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Reconciling Itô and rough path theory

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Multidimensional stochastic differential equations (SDEs)

$$dY = b_t(\omega, Y)dt + \sigma_t(\omega, Y)d\,B(\omega)$$

Brownian motion $B = B(\omega)$, progressive coefficients b, σ.

- Lipschitz data $\rightsquigarrow$ unique strong solution. "Itô map $B \mapsto Y$" (measurable only)
- Ex-1: b, σ function of (t, Y_t): Markov diffusions $\rightsquigarrow$ linear PDEs (FPE, backward equ)
- Ex-2: b, σ function of $(Y_t, \eta_t(\omega))$: controlled Markov diffusions

$$V(s, y) = \sup_\eta \mathbb{E}^{s,y} g(Y_T) \rightsquigarrow \text{non-linear backward equ (HJB)}$$

- Ex-3: b, σ function of $(Y_t, \mathrm{Law}(Y_t))$, McKean-Vlasov SDE, $\rightsquigarrow$non-linear/non-local FPE

Multidimensional rough differential equations (RDEs)

$$(\star): dY = f(Y)d\,\boldsymbol{X}$$

where $\boldsymbol{X} = (X, \mathbb{X})$ is a α-Hölder rough path, $[1/\alpha] = 2$. This means

$$|X_t - X_s| \lesssim |t-s|^{\alpha}, \qquad |\mathbb{X}_{s,t}| \lesssim |t-s|^{2\alpha}, \qquad \delta\mathbb{X}_{s,t,u} = X_{s,t} \otimes X_{t,u}$$

with $\delta\mathbb{X}_{s,t,u} := \mathbb{X}_{s,u} - \mathbb{X}_{s,t} - \mathbb{X}_{t,u}$. [Dav07]: $(\star) \Leftrightarrow \exists$ neglible $J_{s,t} = O(|t-s|^{3\alpha}) = o(t-s)$

$$Y_t - Y_s = f(Y_s)(X_t - X_s) + ((Df)f)(Y_s)\mathbb{X}_{s,t} + J_{s,t}$$

- $C^{1/\alpha}$ data $\rightsquigarrow$ well-posed [Lyo98, Dav07, FV10 ...] also for regular $f = f(t, y)$.

"Controlled" view [Gub04, FH14+20]: contraction in controlled rough path space

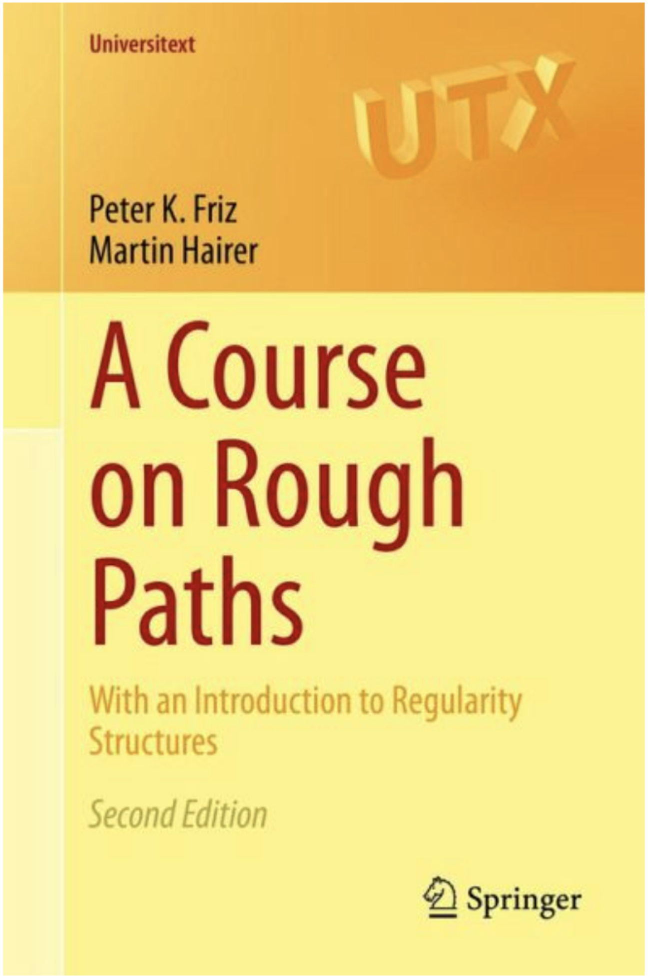
$$\mathfrak{D}_X^{2\alpha} = \{(Y, Y') \colon Y_t - Y_s = Y_s'(X_t - X_s) + O(|t-s|^{2\alpha}), \quad Y_t' - Y_s' = O(|t-s|^{\alpha}) \}$$

