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(Lowest-Order) Discontinuous Petrov-Galerkin Schemes

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(Lowest-Order) Discontinuous Petrov-Galerkin Schemes

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Outline (of Part II and supplements)

dPG FEM

Built-in a posteriori error control

(Flexible geometry)

Nonlinear dPG FEM

Natural adaptive dPG FEM

Reduction of mixed problem

Alternative error estimator

Stability and reduction, estimator reduction

Quasi-orthogonality and plain convergence

Discrete reliability

dPG FEM

Discontinuous Petrov-Galerkin FEM

Given PDE in some weak form

$$b(x, \bullet) = F \quad \text{in } Y$$

with exact solution $x \in X$, the dPG FEM minimizes

$$\| b(\xi_h, \bullet) - F \|_{Y_h^*}$$

amongst all ξ_h in some subspace $X_h \subset X$ and with $Y_h \subset Y$.

Discontinuous Petrov-Galerkin FEM

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Equivalent mixed system [Cohen-Dahmen-Welper 12] seeks $(x_h, y_h) \in X_h \times Y_h$ with

$$\begin{aligned} \langle y_h, \eta_h \rangle_Y + b(x_h, \eta_h) &= F(\eta_h) && \text{for all } \eta_h \in Y_h \\ b(\xi_h, y_h) &= 0 && \text{for all } \xi_h \in X_h \end{aligned} \qquad \text{(M)}$$

Discontinuous Petrov-Galerkin FEM

Equivalence *for all* $\xi_h \in X_h$ (inexact solve) and for some Fortin operator $P : Y \to Y$ is *built-in a posteriori error control*

$$\|x - \xi_h\|_X \approx \underbrace{\|b(\xi_h, \bullet) - F\|_{Y_h^*}}_{\text{error estimator}} + \underbrace{\|F \circ (1 - P)\|_{Y^*}}_{\text{data approximation}}$$

[C-Demkowicz-Gopalakrishnan SINUM 14]

Lowest-order Primal dPG

First-order system for Poisson model problem

$$f + \operatorname{div} p = 0 \quad \text{and} \quad p = \nabla u \quad \text{in } \Omega \qquad \text{for } p \in H(\operatorname{div}, \Omega) \text{ and } u \in H^1_0(\Omega).$$

Primal dPG FEM utilizes $b(u, t; v) = a_{NC}(u, v) - \langle t, v \rangle_{\partial\mathcal{T}}$ and $F(v) = (f, v)_{L^2(\Omega)}$ for $(u, t) \in X$ and $v \in Y$ with

$$X_h = S^1_0(\mathcal{T}) \times P_0(\mathcal{E}) \subset X = H^1_0(\Omega) \times H^{-1/2}(\partial\mathcal{T})$$
$$Y_h = P_1(\mathcal{T}) \subset Y = H^1(\mathcal{T})$$

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$\|F \circ (1 - P)\|_{Y^*}$ is not h.o.t. as in examples of [C-Demkowicz-Gopalakrishnan 14], but $h_{\mathcal{T}} f$ [C-Hellwig SINUM 16]. This motivates error estimator

$$\eta^2(K) = \|v_1\|^2_{H^1(K)} + h_k^2 \|f\|^2_{L^2(K)}$$

Natural Adaptive Algorithm

Input: $\mathcal{T}_0$, $0 < \theta < 1$

for any level $\ell = 0, 1, 2, \dots$ **do**

Solve (M) on $\mathcal{T}_\ell$ with solution (u_ℓ, t_ℓ, v_ℓ).

Compute error estimator $\eta_\ell(T) = (\|v_\ell\|^2_{H^1(T)} + h_T^2 \|f\|^2_{L^2(T)})^{1/2}$ and $\eta_\ell^2 = \sum_{T \in \mathcal{T}_\ell} \eta_\ell^2(T)$

Mark subset $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell$ of (almost) minimal cardinality $|\mathcal{M}_\ell|$ with

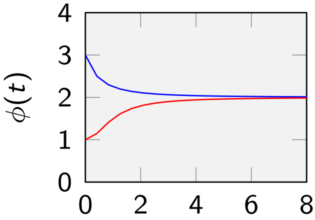
$$\theta\, \eta_\ell^2 \leq \sum_{T \in \mathcal{M}_\ell} \eta_\ell^2(T)$$

Compute smallest regular refinement $\mathcal{T}_{\ell+1}$ of $\mathcal{T}_\ell$ with $\mathcal{M}_\ell \subseteq \mathcal{T}_\ell \setminus \mathcal{T}_{\ell+1}$ by NVB **od**

Output: Sequence $(u_\ell, t_\ell, v_\ell)_{\ell \in \mathbb{N}_0}$, $(\eta_\ell)_{\ell \in \mathbb{N}_0}$, and $(\mathcal{T}_\ell)_{\ell \in \mathbb{N}_0}$

Numerical Example: Convex Energy Minimization

$\Omega \subset \mathbb{R}^n$ polyhedral Lipschitz domain
$\phi \in C^2(0,\infty)$ with $0 < \gamma_1 \leq \phi(t) \leq \gamma_2$
and $0 < \gamma_1 \leq \phi(t) + t\phi'(t) \leq \gamma_2$
$\sigma(A) = \phi(|A|)A$, $A \in \mathbb{R}^n$



$2 + (1+t)^{-2}$ $\quad 2 - (1+t^2)^{-1}$

Nonlinear model problem seeks $u \in H^1_0(\Omega)$ with

$$\int_\Omega \sigma(\nabla u) \cdot \nabla v \, \mathrm{d}x = \int_\Omega fv \, \mathrm{d}x =: F(v) \quad \text{for all } v \in H^1_0(\Omega).$$

