Colloquium du CERMICS

Asymptotics for Critical Nonlinear Dispersive Equations

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10 janvier 2019

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General Setting and Universality Questions

Nonlinear partial differential equations with Hamiltonian and *reversible* in time structure appear in models of wave propagation in physics or geometry.

- In the 80s, basic properties of these equations were established, notably the existence and stability of special solutions called solitons.

- In the 90s, tools from harmonic analysis led to a refined understanding of properties of the corresponding linear equations and how to extend these properties to nonlinear equations. In particular, the notion of criticality appeared.

- The problem of *understanding the asymptotic dynamics related to nonlinear objects (or special solutions)* remained. This question has attracted considerable interest in the last fifteen years, and yet we are just beginning to have a rough picture of the subject.

In this talk, I will consider a few classical models:

- *the mass critical Nonlinear Schrödinger Equation*

$$(cNLS) \qquad i\partial_t u + \partial_x^2 u + |u|^4 u = 0,\ u_{|t=0} = u_0, (t,x) \in [0,T) \times \mathbb{R},$$

- *the mass critical Korteweg–de Vries Equation*

$$(cKdV) \qquad \partial_t u + \partial_x(\partial_x^2 u + u^5) = 0, u_{|t=0} = u_0, (t,x) \in [0,T) \times \mathbb{R}.$$

These equations are called focusing and are expected to have some truly *nonlinear behavior*.

In the 80s/90s, local solutions in time were studied using Strichartz and related estimates on the linear equation, and treating the nonlinear term in a perturbative way (Ginibre/Velo, Kato, Kenig/Ponce/Vega, Bourgain, Cazenave/Weissler...).

For (cNLS) and (cKdV), we have existence and uniqueness of a maximal solution $u(t)$ on $[0,T)$ in L^2 and $H^1 = \{f : f \text{ and } \nabla f \in L^2\}$.
- Either $T = +\infty$ ($u(t)$ is global),
- or $T < +\infty$ ($u(t)$ blows up in finite time). If $u_0 \in H^1, \lim_{t\to T} |\nabla u(t)|_{L^2} = +\infty$.

Further properties:

Conservation laws: for $t \in [0, T)$,

$$M(u(t)) = \int |u|^2(t,x)dx = M(u_0),$$

$$\text{if } u_0 \in H^1, \quad E(u(t)) = \frac{1}{2}\int |\partial_x u|^2(t,x)dx - \frac{1}{6}\int |u|^6(t,x)dx = E(u_0).$$

Focusing is reflected in the $-$ sign in the energy.

Symmetry:

- space/time translation, for (cNLS) phase and Galilean invariance,
- scaling invariance:

if $u(t,x)$ is a solution of (cNLS), then for $\lambda > 0$, $u_\lambda(t,x) = \lambda^{\frac{1}{2}} u(\lambda^2 t, \lambda x)$ also (and for (cKdV), $u_\lambda(t,x) = \lambda^{\frac{1}{2}} u(\lambda^3 t, \lambda x)$).

These transformations leave invariant the L^2 norm of the solution, so that both problems are called *mass critical*.

Classical examples of special solutions:

(i) *Small data result:*
smallness in the critical space implies global existence and scattering (asymptotic linear behavior in time).

Specific for focusing problems:

(ii) *Nonlinear objects* such as:
- for (cKdV): traveling wave $u(t,x) = Q(x-t)$,
- for (cNLS): periodic solution of the form $u(t,x) = e^{it}Q(x)$,

where $Q \in H^1$ solves $Q_{xx} - Q + Q^5 = 0$.

(iii) *Self-similar solution:*
(The typical example of blow-up solution for noncritical eq.) Up to some time-dependent translation and phase:
- for (cKdV), $u(t,x) = \frac{1}{(T-t)^{\frac{1}{6}}} F(\frac{x}{(T-t)^{\frac{1}{3}}})$,
- for (cNLS), $\frac{1}{(T-t)^{\frac{1}{4}}} F(\frac{x}{(T-t)^{\frac{1}{2}}})$.

From the criticality and the conservation laws, we will exclude such self-similar blow-up (in a more general form). *Nonexistence of self-similar blow-up* is a key feature of critical equations compared to noncritical ones and a challenge deeply related to the nature of each equation. This is an essential step toward classification results for the asymptotic behavior of solutions.