Randomize solution to $dY = f(Y)d\boldsymbol{X}$ with Brownian rough path:

$$\boldsymbol{X} \rightsquigarrow \boldsymbol{W} = (W, \mathbb{W}) \text{ with } \mathbb{W}_{s,t} = \int_s^t (W_r - W_s) dW_r$$

$$\rightsquigarrow \quad dY = f(Y)dW(\omega) \quad \text{(SDE)}$$

Itô map factorizes $W \mapsto \boldsymbol{W} \mapsto Y$, and continuous w.r.t. $\|W - \bar{W}\|_\alpha + \|\mathbb{W} - \bar{\mathbb{W}}\|_{2\alpha}$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A Course on Rough Paths
With an Introduction to Regularity Structures
Second Edition
Springer

For Brownian noise (B, W) and another (Lipschitz) coefficient field f consider

$$dY = b_t(\omega, Y)dt + \sigma_t(\omega, Y)d\,B(\omega) + f_t(\omega, Y)dW(\omega). \qquad \text{(dSDE)}$$

Interested in (dSDE) *conditionally on W*. **NB**: $Y|W$ no longer a semimartingale

Studied in 80ies+ (Kunita, Bismut, Davis ...) via stochastic flow transformations (FT).

Idea:

At least for $f = f(Y_t)$, use auxiliary SDE flow to $d\Phi = f(\Phi) \circ d\,W(\omega)$

Consider $Y = \Phi(\tilde{Y})$ and study (Itô) dynamics of $\tilde{Y}$ ...

- Robust Filtering of diffusion processes: (Y, Z) signal-observation pair

$$\begin{aligned} dY_t &= (...)dt + (...)d\,B_t + (...)d\,W_t \qquad \text{(signal)} \\ d\,Z_t &= h(Y_t)dt + W_t \qquad \text{(observation)} \end{aligned}$$

Signal | observation? $\rightsquigarrow$ filter $\mathbb{E}(\varphi(Y_t)|\mathcal{F}_t^Z)$, with $\mathcal{F}_t^Z = \sigma(Z_s: 0 \leqslant s \leqslant t)$

Classical question: M. Clark (1978): Stability of filter w.r.t. observation?

- Positive results (...Clark, Davis, Crisan ...): no correlation, scalar observation

- Counter examples in general case (til rough paths entered the picture ...)

- Dynamics of $Y|Z \leftrightsquigarrow Y|W$?

- Closely related, SPDEs of Zakai type (... Pardoux, Krylov & many ...)

$$\partial_t p = L^\star p + (\Gamma^\star p)\dot{W}$$

(dSDEs appear as stochastic characteristics of such SPDEs)

Doubly SDEs in pathwise control

* Pathwise stochastic control (Lions-Souganidis '98 onwards, Buckdahn-Ma '07 ...)

$$dY = b(\eta_t(\omega), Y)dt + \sigma(\eta_t(\omega), Y)dB + f(Y)dW(\omega)$$

with $\mathcal{F}_t^B$-progressive controls. Leads to random value function

$$V(s, y; \omega) = \mathrm{esssup}_\eta \mathbb{E}^{s,y}(g(Y_T)|\mathcal{F}_T^W),$$

$\rightsquigarrow$ stochastic (viscosity, HJB) PDEs, W-pathwise analysis (only) for constant f [LS98], stochastic FT [BM07], rough PDE view [CFO11] via rough FT

Dynamics of $Y|W$?