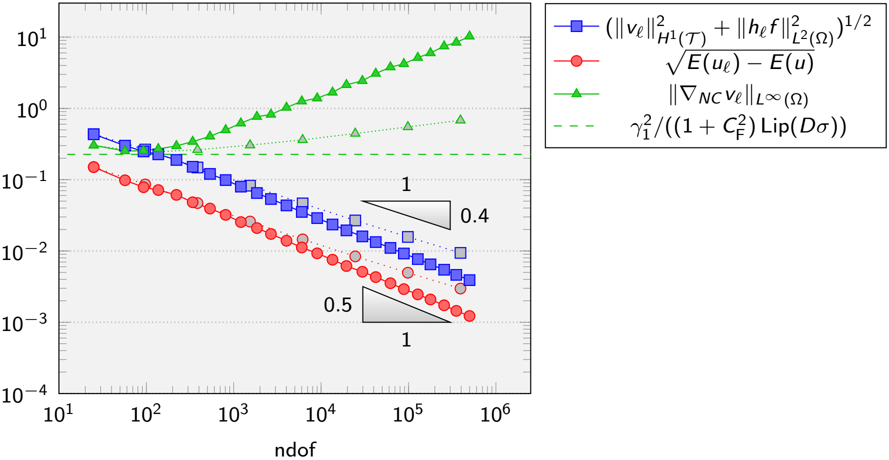
Reduced formulation seeks $(u_h, v_h) \in S^1_0(\mathcal{T}) \times CR^1_0(\mathcal{T})$ and

$$\begin{aligned} a(v_h, w_{CR}) + (\sigma(\nabla u_h), \nabla_{NC} w_{CR})_{L^2(\Omega)} &= F(w_{CR}) \\ (\nabla w_C, D\sigma(\nabla u_h)\nabla_{NC} v_h)_{L^2(\Omega)} &= 0 \end{aligned} \qquad \text{(R)}$$

for all $w_{CR} \in CR^1_0(\mathcal{T})$ and $w_C \in S^1_0(\mathcal{T})$

L-Shaped Domain

$\Omega = (-1,1)^2 \setminus [0,1)^2, \quad u|_{\partial\Omega} \equiv 0, \quad f \equiv 1, \quad \theta = 0.3$



[C-Bringmann-Hellwig-Wriggers 17 *Nonlinear dPG* arXiv:1710.00529]

Reduction of (M) to (R)

Reduction in Primal dPG

Theorem [C-Bringmann-Hellwig-Wriggers 17]

If $(u_C, t_0; v_1) \in X_h \times Y_h$ solves (M), then $v_1 \in CR_0^1(\mathcal{T})$ and $(u_C, v_1) \in S_0^1(\mathcal{T}) \times CR_0^1(\mathcal{T})$ solves

$$\begin{aligned} a_{NC}(v_{CR} + u_C, w_{CR}) + (v_{CR}, w_{CR})_{L^2(\Omega)} &= (f, w_{CR})_{L^2(\Omega)} \\ a_{NC}(w_C, v_{CR}) = 0 \text{ for all } (w_{CR}, w_C) &\in CR_0^1(\mathcal{T}) \times S_0^1(\mathcal{T}) \end{aligned} \tag{R}$$

Conversely, for any solution (u_C, v_{CR}) to this, there exists a unique $t_0 \in P_0(\mathcal{E})$ such that $(u_C, t_0; v_{CR})$ solves (M).

Reduction in Primal dPG

Proof. (M) for primal dPG seeks $(u_C, t_0; v_1) \in S^1_0(\mathcal{T}) \times P_0(\mathcal{E}) \times P_1(\mathcal{T})$ with

$$\begin{aligned}(v_1, w_1)_{L^2(\Omega)} + a_{NC}(v_1 + u_C, w_1) - \langle t_0, w_1\rangle_{\partial\mathcal{T}} &= (f, w_1)_{L^2(\Omega)}\\ a_{NC}(w_C, v_1) - \langle s_0, v_1\rangle_{\partial\mathcal{T}} &= 0\end{aligned}$$

tested with $w_1 \in P_1(\mathcal{T})$, $(w_C, s_0) \in S^1_0(\mathcal{T}) \times P_0(\mathcal{E})$. Since $CR^1_0(\mathcal{T}) = \{w_1 \in P_1(\mathcal{T}) \mid \langle s_0, w_1\rangle_{\partial\mathcal{T}} = 0 \text{ for all } s_0 \in P_0(\mathcal{E})\}$, this is the reduced system.

Conversely, the unique solution (u_C, v_{CR}) to (R) defines $\Lambda = a_{NC}(v_{CR} + u_C, \bullet) - F \in P_1(\mathcal{T})^*$ with $CR^1_0(\mathcal{T}) \subset \ker\Lambda$. This leads to $t_0 \in P_0(\mathcal{E})$ with $\langle t_0, \bullet\rangle_{\partial\mathcal{T}} = \Lambda \in P_1(\mathcal{T})^*$. □

Reduced Problem

For (local) orthogonal projection $Q : L^2(\Omega) \to L^2(\Omega)$ and global parameter $0 \le \alpha \le 1$, consider solution $(v_{CR}, u_C) \in CR_0^1(\mathcal{T}) \times S_0^1(\mathcal{T})$ to

$$
\begin{aligned}
&a_{NC}(v_{CR} + u_C, w_{CR}) + \alpha(Qv_{CR}, w_{CR})_{L^2(\Omega)} = (f, Qw_{CR})_{L^2(\Omega)} \\
&a_{NC}(w_C, v_{CR}) = 0 \text{ for all } (w_{CR}, w_C) \in CR_0^1(\mathcal{T}) \times S_0^1(\mathcal{T})
\end{aligned}
\tag{R}
$$

	Q	α
Primal dPG	id	1
Dual dPG	Π_0	1
Primal Mixed dPG	id	$1/2$
Ultraweak dPG	id	$1/2$
Nonconforming	id	0
Mixed	Π_0	0

Properties of Q

For each $T \in \bigcup \mathbb{T}$, $Q_T : L^2(T) \to L^2(T)$ orthogonal projection and $Q_{\mathcal{T}} : L^2(\Omega) \to L^2(\Omega)$ with

$$(Q_{\mathcal{T}} v)|_T := Q_T v := Q_T(v|_T) \text{ a.e. in } T \in \mathcal{T} \text{ for all } v \in L^2(\Omega)$$

Properties for $Q := Q_{\mathcal{T}}$, $\widehat{Q} := Q_{\widehat{\mathcal{T}}}$ with $\widehat{Q} Q = Q = Q \widehat{Q}$

For $K \in \mathcal{T}$, $\widehat{\mathcal{T}}(K) := \{T \in \widehat{\mathcal{T}} : T \subset K\}$ and $\widehat{w}_{CR} \in CR^1(\widehat{\mathcal{T}}(K))$,

$$\|(Q_{\widehat{\mathcal{T}}} - Q_{\mathcal{T}})\widehat{w}_{CR}\|_{L^2(K)} \le C_{dP} |K|^{1/n} \|\!|\widehat{w}_{CR}\|\!|_{NC(K)}$$

$$(Qv, (1 - \widehat{Q})w)_{L^2(K)} = 0$$

$$(\widehat{Q} - Q)v = 0 \text{ a.e. in } \operatorname{int}\Big(\bigcup(\mathcal{T} \cap \widehat{\mathcal{T}})\Big)$$