Surprisingly, all objects appearing in such classifications so far are simply the ones presented in (i) (ii). Since the 70s, there has been a belief that:
for large global solutions of dispersive equations, the evolution asymptotically *decouples for large time into a sum of modulated solitons and a free radiation term (the soliton resolution conjecture)*.
In PDE, this is known in only two situations: the integrable case (where nonlinear eq. can be reduced to linear eq.) and the parabolic case (where time irreversibility is natural).

The problem has two parts: *construction of examples and classification*. These questions are linked to understanding the nature of dispersion at infinity in space and its coupling with the nonlinear dynamics.

(A) *Interaction of nonlinear/linear dynamics*: Typically, these are dynamics which are (up to modulation) asymptotic to a nonlinear object *(asymptotic stability or bubbling solution with a universal profile).* In the examples, these dynamics are degenerate and unstable (more degenerate directions than those given by symmetries), and behavior of initial data at infinity is essential. A formal understanding of these dynamics and rigorously establishing them was a challenge that required new ideas related to *nonlinear properties*.

(B) *Interaction of nonlinear objects/nonlinear dynamics*: Interaction between (spatially) decoupled nonlinear objects.

(C) *General decomposition and interaction*: General case of not prepared, large data.
General challenge: find a nonintegrable dispersive situation with *soliton resolution*.

Main type (A) problems:
- *Understand blow-up behavior for mass critical NLS,*
- *Prove and understand blow-up for mass critical KdV.*
Type (B),(C) problems will be mentioned briefly.

Strategy: see that deep knowledge of *dispersion* is related to a *monotonicity formula* (decreasing quantity up to lower order terms) and gives *irreversibility.*

The Nonlinear Schrödinger Equation

- From an obstructive identity related to a conformal invariance (70's),

if $E(u_0) < 0$ and $u_0 \in \Sigma = H^1 \cap \{xu_0 \in L^2\}$, then $T < +\infty$ (blow-up),

(no information on how) and we have the explicit blow-up solution ($T = 0$)

$$S(t,x) = \frac{1}{|t|^{\frac{1}{2}}} Q\left(\frac{x}{|t|}\right) e^{i\frac{|x|^2}{4t} + \frac{i}{|t|}} \text{ with } |S|_{L^2} = |Q|_{L^2}\ ,\ |\nabla S(t)|_{L^2} \sim \frac{1}{|t|}.$$

- Variational arguments yield that no blow-up occurs for $|u_0|_{L^2} < |Q|_{L^2}$.

- The following initial result containing a rigidity notion for Hamiltonian dynamics and notion of *nondispersive solution* was proved:

Theorem (Dynamical characterization of S and Q, Merle, 92)

Let $u_0 \in H^1$, $|u_0|_{L^2} = |Q|_{L^2}$ then:
- either u is equal to S or to $Q(x)e^{it}$, up to the symmetries of the equation,
- or u is global, and scatters as $t \to \pm\infty$ if $u_0 \in \Sigma$.

The next challenge was to understand the dynamics for $u_0 \in H^1$ such that

$$|Q|_{L^2}^2 < |u_0|_{L^2}^2 < |Q|_{L^2}^2 + \alpha^* \quad \text{with} \quad 0 < \alpha^* \ll 1. \tag{1}$$

We now consider dynamics close to Q up to renormalization.

- A direct approach allowed Bourgain/Wang (Krieger/Schlag) to construct *unstable* (Merle/Raphaël/Szeftel) blow-up solutions such that

$$u(t,x) - S(t,x) \to u^* \ \text{ in } \ H^1 \text{ as } \ t \to 0.$$

- The *generic/stable singularity formation* was open for several decades. In the 80s, formal/numerical works (Landman/Papanicolaou/Sulem/Sulem) suggested a “loglog” correction of the self-similar rate:

$$|\nabla u(t)|_{L^2} \sim \sqrt{\frac{\log|\log(T-t)|}{T-t}}.$$

- Since the linearized problem around Q is very degenerate, the linear theory does not give the formal picture. The idea was to consider, near Q, a localization of self-similar blow-up profile (which barely fails to be in L^2) $\tilde{Q}_{b(t)}$, where $b(t) \in \mathbb{R}$ and *find irreversibility on $b(t)$ from a monotonicity formula.*

Theorem (L^2 critical blow-up 02-05's, Merle, Raphaël)

