$.

Description of the test

- P3 FE with $N_{\text{in}} = 360$ dofs on Γ_{in}.

- TE+POD based on $n_{\text{train}} = 100$.

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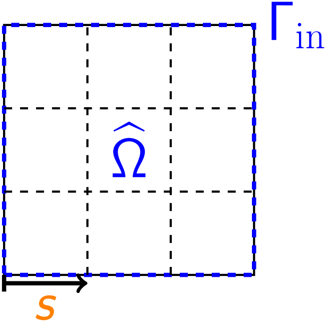
$$\begin{cases} -\nabla \cdot \left(\mu_1 \kappa^{-1} \nabla u_\mu + [\mu_2, \mu_3] u_\mu\right) + \mu_4 u_\mu = 0 \text{ in } U \\ u_\mu\big|_{\partial U = \Gamma_{\rm in}} = g, \end{cases}$$

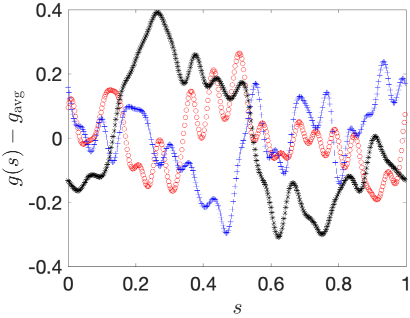
We consider the extracted domain $\widehat{\Omega} = [0.1, 0.2]^2$.

Description of the test

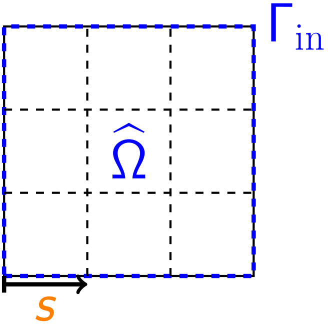
- Maximum relative projection error $E_{\rm max,rel}$ based on $n_{\rm test} = 100$ samples $\{T_{\mu^{(k)}} g^{(k)}\}_k$ with $g^{(k)} \overset{\rm iid}{\sim} p_G^{\rm smooth}(\alpha = 1)$, $\mu^{(k)} \overset{\rm iid}{\sim} \mathrm{Uniform}(\mathcal{P})$.
- Randomized training with $r_{\rm train} = 10$, $g^{(k)} \overset{\rm iid}{\sim} p_G^{\rm smooth}$, $\mu^{(k)} \overset{\rm iid}{\sim} \mathrm{Uniform}(\mathcal{P})$.

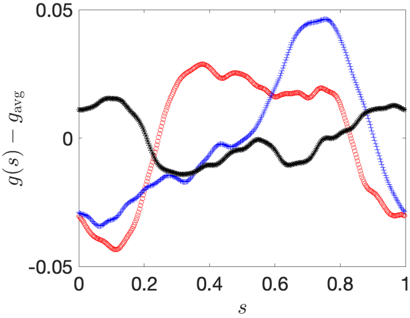
Samples of the datum on Γ_{in} for $\alpha = 1$.



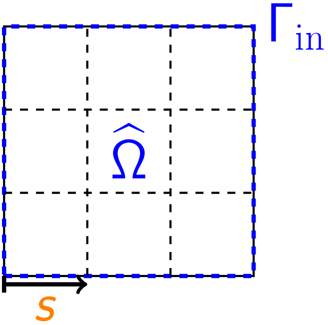


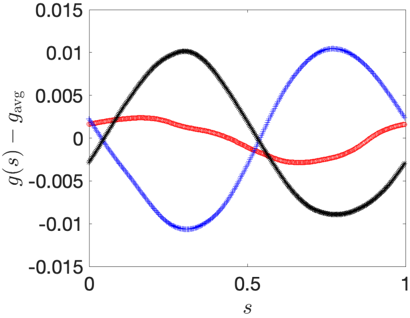
Samples of the datum on $\Gamma_{\rm in}$ for $\alpha = 2$.