Reduction in Ultraweak dPG

The ultraweak dPG utilizes

$$
\begin{aligned}
X &:= L^2(\Omega;\mathbb{R}^n) \times L^2(\Omega) \times H^{-1/2}(\partial\mathcal{T}) \times H^{1/2}(\partial\mathcal{T})\\
X_h &:= P_0(\mathcal{T};\mathbb{R}^n) \times P_0(\mathcal{T};\mathbb{R}^n) \times P_0(\mathcal{E}) \times S^1_0(\mathcal{E}) \subset X\\
Y &:= H(\mathrm{div},\Omega) \times H^1(\mathcal{T})\\
Y_h &:= RT^{NC}(\mathcal{T}) \times P_1(\mathcal{T}) \subset Y
\end{aligned}
$$

and bilinear form

$$
\begin{aligned}
b(r,w,t,s;q,v) &:= (r,q)_{L^2(\Omega)} + (r,\nabla_{NC} v)_{L^2(\Omega)} + (w,\mathrm{div}_{NC}\, q)_{L^2(\Omega)}\\
&\qquad - \langle q\cdot\nu, s\rangle_{\partial\mathcal{T}} - \langle t, v\rangle_{\partial\mathcal{T}}\\
F(q,v) &:= (f,v)_{L^2(\Omega)} \quad \text{for } (r,w,t,s)\in X \text{ and } (q,v)\in Y
\end{aligned}
$$

Reduction in Ultraweak dPG

Theorem

(a) If $(r_0, w_0, t_0, s_C; q_1, v_1) \in X_h \times Y_h$ *solves* (M)*, then* $v_1 \in CR_0^1(\mathcal{T})$ *and* $(u_C, 2v_1) \in S_0^1(\mathcal{T}) \times CR_0^1(\mathcal{T})$ *with* $u_C|_{\partial\mathcal{T}} = s_C$ *solves* (R)
(b) If (u_C, v_{CR}) *solves* (R)*, then there exist unique* $(r_0, w_0, t_0, q_1) \in P_0(\mathcal{T}; \mathbb{R}^n) \times P_0(\mathcal{T}; \mathbb{R}^n) \times P_0(\mathcal{E}) \times RT^{NC}(\mathcal{T})$ *s.t.* $(r_0, w_0, t_0, u_C|_{\partial\mathcal{T}}; q_1, v_{CR})$ *solves* (M)

(a) The second equation of (M) reads

$$(\tau_0, q_1 + \nabla_{NC} v_1)_{L^2(\Omega)} + (u_0, \operatorname{div}_{NC} q_1)_{L^2(\Omega)} - \langle q_1 \cdot \nu, w_C \rangle_{\partial\mathcal{T}} - \langle s_0, v_1 \rangle_{\partial\mathcal{T}} = 0$$

for any $(\tau_0, u_0, s_0, w_C) \in P_0(\mathcal{T}; \mathbb{R}^n) \times P_0(\mathcal{T}; \mathbb{R}^n) \times P_0(\mathcal{E}) \times S_0^1(\mathcal{E})$. Hence $v_1 \in CR_0^1(\mathcal{T})$, $\operatorname{div}_{NC} q_1 = 0$, and $q_1 + \nabla_{NC} v_1 = 0$. An integration by parts shows $a_{NC}(v_1, w_C) = 0$ for any $w_C \in S_0^1(\mathcal{T})$

Reduction in Ultraweak dPG

Given any $w_{CR} \in CR_0^1(\mathcal{T})$, test the first equation of (M) and integrate by parts with $u_C \in S_0^1(\mathcal{T})$,

$$
\begin{aligned}
(f, w_{CR})_{L^2(\Omega)} &= (r_0, -\nabla_{NC} w_{CR} + \nabla_{NC} w_{CR})_{L^2(\Omega)} + (\nabla w_{CR}, \nabla u_C)_{L^2(\Omega)} \\
&\quad - (\Pi_0 q_1, \nabla w_{CR})_{L^2(\Omega)} + (v_1, w_{CR})_{L^2(\Omega)} + (\nabla v_1, \nabla w_{CR})_{L^2(\Omega)} \\
&= a_{NC}(w_{CR}, u_C) + (2v_1, w_{CR})_{L^2(\Omega)}/2 + (\nabla(2v_1), \nabla w_{CR})_{L^2(\Omega)}
\end{aligned}
$$

(b) The choice $q_1 = -\nabla_{NC} v_{CR}/2$ and (R) imply the second equation of (M). As for primal dPG, there exists unique $t_0 \in P_0(\mathcal{E})$ with $\langle t_0, \bullet \rangle_{\partial\mathcal{T}} = a_{NC}(u_C, \bullet) + a_h(v_{CR}, \bullet)/2 - F$ in $P_1(\mathcal{T})$. Choose $r_0 = \nabla_{NC} v_{CR}/2 + \nabla u_C$, $w_0 = \Pi_0 u_C$ to deduce first equation of (M) □

Alternative Error Control

Axioms of Adaptivity at a Glance

Optimal convergence rates for AFEM with estimator $\eta(\mathcal{T}, T)$ for $T \in \mathcal{T} \in \mathbb{T}$ and distance function $\delta(\mathcal{T}, \widehat{\mathcal{T}})$ for refinement $\hat{\mathcal{T}}$ of $\mathcal{T}$ s.t. $\exists \rho_2 < 1$, $\Lambda_k < \infty$ s.t. $\forall \mathcal{T} \in \mathbb{T} \forall \hat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})\ \exists \mathcal{R} \subset \mathcal{T}$ s.t. $\mathcal{T} \setminus \hat{\mathcal{T}} \subseteq \mathcal{R}$ and $|\mathcal{R}| \lesssim |\mathcal{T} \setminus \hat{\mathcal{T}}|$

$$|\eta(\hat{\mathcal{T}}, \mathcal{T} \cap \hat{\mathcal{T}}) - \eta(\mathcal{T}, \mathcal{T} \cap \hat{\mathcal{T}})| \le \Lambda_1 \delta(\mathcal{T}, \hat{\mathcal{T}}) \tag{A1}$$