Assume (1).
(i) Sufficient condition for loglog blow-up*: If $E(u_0) < 0$, or $E(u_0) = 0$ and $u \neq Q$, then u blows up in finite time with the loglog speed*

$$|\nabla u(t)|_{L^2} \sim \sqrt{\frac{\log|\log(T-t)|}{2\pi(T-t)}} \quad as \quad t \to T.$$

(ii) Stability of loglog blow-up*: The set of initial data u_0 such that $u(t)$ blows up in finite time with the loglog speed is open in H^1.*
(iii) Universality of bubble profile and classification of blow-up rate*: If $T < +\infty$, then there exist $(\lambda(t), x(t), \gamma(t))$ and $u^* \in L^2$ such that:*

$$u(t,x) - \frac{1}{\lambda(t)^{\frac{1}{2}}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) e^{i\gamma(t)} \to u^* \quad in \quad L^2,$$

with $x(t) \to x(T)$, $\lambda(t) \sim \frac{1}{|\nabla u(t)|_{L^2}}$ as $t \to T$, and $|\nabla u(t)|_{L^2}$ satisfies either the loglog speed or $|\nabla u(t)|_{L^2} \geq \frac{1}{T-t}$ as $t \to T$.

See also G. Perelman.

The point of view of using monotonicity properties together with specific localization of a self-similar profile, in problems of oscillatory integrals, has been successfully used to give a formal analysis and a rigorous proof of a blow-up regime in the following cases:

(i) *The energy critical wave (Hillairet/Raphaël, Krieger/Schlag/Tataru)*,

(ii) *The energy critical wave maps into the sphere S^2 (Raphaël/Rodnianski , Rodnianski/Sterbenz, Krieger/Schlag/Tataru)*,

(iii) *The energy critical Schrödinger maps into the sphere S^2 (Merle/Raphaël/Rodnianski)*,

(iv) *The energy supercritical Heat, Wave, Schrödinger Equations (Herrero/Velasquez, Merle/Matano, Merle/Raphaël/Rodnianski, Collot, Collot/Merle/Raphael, Merle/Raphae/Szeftel)*
Another range of applicability in the recent years.

Generalized Korteweg–de Vries Equation

(cKdV) admits the same conservation laws and a similar scaling invariance as (cNLS) but no conformal invariance or obstructive identity: the problem of blow-up for (cKdV) is natural and challenging. For $0 < \alpha^* \ll 1$, consider

$$|u_0|_{L^2}^2 < |Q|_{L^2}^2 + \alpha^*. \tag{2}$$

As an application of rigidity (Liouville type theorems), we have:

Theorem (Blow-up by rigidity, Martel, Merle 00's)

(i) Negative energy gives blow-up: If $E(u_0) < 0$, then the solution blows up with T finite or infinite ($|\nabla u(t)|_{L^2} \to \infty$ as $t \to T$).
(ii) Universality of bubbling: There are $\lambda(t) \sim \frac{1}{|\nabla u(t)|_{L^2}}, x(t)$ such that for $A > 0$,

$$u(t,x) - \frac{1}{\lambda(t)^{\frac{1}{2}}} Q\left(\frac{x - x(t)}{\lambda(t)}\right) \to 0 \ \ in \ \ L^2 \ \ for \ \ \{|x - x(t)| < A\lambda(t)\} \ as \ t \to T.$$

Main point of the proof:
If not, we have a contradiction from energy constraints ($E(u_0) < 0$) and the exact asymptotic behavior of the solution (asymptotic stability of Q).

Let $v(t)$ be a new solution such that for $t_n \to \infty$, up to scaling

$$u(t_n + t, x_n + x) \to v(t, x) \text{ locally in } L^2.$$

Monotonicity properties on $u(t)$ break the reversible character of the solution $v(t)$, showing that $v(t)$ is a nondispersive solution (up to modulation has small tails uniformly in time). We then get rigidity: $v(t, x) = Q(x - t)$.
Since $E(Q) = 0$, the conservation of the energy gives a contradiction.

Recently, we understood all solutions as in (2) and their asymptotics:
For initial data with *decay*, near Q, we give a *complete nonlinear finite dimensional description of the dynamical picture* (despite the high degeneracy of Q).
This is the only such situation known. We expect that this picture is *canonical*.

