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École des Ponts
ParisTech

The Cauchy problem treated as a convex optimization problem

Yann Brenier (CNRS, DMA-ENS, PSL)

5 mars 2018

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Yann Brenier
CNRS, DMA-ENS, PSL
en association avec le projet CNRS-INRIA "MOKAPLAN"

COLLOQUIUM,
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$$\partial_t U + \nabla \cdot (F(U)) = 0, \quad U = U(t,x) \in \mathcal{W} \subset \mathbb{R}^m,$$

$$t > 0,\ x \in \mathbb{R}^d,\ \nabla = (\frac{\partial}{\partial x_i})_{i=1}^d, \quad \text{such that } \sum_{\beta=1}^m \partial_\beta \mathcal{E}(W) \partial_\alpha F^{i\beta}(W) = \partial_\alpha Q^i(W),\ \forall W \in \mathcal{W},$$

for some smooth pair $(\mathcal{E}, Q) : \mathcal{W} \to \mathbb{R}^{1+d}$ (with $\mathcal{W}$ open convex and $\mathcal{E}$, called "entropy", strictly convex), which implies $\partial_t(\mathcal{E}(U)) + \nabla \cdot (Q(U)) = 0$, for all smooth solutions U.

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A famous example: the equations of an isothermal gas (Euler, 1755)

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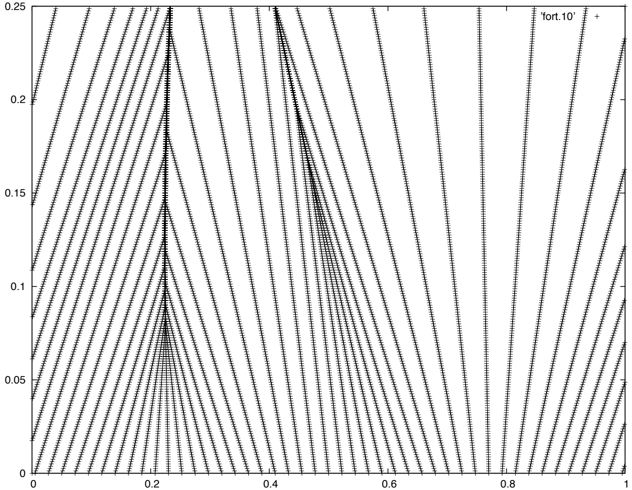
Typically, these systems are locally well-posed, with generic formation of shock waves.



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The problem is not trivial since there may be many weak solutions starting from U_0 which are not entropy-preserving (by "convex integration" à la De Lellis-Székelyhidi).

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$$\inf_{U} \sup_{A} \int_{[0,T]\times\mathbb{T}^d} \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U)$$

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N.B. The supremum in A exactly encodes that U is a weak solution with initial condition U_0, all test functions A acting like Lagrange multipliers.

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leads to a *concave* maximization problem in A, namely

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$$= \sup_A \int_{[0,T]\times\mathbb{T}^d} -K(\partial_t A, \nabla A) - \int_{\mathbb{T}^d} A(0,\cdot)\cdot U_0$$

$$K(E,B) = \sup_{V\in\mathcal{W}\subset\mathbb{R}^m} E\cdot V + B\cdot F(V) - \mathcal{E}(V),\ \ (E,B)\in\mathbb{R}^m\times\mathbb{R}^{m\times d}.$$

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Notice that K is automatically convex.

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For instance, if K were just a square, we would get

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Answer: in our construction, K is convex *degenerate*!

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(*) more precisely if, $\forall\ t, x, V \in \mathcal{W}$, $\mathcal{E}''(V) - (T-t)F''(V) \cdot \nabla(\mathcal{E}'(U(t,x))) > 0$, which requires, in particular, $\mathcal{E}''(V) - TF''(V) \cdot \nabla(\mathcal{E}'(U_0(x))) > 0$.

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...But what about shocks and large T???

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Then, the maximization problem in A simply reads

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with $A = A(t,x) \in \mathbb{R}$ subject to $A(T,\cdot) = 0, \ \ \partial_x A \le 1.$

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Introducing $\rho = 1 - \partial_x A \geq 0$, $q = \partial_t A \in \mathbb{R}$, this problem is equivalent to

$$-\inf\{\int_{[0,T]\times\mathbb{T}} \frac{q^2}{2\rho} - qu_0, \;\; (\rho \geq 0, q) \text{ subject to } \partial_t\rho + \partial_x q = 0, \;\; \rho(T,\cdot) = 1\}.$$

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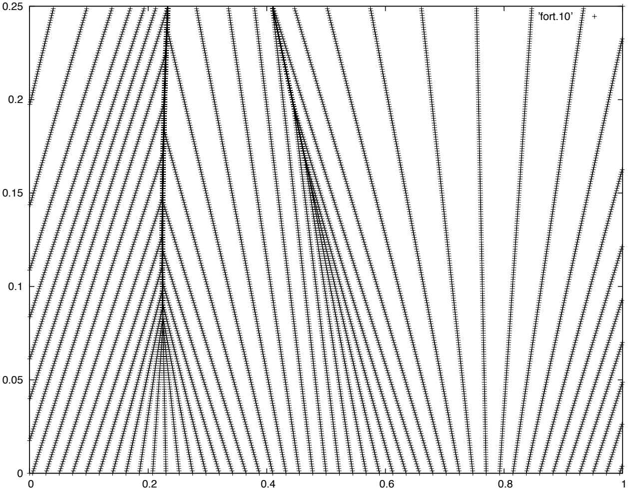
i.e. the "ballistic" version (à la Ghoussoub) of the optimal transport problem with quadratic cost (à la Benamou-B.), and, as well, an elementary "mean-field game".