$$\eta(\hat{\mathcal{T}}, \hat{\mathcal{T}} \setminus \mathcal{T}) \le \rho_2 \eta(\mathcal{T}, \mathcal{T} \setminus \hat{\mathcal{T}}) + \Lambda_2 \delta(\mathcal{T}, \hat{\mathcal{T}}) \tag{A2}$$

$$\delta^2(\mathcal{T}, \hat{\mathcal{T}}) \le \Lambda_3 \eta^2(\mathcal{T}, \mathcal{R}) + \widehat{\Lambda}_3 \eta^2(\hat{\mathcal{T}}) \tag{A3}$$

$$\sum_{k=\ell}^{\infty} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) \le \Lambda_4 \eta_\ell^2 \quad \text{for all } \ell \in \mathbb{N}_0 \tag{A4}$$

and $\mathcal{T}_k, \eta_k := \eta(\mathcal{T}_k, \mathcal{T}_k)$ of AFEM

Alternative Error Estimator

For $K \in \mathcal{T}$ and solution (v_{CR}, u_C) to (R), define

$$\eta^2(K) = |K|^{2/n} \|f - \alpha Q v_{CR}\|^2_{L^2(K)} + |K|^{1/n} \sum_{E \in \mathcal{E}(K)} \|[\nabla_{NC} v_{CR}]_E\|^2_{L^2(E)}$$

Distance function for solutions (v_{CR}, u_C) to (R) and $(\widehat{v}_{CR}, \widehat{u}_C)$ to ($\hat{\text{R}}$) on $\widehat{\mathcal{T}}$

$$\delta^2(\mathcal{T}, \widehat{\mathcal{T}}) := \|\!|\widehat{v}_{CR} - v_{CR}|\!\|^2_{NC} + \alpha \|\widehat{Q}\widehat{v}_{CR} - Q v_{CR}\|^2_{L^2(\Omega)}$$

Theorem (Global reliability)

The solution $(v_{CR}, u_C) \in CR^1_0(\mathcal{T}) \times S^1_0(\mathcal{T})$ *to* (R) *satisfies*

$$C_{rel}^{-1} \|\!|v_{CR}|\!\|^2_{NC} \leq \sum_{E \in \mathcal{E}} |E|^{1/(n-1)} \|[\nabla_{NC} v_{CR}]_E\|^2_{L^2(E)} \leq \eta^2(\mathcal{T})$$

Stability (A1) and Reduction (A2)

Proof of (A1)

Theorem (Stability) $\left|\widehat{\eta}(\mathcal{T}\cap\widehat{\mathcal{T}})-\eta(\mathcal{T}\cap\widehat{\mathcal{T}})\right| \le \Lambda_1 \delta(\mathcal{T},\widehat{\mathcal{T}})$

Proof. $\eta(\mathcal{T}\cap\widehat{\mathcal{T}})$ (resp. $\widehat{\eta}(\mathcal{T}\cap\widehat{\mathcal{T}})$) is Euclidean norm of vector in $\mathbb{R}^m$, $m := (n+2)|\mathcal{T}\cap\widehat{\mathcal{T}}|$, with entries $|T|^{1/(2n)}\|[\nabla_{NC} v_{CR}]_E\|_{L^2(E)}$ and $\|h_{\mathcal{T}}(f-\alpha Q v_{CR})\|_{L^2(T)}$ for any $T \in \mathcal{T}\cap\widehat{\mathcal{T}}$ and $E \in \mathcal{E}(T)$. The reverse triangle inequality in $\mathbb{R}^m$ proves

$$
\begin{aligned}
&|\eta(\mathcal{T}\cap\widehat{\mathcal{T}})-\widehat{\eta}(\mathcal{T}\cap\widehat{\mathcal{T}})|^2 \\
&\qquad \le \sum_{T\in\mathcal{T}\cap\widehat{\mathcal{T}}}\Big(|T|^{1/n}\sum_{E\in\mathcal{E}(T)}(\|[\nabla_{NC} v_{CR}]_E\|_{L^2(E)}-\|[\nabla_{NC}\widehat{v}_{CR}]_E\|_{L^2(E)})^2 \\
&\qquad\qquad + (\|h_{\mathcal{T}}(f-\alpha Q v_{CR})\|_{L^2(T)}-\|h_{\widehat{\mathcal{T}}}(f-\alpha\widehat{Q}\widehat{v}_{CR})\|_{L^2(T)})^2\Big)
\end{aligned}
$$

Proof of (A1)

The reverse triangle inequality in $L^2(T)$ and $h_{\mathcal{T}} \leq h_0$ a.e. on $T \in \mathcal{T} \cap \widehat{\mathcal{T}}$ imply

$$\begin{aligned}\big| \|h_{\mathcal{T}}(f - \alpha Q v_{CR})\|_{L^2(T)} - \|h_{\mathcal{T}}(f - \alpha \widehat{Q}\widehat{v}_{CR})\|_{L^2(T)} \big| \\ \leq \alpha^{1/2} h_0 \|Q v_{CR} - \widehat{Q}\widehat{v}_{CR}\|_{L^2(T)}\end{aligned}$$

Trace inequality for $e := \nabla_{NC} v_{CR} - \nabla_{NC}\widehat{v}_{CR} \in P_0(\widehat{\mathcal{T}};\mathbb{R}^n)$ shows

$$\begin{aligned}\big| \|[\nabla_{NC} v_{CR}]_E\|_{L^2(E)} - \|[\nabla_{NC}\widehat{v}_{CR}]_E\|_{L^2(E)} \big| \leq \|[e]_E\|_{L^2(E)} \\ \lesssim |T|^{-1/(2n)} \|e\|_{L^2(\widehat{\omega}_E)}\end{aligned}$$

The sum over all T in $\mathcal{T} \cap \widehat{\mathcal{T}}$ concludes the proof $\square$

Proof of (A2)

Theorem (Reduction) $\widehat{\eta}(\widehat{\mathcal{T}} \setminus \mathcal{T}) - 2^{-1/(2n)}\eta(\mathcal{T} \setminus \widehat{\mathcal{T}}) \le \Lambda_2\delta(\mathcal{T}, \widehat{\mathcal{T}})$