Local stochastic volatility: consider $\mathcal{F}^W$-progressive volatility process $v_t(\omega)$

$$dY = \ell(t, Y_t) v_t(\omega) \Big(\rho d B + \sqrt{1-\rho^2} dW \Big),$$

example of dSDE of form $dY = \sigma_t(\omega, Y) d B(\omega) + \ell(t, Y_t) d M_t$ with B BM, $M \in \mathcal{M}^{\mathrm{loc}}$.

NB: want minimal assumptions on $v_t(\omega)$, such as to rough volatility models [GJR18, BFG16],

$$v(t, \omega) := \psi\Big(t, \int_0^t (t-s)^{H-1/2} d W \Big), \quad H \in (0, 1/2).$$

Special case $\ell \equiv 1$: $\mathrm{Law}(Y_T | W) = \mathcal{N}(\rho \int_0^T v(t,\omega) d W, (1-\rho^2) \int_0^T v^2(t,\omega) dt)$

Dynamics of $Y | W$? Gaussian, Markov (indep. increments), not a semimartingale!

Case of general leverage function ℓ?

Doubly SDEs and common noise models

Regard W as environmental/common noise for approximating particle systems

$$\begin{aligned} \mathrm{dY} &= b(\omega, y, \mathrm{Law}(Y_t|W))dt \\ &+ \sigma(\omega, y, \mathrm{Law}(Y_t|W))d\,B(\omega) \\ &+ f(\omega, y, \mathrm{Law}(Y_t|W))d\,W(\omega) \end{aligned}$$

Conditional McKean-Vlasov equation, possibly with progressive coefficients.

(cf. Coghi-Flandoli '17, Lacker-Shkolnikov-Zhang '22)

Conditional propagation of chaos, [CF17]. Conditional dynamics?

Dependence on W?

Natural problem: partial robustification ("rough-path-ification"):

$$dY = b_t(\omega, Y)dt + \sigma_t(\omega, Y)d\,B(\omega) + f_t(\omega, Y)d\,W(\omega) \quad (\text{dSDE})$$

$$\rightsquigarrow$$

$$dY^{\boldsymbol{X}} = b_t(\omega, Y^{\boldsymbol{X}})dt + \sigma_t(\omega, Y^{\boldsymbol{X}})d\,B(\omega) + f_t(\omega, Y^{\boldsymbol{X}})d\,\boldsymbol{X} \quad (\text{formal } \mathbf{R}\text{SDE})$$

With $B \perp W$ and $(\dots)$, expect $Y^{\boldsymbol{X}}|_{\boldsymbol{X} \rightsquigarrow (W, \mathbb{W})} = Y$, (rough path) robustness for r.c.d. $Y|W$ **!**

- RSDEs first appeared in filtering. Crisan-Diehl-F-Oberhauser (2013) consider

$$dY = b(Y)dt + \sigma(Y)d\,B(\omega) + f(Y) \circ d\,\boldsymbol{X},$$

with $\boldsymbol{X} \rightsquigarrow$ observation path, $B \rightsquigarrow$ signal noise, and show robustness in $\boldsymbol{X}$.

Formal RSDEs I: the flow transformation (FT) method

Actually, in [CDFO13] we did not give intrinsic meaning to

$$dY = b(Y)dt + \sigma(Y)dB(\omega) + f(Y)d\boldsymbol{X} \quad \text{(formal RSDE)}$$

Instead: rough flow transformations: $d\Phi = f(\Phi)d\boldsymbol{X}$, and define $Y := \Phi(\tilde{Y})$

$$d\tilde{Y} = \tilde{b}(t,\tilde{Y})dt + \tilde{\sigma}(t,\tilde{Y})dB(\omega).$$

Combine rough flow / Itô SDE robustness, see that $Y := \Phi(\tilde{Y})$ is robust in $\boldsymbol{X}$. But:

- No intrinsic (local) RSDE definition (bad for e.g. discrete approximations)
- Not good for jumps (need nice flow of diffeos ...)
- Excessive regularity demands $f \in C_b^{(1/\alpha)^+ +2}$
- Rigid structural assumptions: method fails e.g. for $f = f(y, \mathrm{Law}(Y_t))$

