The Cauchy problem treated as a convex optimization problem

Yann Brenier (CNRS, DMA-ENS, PSL)

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en association avec le projet CNRS-INRIA "MOKAPLAN"

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An important class of evolution PDEs
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\[ \partial_t U + \nabla \cdot (F(U)) = 0, \quad U = U(t, x) \in \mathcal{W} \subset \mathbb{R}^m, \]

for some smooth pair \((\mathcal{E}, Q) : \mathcal{W} \to \mathbb{R}^{1+d}\) (with \(\mathcal{W}\) open convex and \(\mathcal{E}\), called "entropy", strictly convex), which implies \(\partial_t (\mathcal{E}(U)) + \nabla \cdot (Q(U)) = 0\), for all smooth solutions \(U\).
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A famous example: the equations of an isothermal gas (Euler, 1755)

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Typically, these systems are locally well-posed, with generic formation of shock waves.
Inviscid Burgers equation: \( \partial_t u + \partial_x (u^2/2) = 0 \), \( u = u(t, x), \ x \in \mathbb{R}/\mathbb{Z}, \ t \geq 0 \).
Formation of two shock waves. (Vertical axis: \( t \in [0, 1/4] \), horizontal axis: \( x \in \mathbb{T} \).)
A variational approach to the Cauchy problem

Given $U_0$ on $T^d = \mathbb{R}^d / \mathbb{Z}^d$ and $T > 0$, we minimize the entropy among all weak solutions $U$ of the Cauchy pb:

$$\inf_{U} \int_{[0,T] \times \mathbb{T}^d} E(U) \, dx \, dt,$$

subject to

$$\frac{\partial}{\partial t} A \cdot U + \nabla F(U) + \int_{T^d} A(0) \cdot U_0 = 0$$

for all smooth $A = A(t,x)$ with $A(T,\cdot) = 0$.

The problem is not trivial since there may be many weak solutions starting from $U_0$ which are not entropy-preserving (by "convex integration" à la De Lellis-Székelyhidi).
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Given $U_0$ on $\mathbb{T}^d = \mathbb{R}^d / \mathbb{Z}^d$ and $T > 0$, we minimize the entropy among all weak solutions $U$ of the Cauchy pb:

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\inf_{U} \mathbb{Z}_{[0, T]} \left[ \mathbb{Z} \cdot \left[ \frac{\partial}{\partial t} A \cdot U + r \cdot A \cdot F(U) + \mathbb{Z} \cdot A \left( 0, \cdot \right) \cdot U_0 \right] = 0 \right] \\
\text{subject to } \mathbb{Z}_{[0, T]} \cdot A \cdot U \in \mathcal{W}^\infty_{\mathbb{R}^{m}} \\
\text{for all smooth } A = A(t, x) \in \mathbb{R}^{m} \text{ with } A(T, \cdot) = 0.
\]
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The resulting saddle-point problem

\[
\begin{align*}
\inf_{U} & \sup_{A} Z \left[ 0, T \right] \cdot T dE(U) \\
& A \cdot U \cdot F(U) \\
& Z T dA \left[ 0, \cdot \right] \cdot U_0
\end{align*}
\]

where \( A = A(t, x) \) is smooth with \( A(T, \cdot) = 0 \). Here \( U_0 \) is the initial condition and \( T \) the final time.

N.B. The supremum in \( A \) exactly encodes that \( U \) is a weak solution with initial condition \( U_0 \), all test functions \( A \) acting like Lagrange multipliers.
The resulting saddle-point problem

\[
\inf_U \sup_A \int_{[0,T] \times \mathbb{T}^d} \mathcal{E}(U) - \partial_t A \cdot U - \nabla A \cdot F(U) \\
- \int_{\mathbb{T}^d} A(0, \cdot) \cdot U_0
\]

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leads to a concave maximization problem in $A$, namely

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$$

$$
= \sup_A \int_{[0, T] \times \mathbb{T}^d} -K(\partial_t A, \nabla A) - \int_{\mathbb{T}^d} A(0, \cdot) \cdot U_0
$$

$$
K(E, B) = \sup_{V \in \mathcal{W} \subset \mathbb{R}^m} E \cdot V + B \cdot F(V) - \mathcal{E}(V), \ (E, B) \in \mathbb{R}^m \times \mathbb{R}^{m \times d}.
$$
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leads to a \textit{concave} maximization problem in $A$, namely

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$$

Notice that $K$ is automatically convex.
Here is the paradox!

How a convex optimization problem could be compatible with a well-posed evolution problem?

\[ \sup_{A \in [0, T]} \langle \frac{\partial}{\partial t} A \rangle^{2} \| r_{A} \|^{2} \leq \int_{0}^{T} d A(0, \cdot) \cdot U_{0} \]

Answer: in our construction, \( K \) is convex degenerate!
Here is the paradox!