Proof. Any $K \in \mathcal{T} \setminus \widehat{\mathcal{T}}$, $\widehat{T} \in \widehat{\mathcal{T}}(K)$ satisfy $|\widehat{T}| \le |K|/2$. Any $\widehat{E} \in \widehat{\mathcal{E}}(\widehat{T})$ with $\widehat{E} \not\subseteq \partial K$ satisfies $[\nabla_{NC} v_{CR}]_{\widehat{E}} \equiv 0$. Young's inequality shows

$$\begin{aligned}
\widehat{\eta}^2(\widehat{\mathcal{T}}(K)) \le \sum_{\widehat{T}\in\widehat{\mathcal{T}}(K)} \Big(&(1+1/\lambda)|\widehat{T}|^{2/n}\alpha\|\widehat{Q}\widehat{v}_{CR} - Qv_{CR}\|^2_{L^2(\widehat{T})} \\
&+ (1+\lambda)2^{-2/n}\|h_{\widehat{\mathcal{T}}}(f - \alpha Q v_{CR})\|^2_{L^2(\widehat{T})} \\
&+ |\widehat{T}|^{1/n}(1+1/\lambda)\sum_{\widehat{E}\in\widehat{\mathcal{E}}(\widehat{T})} \|[\nabla_{NC}\widehat{v}_{CR} - \nabla_{NC}v_{CR}]_{\widehat{E}}\|^2_{L^2(\widehat{E})} \\
&+ |K|^{1/n}2^{-1/n}(1+\lambda)\sum_{\widehat{E}\in\widehat{\mathcal{E}}(\widehat{T})} \|[\nabla_{NC}v_{CR}]_{\widehat{E}}\|^2_{L^2(\widehat{E})}\Big)
\end{aligned}$$

Proof of (A2)

The same arguments as before show

$$\sum_{K\in\mathcal{T}\setminus\widehat{\mathcal{T}}}\sum_{\widehat{T}\in\widehat{\mathcal{T}}(K)}|\widehat{T}|^{1/n}\sum_{\widehat{E}\in\widehat{\mathcal{E}}(\widehat{T})}\|[\nabla_{NC}\widehat{v}_{CR}-\nabla_{NC}v_{CR}]_{\widehat{E}}\|^2_{L^2(\widehat{E})}$$
$$\leq\sum_{K\in\mathcal{T}\setminus\widehat{\mathcal{T}}}\sum_{\widehat{T}\in\widehat{\mathcal{T}}(K)}\sum_{\widehat{E}\in\widehat{\mathcal{E}}(\widehat{T})}\|\nabla_{NC}\widehat{v}_{CR}-\nabla_{NC}v_{CR}\|^2_{L^2(\widehat{\omega}_E)}$$
$$\lesssim\|\nabla_{NC}\widehat{v}_{CR}-\nabla_{NC}v_{CR}\|_{L^2(\Omega)}$$

The combination of the aforementioned estimates with optimal $\lambda>0$ concludes the proof □

(A1)-(A2) & Dörfler Marking Imply (A12)

Theorem (estimator reduction in AFEM)

There exist $0 \leq \varrho_{12} < 1$ *and* $\Lambda_{12} < \infty$ *such that for any* $1 - \theta(1 - \varrho_2^2) < \varrho_{12} < 1$, *(A1)-(A2) & Dörfler marking with bulk parameter* $0 < \theta \leq 1$ *imply*

$$\eta_{\ell+1}^2 \leq \varrho_{12}\, \eta_\ell^2 + \Lambda_{12}\, \delta^2(\mathcal{T}_{\ell+1}, \mathcal{T}_\ell) \qquad \text{(A12)}$$

Proof. Let $\lambda > 0$ satisfy $1 - \theta(1 - \varrho_2^2) = \varrho_{12}/(1 + \lambda)$. (A1) leads to

$$\begin{aligned} \eta^2(\mathcal{T}_{\ell+1}, \mathcal{T}_{\ell+1} \cap \mathcal{T}_\ell) &\leq \big(\eta(\mathcal{T}_\ell, \mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}) + \Lambda_1 \delta(\mathcal{T}_{\ell+1}, \mathcal{T}_\ell)\big)^2 \\ &\leq (1 + \lambda)\eta^2(\mathcal{T}_\ell, \mathcal{T}_\ell \cap \mathcal{T}_{\ell+1}) + (1 + 1/\lambda)\Lambda_1^2 \delta^2(\mathcal{T}_{\ell+1}, \mathcal{T}_\ell) \end{aligned}$$

Proof of (A12)

The same argument with (A2) leads to

$$\eta^2(\mathcal{T}_{\ell+1},\mathcal{T}_{\ell+1}\backslash\mathcal{T}_\ell)\le\varrho_2^2(1+\lambda)\eta^2(\mathcal{T}_\ell,\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1})+(1+1/\lambda)\Lambda_2^2\delta^2(\mathcal{T}_{\ell+1},\mathcal{T}_\ell)$$

Combine the previous estimates with the decomposition

$$\begin{aligned}\eta_{\ell+1}^2&=\eta^2(\mathcal{T}_{\ell+1},\mathcal{T}_{\ell+1}\cap\mathcal{T}_\ell)+\eta^2(\mathcal{T}_{\ell+1},\mathcal{T}_{\ell+1}\backslash\mathcal{T}_\ell)\\&\le(1+\lambda)\underbrace{\left(\eta^2(\mathcal{T}_\ell,\mathcal{T}_\ell\cap\mathcal{T}_{\ell+1})+\varrho_2^2\eta^2(\mathcal{T}_\ell,\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1})\right)}_{(*):=\eta_\ell^2-(1-\varrho_2^2)\eta^2(\mathcal{T}_\ell,\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1})}+\Lambda_{12}\delta^2(\mathcal{T}_{\ell+1},\mathcal{T}_\ell)\end{aligned}$$

The Dörfler marking guarantees $\theta\eta_\ell^2\le\eta^2(\mathcal{T}_\ell,\mathcal{T}_\ell\backslash\mathcal{T}_{\ell+1})$ and so

$$(*)\le\left(1-\theta(1-\varrho_2^2)\right)\eta_\ell^2=\varrho_{12}\eta_\ell^2/(1+\lambda)\quad\square$$

Plain Convergence

Convergence $\eta_k \to 0$ as $k \to \infty$ from AFEM

Theorem (plain convergence)

(A12) and (A4) imply that $\Lambda := (1 + \Lambda_{12}\Lambda_4)/(1 - \varrho_{12}) < \infty$ *satisfies*

$$\sum_{k=\ell}^{\infty} \eta_k^2 \le \Lambda \eta_\ell^2 \quad \textit{for all } \ell \in \mathbb{N}_0$$