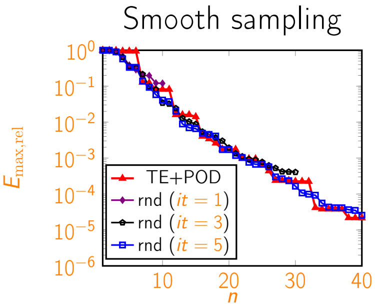
Samples of the datum on Γ_{in} for $\alpha = 3$.

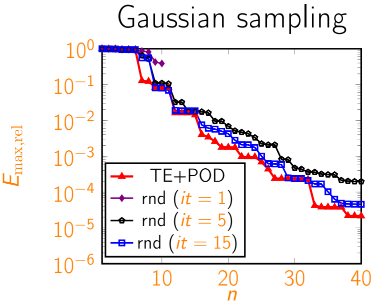




Out-of-sample performance

Gaussian sampling: $g \sim p_G^{\mathrm{gauss}} \Leftrightarrow g(x) = \sum_{i \in I_{\mathrm{dir}}} c_i \phi_i^{\mathrm{fe}}(x)$





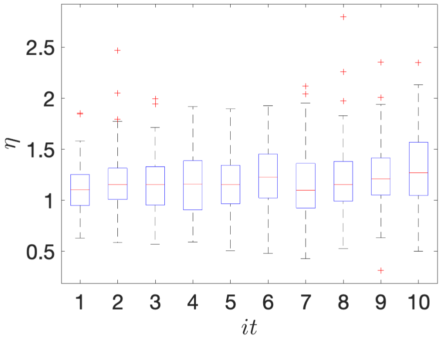
- Randomized training performs similarly to TE+POD after 5 iterations.
- Smooth samping is (slightly) more effective than Gaussian sampling at early iterations.

We compare estimates $\mathbb{E}_{\mu\sim p_\mu, g\in p_G}\left[\frac{\|\Pi_{\mathcal{Z}_n^\perp} T_\mu(g)\|_{\mathcal{Y}}}{\|T_\mu(g)\|_{\mathcal{Y}}}\right]$ computed during the iterations of the randomized algorithm with the estimates on the $n_{\text{test}} = 100$ datapoints.

Randomized training based on smooth sampling ($\alpha = 1$).

Effectivity $\eta = \frac{\widehat{E}_{r_{\text{train}}=10}}{\widehat{E}_{n_{\text{test}}=100}}$.

Results over 100 independent runs.



Randomized sampling for nonlinear PDEs

- Numerical results

Towards the extension to nonlinear PDEs

Objective: extend the localized training procedure to nonlinear PDEs.

- Error estimation and oversampling can be extended *as is*.
- How should we choose p_G?

Theoretical questions

- Compactness of the solution manifold[1].

Caccioppoli's ineq.

- Constructive optimal approximations[2]

[1] Compactness of the transfer operator is key for approximability of the nonlinear set of functions, and ultimately justifies oversampling.

[2] Generalization of transfer eigenproblems to nonlinear PDEs.

Choice of p_G for the model problem

Model pb: $-\nabla \cdot (\kappa_\mu(u_\mu)\nabla u_\mu) = f_\mu \text{ in } \Omega,\ u_\mu\big|_{\partial\Omega} = 0$

$$\kappa_\mu(u)\big|_{\Omega_i} = \frac{36}{\mu_2^{(i)}}\left(\frac{u(1-u)}{u^3 + \frac{12}{\mu_2^{(i)}}(1-u)^3}\right)^2 + \mu_1^{(i)},\ i = 1, 2, \ldots$$

$$f_\mu(x) = 100e^{\left(-100\frac{\|x - x_{\mathrm{c,i^\star}}\|_2^2}{2}\right)}\mathbb{1}_{\Omega_i^\star}(x),\quad x_{\mathrm{c,i^\star}} := \frac{1}{\Omega_i^\star}\int_{\Omega_i^\star} x\,dx.$$

Obs (I): $u_\mu \in [0,1) \Rightarrow$ we assume that $g(x) \in [0, \bar{u}_{\max}]$

Obs (II): $u_\mu|_{\Gamma_{\mathrm{in}}}$ is expected to be smooth (at least $H^{1/2}$) .