[DOR15]: Brownian motion $B(\omega)$ and rough path $\boldsymbol{X} = (X, \mathbb{X})$ have joint (random) rough path lift: $J(\omega) = (B(\omega), X)$, second level $\mathbb{J}(\omega)$: Itô and IBP provide "missing" interated integrals

$$\mathbb{J}(\omega) = \left(\int B dB, \int X \, dB, \int B \, dX, \mathbb{X} \right), \qquad \rightsquigarrow \boldsymbol{J}(\omega) := (J(\omega), \mathbb{J}(\omega))$$

Ignoring drift $b(.)$ for brevity, this suggests to understand (formal RPDE) as

$$dY = (\sigma, f)(Y) d\, \boldsymbol{J}(\omega) \quad \leftarrow \text{bona fide random RDE.}$$

The so-defined Y is again robust in $\boldsymbol{X}$. But:

- Excessive (RDE) regularity demands for Itô-vector field, $\sigma \in C^{(1/\alpha)}$ rather than Lipschitz
- Rigid structural assumptions: method fails e.g. for progressive $\sigma = \sigma(y, \eta_t(\omega))$
- Lost Itô / martingales benefits. (E.g. Integrability!)

Rough SDEs: the real thing

Intrinsic understanding and analysis of

$$dY = b_t(\omega, Y)dt + \sigma_t(\omega, Y)d\,B(\omega) + f_t(\omega, Y)d\,\boldsymbol{X} \quad \text{(bona fide RSDE)}$$

based on spaces of stochastic controlled rough paths (SCRP),

built on (some extension of) **stochastic sewing** [Lê20].

- **Basically removes all deficiencies of previous method.**

  + well-posedness under optimal regularity assumption $(b, \sigma$ Lipschitz, $f \in C^{1/\alpha})$

  + cover progressive, non-smoothness dependence (controlled, McKean-Vlasov SDEs ...)

  + martingale structures / estimates still possible

K. Lê's stochastic sewing, taken from [FH20]

Proposition 4.19 (Stochastic Sewing Lemma). *Let* $(s,t) \mapsto \Xi_{s,t} \in L^2_t$ *for* $0 \le s \le t \le T$ *be continuous (viewed as a map with values in* L^2*) with* $\Xi_{t,t} = 0$ *for all* t*. Suppose that there are constants* $\Gamma_1, \Gamma_2 \ge 0$ *and* $\varepsilon_1, \varepsilon_2 > 0$ *such that for all* $0 \le s \le u \le t \le T$*,*

$$\|\delta\Xi_{sut}\|_{L^2} \le \Gamma_1 |t-s|^{\frac{1}{2}+\varepsilon_1} . \tag{4.36}$$

and

$$\|\mathbf{E}_s \delta\Xi_{sut}\|_{L^2} \le \Gamma_2 |t-s|^{1+\varepsilon_2}, \tag{4.37}$$

Then there exists a unique continuous (again as a map $[0,T] \to L^2$*) process* $t \mapsto X_t \in L^2_t$ *with* $X_0 = 0$ *and a suitable constant* C *such that, for all* $0 \le s \le t \le T$*,*

$$\|X_t - X_s - \Xi_{s,t}\|_{L^2} \le C\Gamma_1 |t-s|^{\frac{1}{2}+\varepsilon_1} + C\Gamma_2 |t-s|^{1+\varepsilon_2} \tag{4.38}$$

and

$$\|\mathbf{E}_s(X_t - X_s - \Xi_{s,t})\|_{L^2} \le C\Gamma_2 |t-s|^{1+\varepsilon_2} . \tag{4.39}$$

Assumptions imposed to solve

$$dY = b_t(\omega, Y)dt + \sigma_t(\omega, Y)dB + f_t(\omega, Y)d\boldsymbol{X}:$$

- Coefficients b, σ, f are progressive
- b, σ are bounded Lipschitz, uniformly in (t, ω)
- $f_t(Y)$ is a **controlled vector field** [stochastic controlled extension to $f_t(\omega, Y)$ possibel]
  - $f \in C_b^{1/\alpha^+}$ in space
  - Controlled time-dependence possible: $\exists f'(y)$:

$$R^f_{s,t}(.) = f_t(.) - f_s(.) - f'_s(.)X_{s,t} = O(|t-s|^{2\alpha}) \quad \text{in } C_b^\gamma$$