Proof. Write $\delta^2_{k,k+1} \equiv \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1})$ and recall (A12) $\eta^2_{k+1} \le \varrho_{12}\eta_k^2 + \Lambda_{12}\delta^2_{k,k+1}$ and deduce

$$\sum_{k=\ell}^{\ell+m} \eta_k^2 \le \sum_{k=\ell}^{\ell+m+1} \eta_k^2 \le \eta_\ell^2 + \varrho_{12} \sum_{k=\ell}^{\ell+m} \eta_k^2 + \Lambda_{12} \sum_{k=\ell}^{\ell+m} \delta^2_{k,k+1}$$

Utilize $\varrho_{12} < 1$ and (A4) for the last sum to prove

$$(1 - \varrho_{12}) \sum_{k=\ell}^{\ell+m} \eta_k^2 \le (1 + \Lambda_{12}\Lambda_4)\, \eta_\ell^2 \quad \square$$

R-Linear Convergence on Each Level

Theorem

(A12), (A4) and Λ from plain convergence lead to $q := 1 - 1/\Lambda < 1$ with

$$\eta_{\ell+m}^2 \le q^m \Lambda \, \eta_\ell^2 \quad \textit{for all } \ell,\, m \in \mathbb{N}_0$$

Proof. Rewrite plain convergence theorem as

$$\sigma_\ell^2 := \sum_{k=\ell}^{\infty} \eta_k^2 \le \Lambda \, \eta_\ell^2$$

Then

$$\Lambda^{-1}\sigma_\ell^2 + \sigma_{\ell+1}^2 \le \eta_\ell^2 + \sum_{k=\ell+1}^{\infty} \eta_k^2 = \sigma_\ell^2$$

This is $\sigma_{\ell+1}^2 \le q\sigma_\ell^2$, and successively, $\sigma_{\ell+m}^2 \le q^m \sigma_\ell^2$ for all $m \in \mathbb{N}_0$. Consequently

$$\sigma_{\ell+m}^2 \le q^m \sigma_\ell^2 \le q^m \Lambda \eta_\ell^2 \quad \square$$

Proof of (A4)

For all $\ell \in \mathbb{N}_0$, output $\mathcal{T}_k, \eta_k := \eta(\mathcal{T}_k, \mathcal{T}_k)$ of AFEM satisfies

$$\sum_{k=\ell}^{\infty} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) \leq \Lambda_4 \eta_\ell^2$$

Proof of (A4)

Theorem (Quasi-Orthogonality) $\forall \varepsilon > 0\ \exists \delta > 0\ \forall \alpha h_0 \leq \delta\ \exists \Lambda_4$ $\forall \ell, m \in \mathbb{N}_0$

$$\sum_{k=\ell}^{\ell+m} \delta^2(\mathcal{T}_k, \mathcal{T}_{k+1}) \leq \Lambda_4 \eta_\ell^2 + \varepsilon \sum_{k=\ell}^{\ell+m} \eta^2(\mathcal{T}_k) \tag{A4$_\varepsilon$}$$

Proof. *Step 1.* For solution (v_k, u_k) to (R) on $\mathcal{T}_k$, $Q_k := Q_{\mathcal{T}_k}$, etc., define

$$E_k := -\frac{1}{2}(f, Q_k v_k)_{L^2(\Omega)}$$

Then algebra on (R) w.r.t. $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ shows for nonconforming interpolation $I_{NC(k)}$

$$\frac{1}{2}\delta_{k,k+1}^2 + E_k - E_{k+1} = (f - \alpha Q_k v_k, Q_{k+1} v_{k+1} - Q_k I_{NC(k)} v_{k+1})_{L^2(\Omega)}$$

Proof of (A4)

Step 2. General properties of Q_k and Q_{k+1}, $\Omega_k := \operatorname{int}(\bigcup(\mathcal{T}_k \setminus \mathcal{T}_{k+1}))$ plus standard estimates for I_{NC} (with explicit constants κ_{NC} and C_{dP} in [C-Hellwig 17]) imply

$$\frac{1}{4}\delta_{k,k+1}^2 + E_k - E_{k+1} \leq \kappa_{NC}^2 \|h_k(Q_{k+1}f - \alpha Q_k v_k)\|_{L^2(\Omega_k)}^2 + C_{dP}\|h_k(Q_{k+1} - Q_k)f\|_{L^2(\Omega)} \|\!| v_{k+1} |\!\|_{NC}$$

Step 3. For all $0 < \lambda \leq 2$,

$$\begin{aligned}\|h_k(Q_{k+1}f - \alpha Q_k v_k)\|_{L^2(\Omega_k)}^2/4 \leq{}& (1+\lambda)\|h_k(Q_{k+1}f - \alpha Q_k v_k)\|_{L^2(\Omega)}^2 \\ &+ (1+\lambda^{-1})h_0^2\alpha^2\|Q_{k+1}v_{k+1} - Q_k v_k\|_{L^2(\Omega)}^2 \\ &- \|h_{k+1}Q_{k+1}(f - \alpha v_{k+1})\|_{L^2(\Omega)}^2\end{aligned}$$

Proof is based on $2h_{k+1}^2 \leq h_k^2$ a.e. in $\Omega_k = \operatorname{int}(\bigcup(\mathcal{T}_k \setminus \mathcal{T}_{k+1}))$ with triangle and Young inequalities

Step 4. The combination of Steps 2 and 3 and their sum with orthogonalities of Q and Young's inequality lead for $0 \le \delta \le 1$ to

$$
\begin{aligned}
\frac{1}{4}\sum_{k=\ell}^{\ell+m} \delta_{k,k+1}^2 \le\ & E_{\ell+m+1} - E_\ell + 4\kappa_{NC}^2 \| h_\ell Q_\ell (f - \alpha v_\ell)\|_{L^2(\Omega)}^2 \\
& + \delta \sum_{k=\ell+1}^{\ell+m+1} |||v_k|||_{NC}^2/2 + 4\kappa_{NC}^2 \lambda \sum_{k=\ell}^{\ell+m} \| h_k Q_k (f - \alpha v_k)\|_{L^2(\Omega)}^2 \\
& + (4\kappa_{NC}^2(1+\lambda) + C_{dP}^2/(2\delta)) \sum_{k=\ell}^{\ell+m} \| h_k (Q_{k+1} - Q_k) f\|_{L^2(\Omega)}^2 \\
& + 4\kappa_{NC}^2 (1+\lambda^{-1}) h_0^2 \alpha^2 \sum_{k=\ell}^{\ell+m} \| Q_{k+1} v_{k+1} - Q_k v_k \|_{L^2(\Omega)}^2
\end{aligned}
$$