How a convex optimization problem could be compatible with a well-posed evolution problem? For instance, if $K$ were just a square, we would get

$$\sup_A \int_{[0,T] \times \mathbb{T}^d} -|\partial_t A|^2 - |\nabla A|^2 - \int_{\mathbb{T}^d} A(0, \cdot) \cdot U_0$$

which would correspond to an ill-posed equation for $A$:

$$\partial_{tt}^2 A + \Delta A = 0.$$
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which would correspond to an ill-posed equation for $A$:

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Answer: in our construction, $K$ is convex degenerate!
A consistency result

Theorem: If \( U \) is a smooth solution to the Cauchy problem and \( T \) is not too large (*), then \( U \) can be recovered from the concave maximization problem which admits \( A(t, x) = (t T) E_0(U(t, x)) \) as solution.

(*) more precisely if, for all \( t, x, V \in \mathbb{R}^2, E(V) \cdot r(E_0(U(t, x))) > 0 \), which requires, in particular, \( E(V) T F(V) \cdot r(E_0(U_0(x))) > 0 \).

...But what about shocks and large \( T \)??

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Cauchy by convex optimization
CERMICS, 5 Mars 2018
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Theorem: If $U$ is a smooth solution to the Cauchy problem and $T$ is not too large
A consistency result

**Theorem:** If $U$ is a smooth solution to the Cauchy problem and $T$ is not too large (*), then $U$ can be recovered from the concave maximization problem which admits $A(t, x) = (t - T)E'(U(t, x))$ as solution.

(*) more precisely if, $\forall t, x, V \in \mathcal{W}, E''(V) - (T - t)F''(V) \cdot \nabla(E'(U(t, x))) > 0$, which requires, in particular, $E''(V) - TF''(V) \cdot \nabla(E'(U_0(x))) > 0$. 
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...But what about shocks and large $T$???
Let us look at the simple Burgers equation!
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Then, the maximization problem in $A$ simply reads

$$
\sup_A \int_{[0,T] \times \mathbb{T}} - \frac{(\partial_t A)^2}{2(1 - \partial_x A)} - \int_{\mathbb{T}} A(0, \cdot) u_0.
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with $A = A(t, x) \in \mathbb{R}$ subject to $A(T, \cdot) = 0$, $\partial_x A \leq 1$. 


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with $A = A(t, x) \in \mathbb{R}$ subject to $A(T, \cdot) = 0$, $\partial_x A \leq 1$.

Introducing $\rho = 1 - \partial_x A \geq 0$, $q = \partial_t A \in \mathbb{R}$, this problem is equivalent to

$$
- \inf\{ \int_{[0,T] \times \mathbb{T}} \frac{q^2}{2\rho} - qu_0, \ (\rho \geq 0, q) \text{ subject to } \partial_t \rho + \partial_x q = 0, \ \rho(T, \cdot) = 1 \}.
$$
Let us look at the simple Burgers equation!

Then, the maximization problem in $A$ simply reads

$$\sup_{A} \int_{[0,T] \times \mathbb{T}} -\frac{(\partial_t A)^2}{2(1 - \partial_x A)} - \int_{\mathbb{T}} A(0, \cdot) u_0.$$ 

with $A = A(t, x) \in \mathbb{R}$ subject to $A(T, \cdot) = 0$, $\partial_x A \leq 1$.

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i.e. the "ballistic" version (à la Ghoussoub) of the optimal transport problem with quadratic cost (à la Benamou-B.), and, as well, an elementary "mean-field game".
Inviscid Burgers equation: \( \partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0, \ u = u(t, x), \ x \in \mathbb{R}/\mathbb{Z}, \ t \geq 0. \)

Formation of two shock waves. (Vertical axis: \( t \in [0, 1/4] \), horizontal axis: \( x \in \mathbb{T} \).)
Numerics: 2 lines of code differ from a standard (Benamou-B.) OT solver!
Inviscid Burgers equation: \( \partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0, \ u = u(t, x), \ x \in \mathbb{R}/\mathbb{Z}, \ t \geq 0. \)

Recovery of the solution at time \( T=0.1 \) by convex optimization.

Observe the formation of a first vacuum zone as the first shock has formed.
Inviscid Burgers equation: $\partial_t u + \partial_x (u^2/2) = 0$, $u = u(t, x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.

Recovery of the solution at time $T=0.16$ by convex optimization.

Observe the formation of a second vacuum zone as the second shock has formed.
Inviscid Burgers equation: $\partial_t u + \partial_x (u^2/2) = 0$, $u = u(t, x)$, $x \in \mathbb{R}/\mathbb{Z}$, $t \geq 0$.

Recovery of the solution at time $T=0.225$ by convex optimization.

Observe the extension of the two vacuum zones.
Analogy with mountain climbing: going from Everest to Lhotse without following the crest! (Credit to Th. Gallouët.)
Merci de votre attention!
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