Example: $f_t(y) = g(y, z_t)$ with nice g, and X-controlled rough path (z, z').Then

$$f'_t(y) = D_2 g(y, z_t) z'_t$$

Expand rough integral part only! Let $\alpha \in (1/3, 1/2)$. With $F = (Df)f$,

$$
\begin{aligned}
Y_t - Y_s &= \int_s^t b_r(Y_r)dr + \int_s^t \sigma_r(Y_r)dB_r(\omega) + \int_s^t f_r(Y_r)d\boldsymbol{X}_r \\
&= \int_s^t b_r(Y_r)dr + \int_s^t \sigma_r(Y_r)dB_r(\omega) \\
&\quad + f_s(Y_s)(X_t - X_s) + \{F_s(Y_s) + f_s'(Y_s)\}\mathbb{X}_{s,t} + J_{s,t}
\end{aligned}
$$

Stochastic remainder $J_{s,t} = O(|t-s|^{2\alpha})$, not negligible. Think (hidden) terms like

$$
\{\mathcal{F}_s\text{-measurable}\} \times \int_s^t (B_r - B_s)\, dX_r \neq o(t-s)
$$

Idea: only require $\mathbb{E}_s J_{s,t} = o(t-s)$ where $\mathbb{E}_s(.) = \mathbb{E}(\cdot|\mathcal{F}_s)$.

With all o-estimates understood in moment norms, this is our definition of RSDE solution.

Theorem 1. *(F-Hocquet-Lê'21) Under (last slide) assumptions, $\exists!$ RSDE solution.*

RSDE solution as stochastic controlled rough paths

Controlled rough path perspective, with $\alpha > 1/3$

$$R^Y_{s,t} \;:=\; Y_t - Y_s - f_s(Y_s)(X_t - X_s) = \int_s^t \sigma_r(Y_r) dB_r(\omega) + O(|t-s|^{2\alpha}).$$

So that $R^Y_{s,t} \neq O(|t-s|^{2\alpha})$ ... but **(!)** $\mathbb{E}_s R^Y_{s,t} = O(|t-s|^{2\alpha})$.

We say that Y is **stochastically controlled** by X with $Y'_s = f_s(Y_s)$.

As before, the O-estimates are understood in moment norms.

RSDE solutions by contraction in stochastic controlled rough path space?

Problem: SCRP space in L_m scale not invariant under composition with regular functions.

Idea: $L_{m,n}$ scale. Composed SCRP in $L_{m,n/2}$ scale. Take $n = \infty$ to close the loop.

In [F & Zorin-Kranich, AOP 22] we introduced rough semimartingales (RSM):

These are adapted cadlag processeses of the form

$$(M(\omega) + Y(w), Y'(w))$$

with $M \in \mathcal{M}_{\mathrm{loc}}$ and ω-wise X-controlled (Y, Y') in p-variation rough path scale.

Quantitative theory: L_m-norms of p-rough path norms (roots: Lépingle BDG).

(!) S.c.r.p. admit similar decomposition in (Hölder) martingale + $L_{m,n}$-valued c.r.p

Definition 2. *Call Y a **solution to RSDE***

$$dY = b_t(\omega, Y)dt + \sigma_t(\omega, Y)dB + f_t(\omega, Y)d\boldsymbol{X}$$

iff $(Y, f(Y))$ is s.c.r.p. ($L_{2,\infty}$) and equation holds in (well-defined) integral sense.

Equivalently,

$$\begin{aligned} J_{s,t} \ := \ & \delta Y_{s,t} - \int_s^t b_r(Y_r)dr - \int_s^t \sigma_r(Y_r)dB_r(\omega) \\ & - f_s(Y_s)\delta X_{s,t} - \{(Df_s(Y_s))f_s(Y_s)\delta X_{s,t} + f_s'(Y_s)\}\mathbb{X}_{s,t} \end{aligned}$$

satisfies

$$\|(\mathbb{E}_s|J_{s,t}|^2)^{1/2}\|_\infty = o(t-s)^{1/2}, \qquad \|\mathbb{E}_s J_{s,t}\|_\infty = o(t-s)$$