Proof of (A4)

Step 5. Corollary on global reliability and orthogonality of Q_k implies

$$C_{\text{rel}}^{-1} \|\!| v_k |\!\|_{NC}^2 + \| h_k Q_k (f - \alpha v_k) \|_{L^2(\Omega)}^2 + \| h_k (f - Q_k f) \|_{L^2(\Omega)}^2 \leq \eta_k^2$$

Step 6. Let $\varepsilon := 4 \max\{\delta C_{\text{rel}}, 8\kappa_{NC}^2 \lambda\}$ and $4\kappa_{NC}^2 (1 + \lambda^{-1}) h_0^2 \alpha^2 \leq 1/8$,

$$\begin{aligned} \frac{1}{8} \sum_{k=\ell}^{\ell+m} \delta_{k,k+1}^2 \leq \|\!| v_\ell |\!\|_{NC}^2 / 2 + 4\kappa_{NC}^2 \| h_\ell Q_\ell (f - \alpha v_\ell) \|_{L^2(\Omega)}^2 + \frac{\varepsilon}{8} \sum_{k=\ell}^{\ell+m} \eta_k^2 \\ - 4\kappa_{NC}^2 \lambda \sum_{k=\ell}^{\ell+m} \| h_k (f - Q_k f) \|_{L^2(\Omega)}^2 \\ + (4\kappa_{NC}^2 (1 + \lambda) + C_{dP}^2 / (2\delta)) \sum_{k=\ell}^{\ell+m} \| h_k (Q_{k+1} - Q_k) f \|_{L^2(\Omega)}^2 \end{aligned}$$

Step 7. Pythagoras theorem implies that the last sum $\lesssim \eta_\ell^2$ $\square$

(A12)+(A4$_\varepsilon$) for Small $\varepsilon \Rightarrow$ (A4)

Theorem

(A1)-(A2) and (A4$_\varepsilon$) for $0 < \varepsilon < (1 - \varrho_{12})/\Lambda_{12}$ imply (A4)

Proof. (A12) implies

$$\sum_{k=\ell}^{\ell+m} \eta_k^2 \leq \eta_\ell^2 + \varrho_{12} \sum_{k=\ell}^{\ell+m} \eta_k^2 + \Lambda_{12} \sum_{k=\ell}^{\ell+m} \delta(\mathcal{T}_k, \mathcal{T}_{k+1})$$

(A4$_\varepsilon$) proves

$$(1 - \varrho_{12} - \Lambda_{12}\varepsilon) \sum_{k=\ell}^{\ell+m} \eta_k^2 \leq (1 + \Lambda_{12}\Lambda_{4(\varepsilon)})\eta_\ell^2$$

This plus (A4$_\varepsilon$) conclude the proof □

(A1)-(A4) imply optimal convergence rates

Outline of Optimality Analysis I

(A12) Estimator reduction $\eta_{\ell+1}^2 \leq \rho_{12}\eta_\ell^2 + \Lambda_{12}\delta_{\ell,\ell+1}^2$

Convergence from

$$\sum_{k=\ell}^{\infty} \eta_k^2 \leq \Lambda \eta_\ell^2 \quad \text{and then} \quad \sum_{k=0}^{\ell-1} \eta_k^{-1/s} \lesssim \eta_\ell^{-1/s}$$

Quasimonotonicity $\quad \eta^2(\hat{\mathcal{T}}) \leq \Lambda_7 \eta^2(\mathcal{T})$

Comparison Lemma: Given $\mathcal{T}_\ell$, $0 < \kappa < 1$,

$$M := \sup_{N \in \mathbb{N}_0} (1+N)^s \min_{\mathcal{T} \in \mathbb{T}(N)} \eta(\mathcal{T})$$

there exist $\hat{\mathcal{T}}_\ell$ and $0 < \theta_0 < 1$ s.t.

(a) $\eta(\hat{\mathcal{T}}_\ell) \leq \kappa \eta(\mathcal{T}_\ell)$

(b) $\kappa \eta_\ell |\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell|^s \lesssim M$

(c) $\theta_0 \eta_\ell^2 \leq \eta_\ell^2(\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell)$

Outline of Optimality Analysis II

$\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell$ satisfies the bulk criterion for $\theta \leq \theta_0$ by (c). This implies

$$|\mathcal{M}_\ell^\star| \leq |\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell|$$

with the optimal set $\mathcal{M}_\ell^\star$ of marked cells in AFEM. The utilized set $\mathcal{M}_\ell$ of marked cells is almost minimal: $\exists 0 < \Lambda_{\mathrm{opt}} < \infty \, \forall \ell \in \mathbb{N}_0$,

$$|\mathcal{M}_\ell| \leq \Lambda_{\mathrm{opt}} |\mathcal{M}_\ell^\star| \leq \Lambda_{\mathrm{opt}} |\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell|$$

Recall $M := \sup_{N \in \mathbb{N}_0} (1+N)^s \min_{\mathcal{T} \in \mathbb{T}(N)} \eta(\mathcal{T})$ and from (b) deduce

$$|\mathcal{T}_\ell \setminus \hat{\mathcal{T}}_\ell| \lesssim \left(\frac{M}{\kappa \eta_\ell} \right)^{1/s} \approx M^{1/s} \eta_\ell^{-1/s}$$

Recall closure overhead control and combine with aforementioned estimates for

$$|\mathcal{T}_\ell| - |\mathcal{T}_0| \leq C_{BDV} \sum_{j=0}^{\ell-1} |\mathcal{M}_j| \lesssim M^{1/s} \sum_{j=0}^{\ell-1} \eta_j^{-1/s} \lesssim M^{1/s} \eta_\ell^{-1/s} \qquad \square$$

Proof of (A3)

Proof of (A3)

Theorem (Discrete reliability) $\delta^2(\mathcal{T},\widehat{\mathcal{T}}) \leq \Lambda_3\big(\eta^2(\mathcal{R}) + \alpha h_0^2 \widehat{\eta}^2\big)$