$\Rightarrow$ smooth sampling.

Choice of p_G: internal component

We consider the following sampling algorithm:

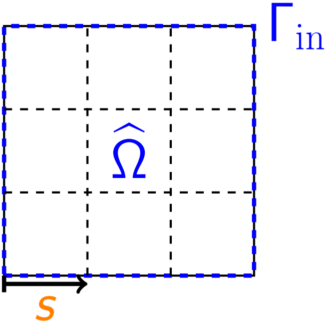
1. Draw $\mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}} \in \mathbb{R}^{N_{\mathrm{f}}}$ s.t. $c_k^{\mathrm{re}}, c_k^{\mathrm{im}} \overset{\mathrm{iid}}{\sim} \mathcal{N}(0,1)$.
2. Draw $X_1, X_2 \overset{\mathrm{iid}}{\sim} \mathrm{Uniform}(0, \bar{u}_{\max})$, set $a = \min\{X_1, X_2\}$, $b = \max\{X_1, X_2\}$.
3. Set $g' = \mathtt{Real}\left[g_{\mathrm{co}}(\cdot; \mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}})\right]$.
4. Set $g = a + \dfrac{b-a}{\max g' - \min g'}(g' - \min g')$.

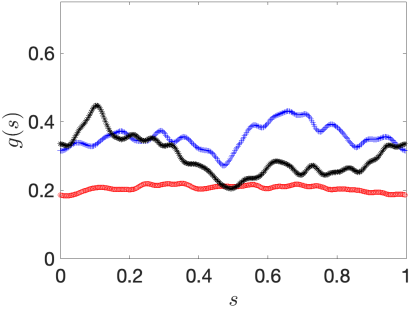
Memo: $g_{\mathrm{co}}(s; \mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}}) = \displaystyle\sum_{k=0}^{N_{\mathrm{f}}-1} \frac{c_k^{\mathrm{re}} + \mathtt{i}\, c_k^{\mathrm{im}}}{\sqrt{1 + (2\pi k)^{2\alpha}}}\, e^{2\pi k s \mathtt{i}},.$

Choice of p_G: internal component (visualization)

- p_G encodes the fact that $Im[g] \subset [0, \bar{u}_{\max}]$.
- α encodes prior knowledge on the regularity of g.

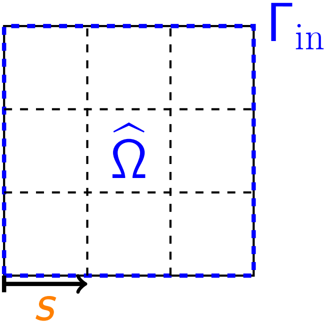
Samples of the datum on Γ_{in} for $\alpha = 1$ ($\bar{u}_{\max} = 0.5$).

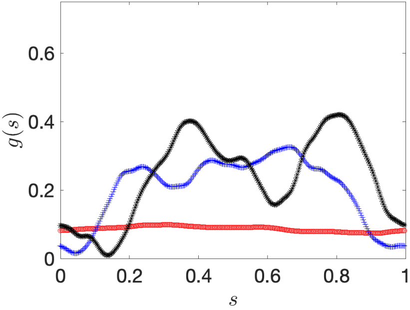




- p_G encodes the fact that $Im[g] \subset [0, \bar{u}_{\max}]$.
- α encodes prior knowledge on the regularity of g.