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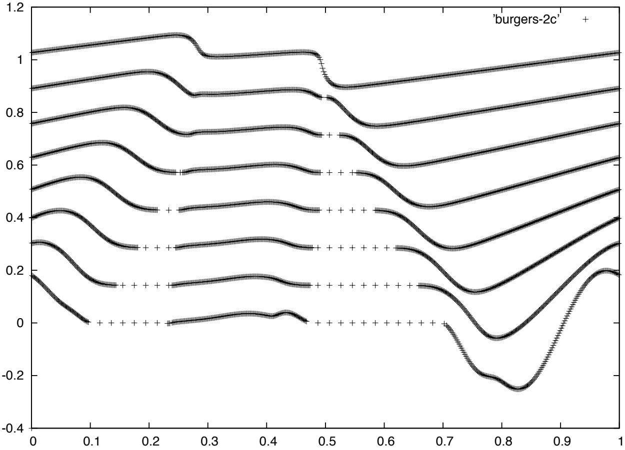
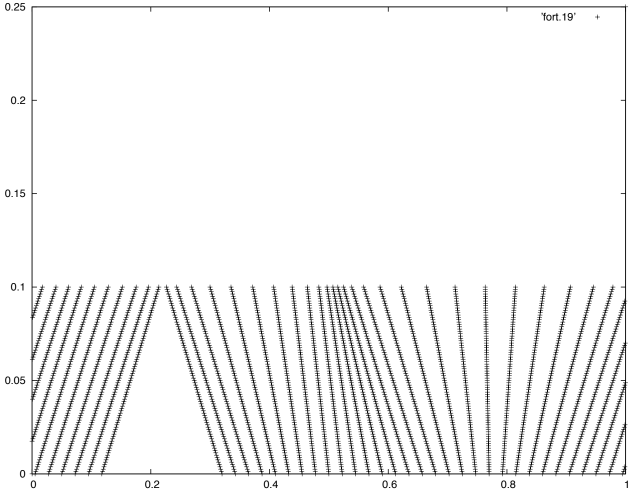


Numerics: 2 lines of code differ from a standard (Benamou-B.) OT solver!



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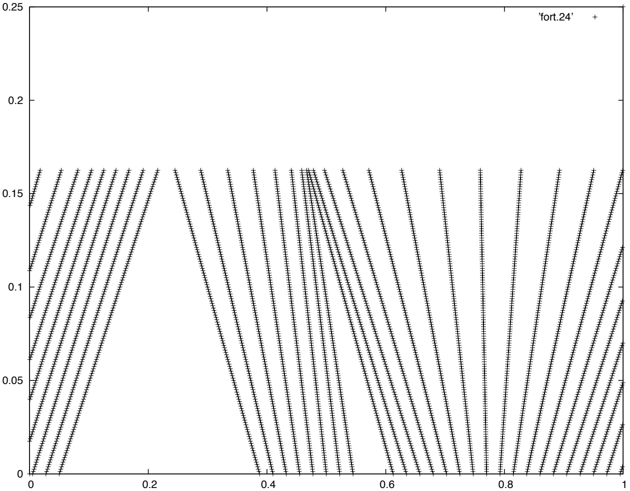
Recovery of the solution at time T=0.1 by convex optimization.
Observe the formation of a first vacuum zone as the first shock has formed.



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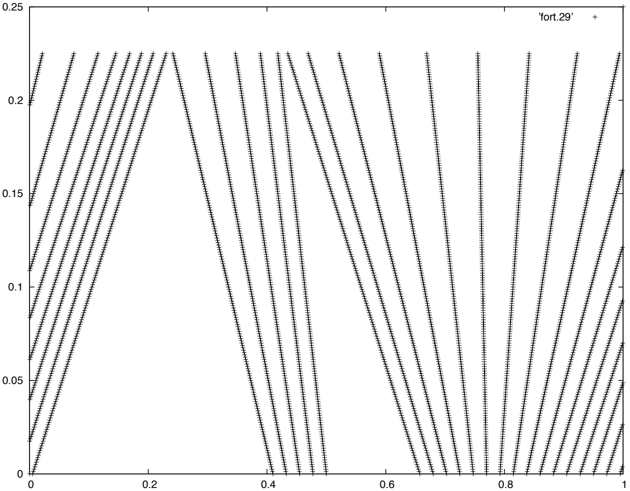
Recovery of the solution at time T=0.16 by convex optimization.
Observe the formation of a second vacuum zone as the second shock has formed.



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Recovery of the solution at time T=0.225 by convex optimization.
Observe the extension of the two vacuum zones.

Analogy with mountain climbing:
going from Everest to Lhotse without following the crest! (Credit to Th. Gallouët.)

Merci de votre attention!

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Pour plus de détails, voir Y.B. ArXiv Oct. 2017.