Theorem 3. *(Strong existence, uniqueness, stability). Under earlier assumptions there exists a unique $L_{2,\infty}$-solution Y to RSDE.*

Let $\bar{Y}$ solve another RSDE (on same $\mathbb{P}$-space, under same regularity assumptions)

$$d\bar{Y} = \bar{b}_t(\omega, \bar{Y})dt + \bar{\sigma}_t(\omega, \bar{Y})dB + \bar{f}_t(\omega, \bar{Y})d\bar{\boldsymbol{X}}.$$

Then

$$\|Y - \bar{Y}\|_{\alpha;2} + \|\mathbb{E}.R^Y - \mathbb{E}.R^{\bar{Y}}\|_{2\alpha;2} \lesssim (I) + (II) + (III)$$

where

$$\begin{aligned}
(I) &= \|Y_0 - \bar{Y}_0\|_2 + \|X - \bar{X}\|_\alpha + \|\mathbb{X} - \bar{\mathbb{X}}\|_{2\alpha} \\
(II) &= \sup_t \left\| \sup_y |\sigma_t(y) - \bar{\sigma}_t(y)| \right\|_2 + \sup_t \left\| \sup_y |b_t(y) - \bar{b}_t(y)| \right\|_2 \\
(III) &= d_{\alpha,2;X,\bar{X}}(f, f'; \bar{f}, \bar{f}')
\end{aligned}$$

Application 1: Robust filtering revisited

We can go back to Crisan-Diehl-F-Oberhauser (2013) and obtain local Lipschitz continuity of the filter under (significantly) weakened structural and regularity assumptions.

Intrinsic formulation (instead of flow transformations) opens door to relate robust filtering in continuous time to discrete time formulations.

Intrinsic existence/uniqueness for rough Zakai requation etc etc

Consider (non-anticipatingly) controlled rough SDE

$$dY = b(Y, \eta_t(w))dt + \sigma(Y, \eta_t(w))dB(\omega) + f(Y)\, d\mathbf{X}$$

with value function $\mathcal{V}(s, y; \boldsymbol{X}) = \sup_{\eta(.)} \mathbb{E}^{s,y}\big(g(Y_T) + \int_s^T \ell(t, Y_t; \eta_t) dt \big)$.

(Upon randomization $\boldsymbol{X} \rightsquigarrow W(\omega)$ the following answers open Q from [BM07], also Cardaliaguet-Seeger '21).

Theorem 4. *Assuming (uniform) bounded/Lipschitz data, have, with* $\lambda_g \equiv \mathrm{Id}$*,*

$$\begin{aligned} |\bar{\mathcal{V}}(\bar{t}, \bar{y}, \bar{\boldsymbol{X}}) - \mathcal{V}(t, y, \boldsymbol{X})| &\lesssim |(\bar{g} - g)_+|_\infty \\ &+ \lambda_g \left(|T - t \vee \bar{t}|^\alpha (|(b, \sigma, \ell) - (\bar{b}, \bar{\sigma}, \bar{\ell})|_\infty + \rho_\alpha(\bar{\boldsymbol{X}}, \boldsymbol{X})) + |t - \bar{t}|^\alpha + |y - \bar{y}| \right). \end{aligned}$$

Corollary 5. *(Rough-pathwise Dynamic Programming)*

$$\mathcal{V}(s,y;\boldsymbol{X}) = \sup_{\eta(.)} \mathbb{E}^{s,y}\left(\mathcal{V}(s+h, Y_{s+h}; \boldsymbol{X}) + \int_s^{s+h} \ell(t, Y_t; \eta_t) dt \right)$$

Proof: W.l.o.g $\boldsymbol{X}$ *geometric rough path, i.e. limit of smooth* X^ε*. Use classical dynamic programming for* $\mathcal{V}^\varepsilon(s,y) := \mathcal{V}(s,y;X^\varepsilon)$ *and pass to limit, using continuity of* $\boldsymbol{X} \mapsto \mathcal{V}$*.*