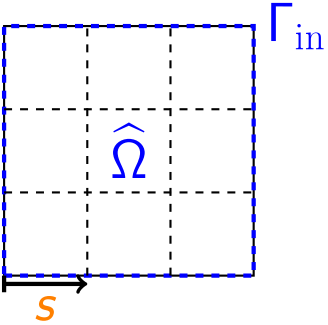
Samples of the datum on Γ_{in} for $\alpha = 2$ ($\bar{u}_{\max} = 0.5$).

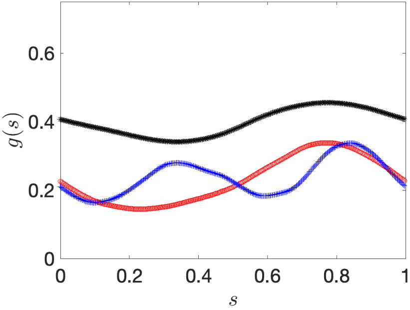




- p_G encodes the fact that $Im[g] \subset [0, \bar{u}_{\max}]$.
- α encodes prior knowledge on the regularity of g.

Samples of the datum on Γ_{in} for $\alpha = 3$ ($\bar{u}_{\max} = 0.5$).



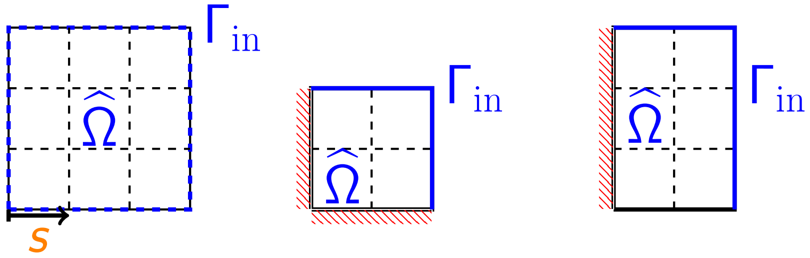


Randomized sampling for nonlinear PDEs

- Numerical results

We perform localized training based on $n_{\mathrm{train}} = 10^2$ simulations for each component to build $\mathcal{Z}^{\mathrm{co}}, \mathcal{Z}^{\mathrm{int}}, \mathcal{Z}^{\mathrm{ed}}$.

Oversampling domains: each subregion is characterized by a value of $\mu \in \mathcal{P}$; source term is activated with probability $p = 0.5$.

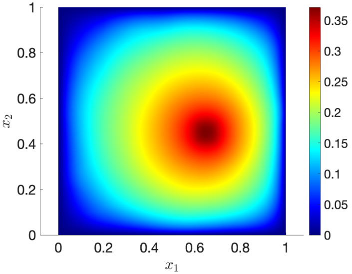


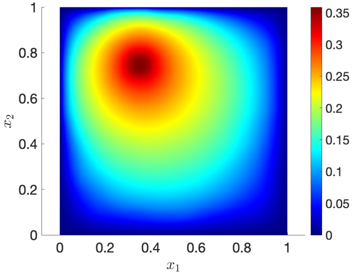
We compute n_{test} global solutions for N_{dd} subdomains and we extract solution in each element.

$$n_{\text{test}} = 15,\ N_{\text{dd}} = 100,\ \Omega = (0,1)^2$$

We use the datasets of local solutions to measure performance of localized training .

$$n_{\text{test}}^{\text{int}} = 960,\ n_{\text{test}}^{\text{co}} = 60,\ n_{\text{test}}^{\text{ed}} = 480$$





Error measures

- Average H^1 relative projection error at component level.
- Average L^2 relative global projection error.

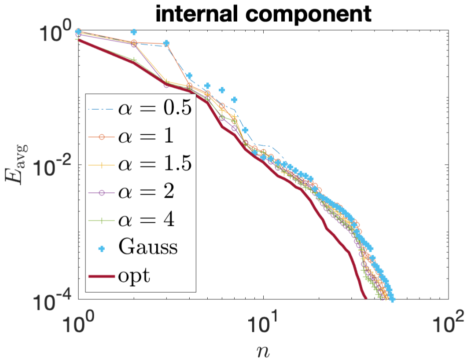
We compare performance of

- smooth sampling for various α;
- Gaussian sampling with clipping $\max\left(\min\left(g, \bar{u}_{\max}\right), 0\right)$;
- **benchmark:** POD space based on test set "`opt`" [3].