Proof. Recall general notation $\mathcal{T},\widehat{\mathcal{T}} \in \mathbb{T}(\mathcal{T})$, $Q = Q_{\mathcal{T}}$, $\widehat{Q} = Q_{\widehat{\mathcal{T}}}$, etc.
Step 1. Minor generalization of [C-Gallistl-Schedensack SINUM 13] provides $v_{CR}^* \in CR_0^1(\widehat{\mathcal{T}})$ with $v_{CR} = I_{NC} v_{CR}^*$ and $\|v_{CR} - v_{CR}^*\|_{NC} \lesssim \eta(\mathcal{R})$ for $\mathcal{R} := \{K \in \mathcal{T} : \exists T \in \mathcal{T} \setminus \widehat{\mathcal{T}}, \operatorname{dist}(K, T) = 0\}$
Step 2. Algebra with (R) and ($\hat{\text{R}}$) leads to

$$\begin{aligned}\delta^2(\mathcal{T},\widehat{\mathcal{T}}) &= a_{NC}(v_{CR} - v_{CR}^*, v_{CR} - \widehat{v}_{CR}) \\ &\quad + (f - \alpha Q v_{CR}, \widehat{Q}(\widehat{v}_{CR} - v_{CR}^*) + Q(v_{CR} - I_{NC}\widehat{v}_{CR}))_{L^2(\Omega)} \\ &\quad + a_{NC}(\widehat{u}_C, v_{CR}^*) + \alpha(Q v_{CR} - \widehat{Q}\widehat{v}_{CR}, Q v_{CR} - \widehat{Q} v_{CR}^*)_{L^2(\Omega)}\end{aligned}$$

Proof of (A3)

Step 3. $a_{NC}(v_{CR}, \widehat{u}_C) \lesssim \eta(\mathcal{R}) \| \widehat{u}_C - u_C \|$
The proof utilizes Scott-Zhang quasi-interpolation, piecewise integration by parts and trace inequality

Step 4. with (R) and (R̂), error estimates of the non-conforming interpolation I_{NC} and Step 3 imply $\| \widehat{u}_C - u_C \| \lesssim \delta + \eta(\mathcal{R})$

Step 5. $a_{NC}(\widehat{u}_C, v_{CR}^*) \lesssim \eta(\mathcal{R})(\delta + \eta(\mathcal{R}))$
The proof combines properties of Q and $\widehat{Q}$, the nonconforming interpolation I_{NC} and triangle inequalities with the steps 3 and 4

Proof of (A3)

Step 6 utilizes orthogonalities, involves v_{CR}^* of Step 1 and proves

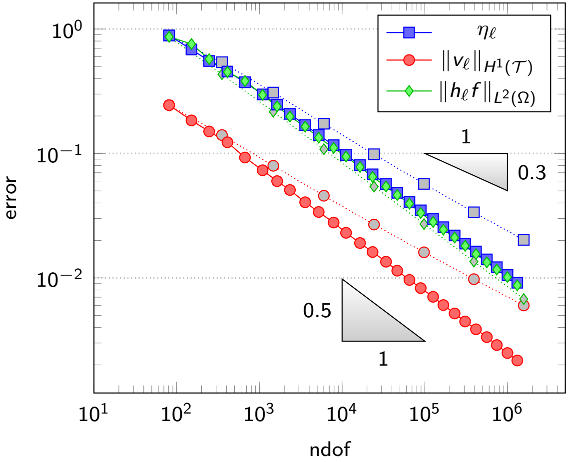
$$\begin{aligned}
&\alpha((\widehat{Q}-Q)\widehat{v}_{CR},(\widehat{Q}-Q)v_{CR})_{L^2(\Omega)}\\
&\quad=\alpha((\widehat{Q}-Q)\widehat{v}_{CR},(\widehat{Q}-Q)(v_{CR}-\widehat{v}_{CR}))_{L^2(\Omega)}+\alpha\|(\widehat{Q}-Q)\widehat{v}_{CR}\|_{L^2(\Omega)}^2\\
&\quad\le\alpha(1+C_{dP}^2h_0^2)\|(\widehat{Q}-Q)\widehat{v}_{CR}\|_{L^2(\Omega)}^2\\
&\qquad\qquad+C_{dP}^{-2}\|h_{\mathcal{T}}^{-1}(\widehat{Q}-Q)(\widehat{v}_{CR}-v_{CR})\|_{L^2(\Omega)}^2/4\\
&\quad\le|\!|\!|\widehat{v}_{CR}-v_{CR}|\!|\!|_{NC}^2/4+\alpha(1+C_{dP}^2h_0^2)C_{dP}^2h_0^2|\!|\!|\widehat{v}_{CR}|\!|\!|_{NC}^2
\end{aligned}$$

Step 7. The combination with $\delta^2(\mathcal{T},\widehat{\mathcal{T}})$ in Step 2 conclude the proof. □

Numerical Examples

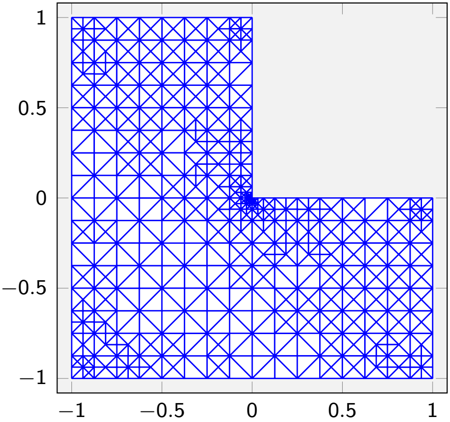
L-Shaped Domain with Adaptive Refinement by η

$$\Omega = (-1,1)^2 \setminus [0,1)^2, \quad u|_{\partial\Omega} \equiv 0, \quad f \equiv 1, \quad \theta = 0.3$$



L-Shaped Domain with Adaptive Refinement by η

$$\Omega = (-1,1)^2 \setminus [0,1)^2, \quad u|_{\partial\Omega} \equiv 0, \quad f \equiv 1, \quad \theta = 0.3$$



triangulation for $\ell = 7$

Conclusion

Conclusion

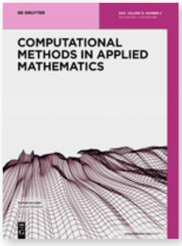
Convergence of natural adaptive dPG FEM possibly *not* understood

Optimal rates are observed but not at all justified theoretically

Reduction of mixed system for dPG methods

Alternative refinement indicators lead to optimal convergence rates

Thank you for your attention!



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