Corollary 6. *Solutions to HJB equations, with* $(\star) = |_{(s,y)}\,\mathrm{resp}$

$$-\partial_s \mathcal{V}^\varepsilon = H(s,y,D\mathcal{V}^\varepsilon(\star), D^2\mathcal{V}^\varepsilon(\star)) + \langle D\mathcal{V}^\varepsilon(\star), f(\star)\rangle \dot{X}_s^\varepsilon, \qquad \mathcal{V}^\varepsilon(T,\cdot) \equiv g$$

converges uniformly to $\mathcal{V}(\cdot,\cdot;\boldsymbol{X})$ *as* $X^\varepsilon \to \boldsymbol{X}$ *in rough path sense, previous assumptions [CFO11] significantly relaxed!*

Classical McKean-Vlasov with common (rough) noise. Look for $Y = Y^{\boldsymbol{X}}$ s.t.

$$dY_t = b(Y_t, \mu_t)dt + \sigma(Y_t, \mu_t)d\,B_t(\omega) + g(Y_t, \mu_t)\,d\mathbf{X}$$

with $\mu_t = \mathrm{Law}(Y^{\boldsymbol{X}}) =: \mathrm{Law}(Y_t; \boldsymbol{X})$, $\quad Y_0 \in L_2$.

Assume

- b, σ bounded Lipschitz on $\mathbb{R}^d \times \mathcal{W}_2(\mathbb{R}^d)$, with 2-Wasserstein space
- Regularity of g described through Lions lift $\hat{g}(y, \eta) := g(y, \mathrm{Law}(\eta))$:

$$\hat{g} \in C_b^{1/\alpha^+} \text{ on } \mathbb{R}^d \times L_2(\Omega; \mathbb{R}^d)$$

Theorem 7. *Under above assumption, have unique solution. Also propagation of chaos.*

Remark 8. Comparison with Cass-Lyons, Bailleul-Catellier-Delarue ...

Elements of proof

Rewrite with Lions lift as

$$dY_t = b(Y_t, \mu_t)dt + \sigma(Y_t, \mu_t)dB_t(\omega) + \hat{g}(Y_t, Y_t(.))\, d\mathbf{X}, \quad \mu_t = \mathrm{Law}(Y_t), \quad Y_0 \in L_2$$

Two-step method:

(i) for a fixed SCRP (η, η') solve equation

$$dY_t = b(Y_t, \mu_t)dt + \sigma(Y_t, \mu_t)dB_t(\omega) + \hat{g}(Y_t, \eta_t(.))\, d\mathbf{X}, \quad \mu_t = \mathrm{Law}(\eta_t), \quad Y_0 \in L_2$$

which is indeed a rough SDE as previously presented with (deterministic) controlled

$$f_t\,(y) = \hat{g}(y, \eta_t(.)), \quad f_t'\,(y) = D_2\hat{g}(y, \eta_t(.))\eta_t'(.)$$

Example: $g(y, \mu) = \int h(y, z)\mu(dz)$, $\hat{g}(y, \eta(.)) = \mathbb{E}h(y, \eta) = \int h(y, \eta(w))\mathbb{P}(d\omega)$ and

$$f_t(y) = \mathbb{E}h(y, \eta_t), \;\; f_t'(y) = \mathbb{E}[D_2 h(y, \eta_t)\eta_t'].$$

(ii) Fixed point for $(\eta, \eta') \mapsto (Y^\eta, f^\eta(Y^\eta))$ is solution to (rough) McKean-Vlasov equation.

Outlook and references

- Strong and weak approximation schemes for RSDE. Rates!
- Doubly BSDEs $\Rightarrow$ rough BSDEs (cf. stability results Diehl-F '12 using rough FT)
- Partial Malliavin calculus, heat-kernel expansions, revisited
- Financial modelling (cf. my Bachelier Lectures in April)
- Non-linear / non-local FPE with rough (common) noise
- Controlled McKean-Vlasov SDEs with rough (common) noise
- $\rightsquigarrow$ mean field games? $\rightsquigarrow$ singular SPDEs? $\rightsquigarrow$ Stochastic reconstruction (H. Kern)

F, Hoquet, Lê: Rough stochastic differential equations, arXiv '21/22

F, Zorin-Kranich: Rough semimartingales ... AoP ('23)

F, Hoquet, Lê: McKean-Vlasov: MKV SDEs with rough common noise (in preparation)