[3] Not computable for practical applications due to the impossibility to perform global solves.

Results: local errors for non-overlapping domains

- Randomized sampling provides accurate spaces for moderate n.
- Gaussian sampling is inaccurate for edge and corner components.
- Performance is nearly independent of α.



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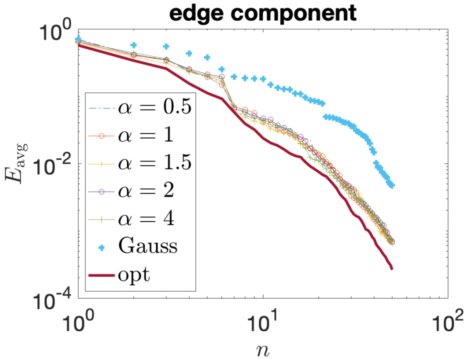

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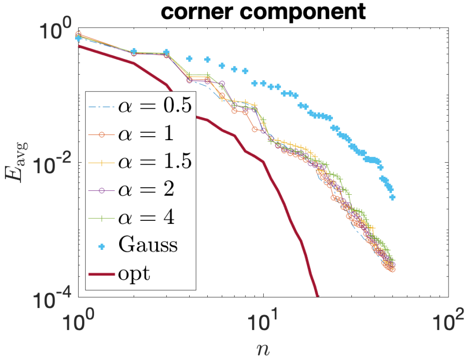
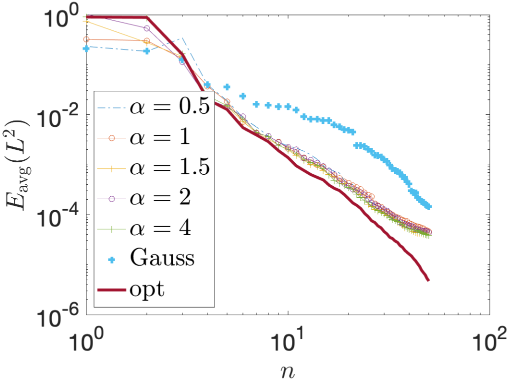


Relative error is below 10^{-4} for $n = 30$.

Total number of degrees of freedom: ROM $n \cdot 100$, FOM 90601.



Conclusions

Summary

Component-based model reduction offers the opportunity to address large-scale parameterized problems.

topology changes, high-dimensional parameterization

Tasks: localized training and domain decomposition.

Summary

Component-based model reduction offers the opportunity to address large-scale parameterized problems.

topology changes, high-dimensional parameterization

Tasks: localized training and domain decomposition.

In this talk, we discussed localized training for steady PDEs:

- transfer operator and optimal approximations;
- randomized training based on random samples of BCs;
- probabilistic error estimation.

Ongoing work and perspectives

Extension to nonlinear problems requires major advances to state-of-the-art methods.

Localized training

- Problem-aware sampling distribution of BCs.
- Online basis enrichment. Schlinder, Ohlberger, 2015

Domain decomposition

- Partition-of-unity overlapping methods. Melenk, Babuska, 1996
- Hyper-reduction.

Thank you for your attention!

For more information, visit the website:

math.u-bordeaux.fr/~ttaddei/ .

Backup slides

- More plots

Backup slides

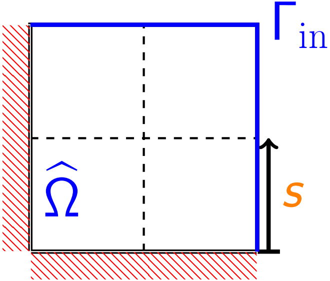
- More plots

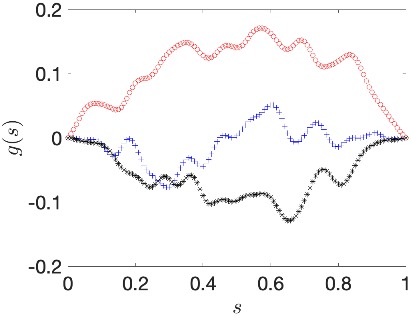
Choice of the boundary distribution: visualization (corner)

$$g(s;\mathbf{c}^{\mathrm{re}},\mathbf{c}^{\mathrm{im}}) \;=\; \texttt{Real}\left[g_{\mathrm{co}}(0.7\cdot s;\mathbf{c}^{\mathrm{re}},\mathbf{c}^{\mathrm{im}})\right]s(1-s).$$

- g_{co} is not periodic in $(0,1)$.
- $g_{\mathrm{co}}(s)=0$ for $s\in\{0,1\}$.

Samples of the datum on Γ_{in} for $\alpha=1$.





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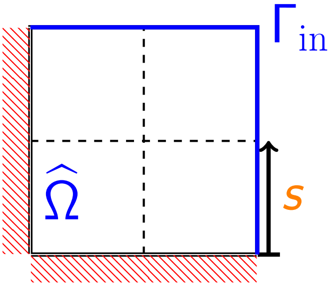
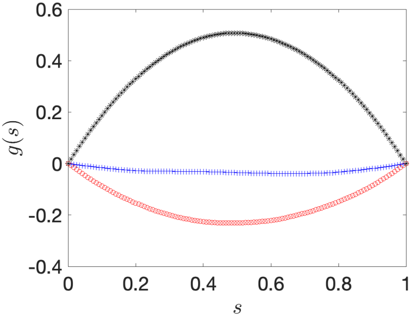
Samples of the datum on Γ_{in} for $\alpha=2$.





Choice of the boundary distribution: visualization (corner)

$$g(s; \mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}}) = \texttt{Real}\left[g_{\mathrm{co}}(0.7 \cdot s; \mathbf{c}^{\mathrm{re}}, \mathbf{c}^{\mathrm{im}})\right] s(1-s).$$

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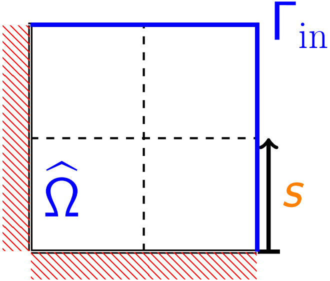
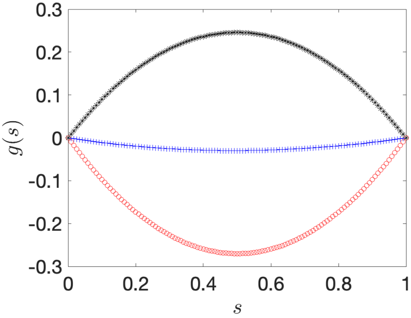
Samples of the datum on Γ_{in} for $\alpha = 3$.

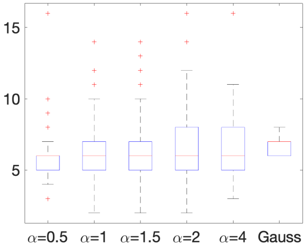




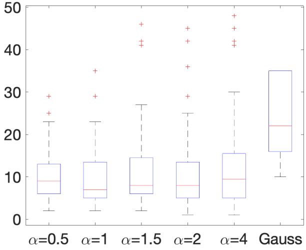
For $\bar{u}_{\max} = 0.5$, Newton solver with line search converges for all training points.

For $\bar{u}_{\max} = 0.75$, Newton solver with line search fails to converge 1% of the times for smooth sampling $\alpha = 0.5$ and 30% of the times for Gaussian sampling.

Results for the edge component:



$\bar{u}_{\max} = 0.5$



$\bar{u}_{\max} = 0.75$