Consider the set of initial data for $0 < \alpha_0 \ll \alpha^* \ll 1$,

$$\mathcal{A} = \{u_0 = Q + \epsilon_0 \text{ with } |\epsilon_0|_{H^1} < \alpha_0 \text{ and } \int_{x>0} x^{10}\epsilon_0^2 < 1\} \subset \mathcal{T}_{\alpha^*}$$

where $\mathcal{T}_{\alpha^*} = \{u \in H^1 \text{ with } \inf_{c_0>0,\, x_0\in\mathbb{R}} |u - Q_{c_0}(. - x_0)|_{L^2} < \alpha^*\}$.

Theorem (Rigidity of the flow in $\mathcal{A}$, Martel, Merle, Raphaël 12)

Let $u_0 \in \mathcal{A}$. Then one of the following scenarios occurs:

(Blow-up): *The solution blows up in finite time $T > 0$ with the universal regime*

$$|u(t)|_{H^1} \sim \frac{\ell(u_0)}{T - t} \text{ as } t \to T, \text{ with } \ell(u_0) > 0.$$

(Soliton): *The solution is global and converges to Q (renormalized).*

(Exit): *The solution leaves the tube $\mathcal{T}_{\alpha^*}$ at some time $0 < t^*(u_0) < +\infty$.*

The scenarios (Blow-up) *and* (Exit) *are stable in $\mathcal{A}$.*

(i) *Universality in the blow-up rate is lost without decay of the initial data* ($u_0 \notin \mathcal{A}$) and the H^1 Martel/Merle theory is optimal for solutions without decay:

- Initial data with slowly decaying tails interacting with the solitary wave lead to blow up solutions $u(t)$ such that $|\partial_x u(t)|_{L^2} \sim (T-t)^{-\nu}$ as $t \to T$, for $\nu > \frac{11}{13}$.

(ii) The long-time dynamics in the (Exit) regime (Martel, Merle, Raphaël):

- We have *existence and uniqueness of a minimal blow-up solution* $\tilde{S}(t)$ ($|\tilde{S}|_{L^2} = |Q|_{L^2}$) which is the surprising generalization of the $S(t)$ dynamics for (cNLS).

- This unstable solution $\tilde{S}(t)$ is the *universal attractor* in L^2 of all solutions in the (Exit) regime (see also Krieger/Nakanishi/Schlag for Klein-Gordon eq).

Example for type (B) problem

We now consider the quartic KdV eq. (no blow-up and *nonintegrable*)

$$\partial_t u + \partial_x(\partial_x^2 u + u^4) = 0, \quad (t,x) \in [0,T) \times \mathbb{R},$$

and *interaction* between two solutions of the form $Q_c(. - ct)$.

Theorem (Martel/Merle/Tsai, Martel (i), Martel/Merle (ii) 10s)

Let $0 < c_1 < c_2$.
(i) 2-solitons at $t = +\infty$: *There exists a unique solution* $U_{c_1,c_2}(t)$ *such that*

$$\lim_{t\to+\infty} \left| U_{c_1,c_2}(t) - [Q_{c_1}(. - c_1 t) + Q_{c_2}(. - c_2 t)] \right|_{H^1} = 0.$$

(ii) behavior at $t = -\infty$: *For a* $\delta > 0$, *if* $\frac{c_1}{c_2} < \delta$ *or* $\frac{3}{4} < \frac{c_1}{c_2} < 1$, *then* $U_{c_1,c_2}(t)$ *is not a 2-soliton as* $t \to -\infty$ *and "has a dispersive part".*

Thus, *collisions are inelastic*. In the *integrable* case, we again recover a 2-soliton as $t \to -\infty$ using the explicit formula (*elastic collision*).

Example for type (C) problem

Consider the *focusing energy critical Nonlinear Wave eq.* in 3 dimensions with radial data:

$$(ecNLW) \qquad \partial_t^2 u = \Delta u + u^5, \quad (u_{|t=0}, \partial_t u_{|t=0}) = (u_0, u_1), \quad (t, x) \in [0, T) \times \mathbb{R}^3.$$

- (ecNLW) has properties similar to those of (cNLS) in L^2, in the energy space: $(u(t), \partial_t u(t)) \in \dot{H}^1 \times L^2$ where $\dot{H}^1 = \{f : \nabla f \in L^2\}$. There is a unique solution $u(t)$ on $[0, T)$ in $\dot{H}^1 \times L^2$ and the scaling of the eq. leaves invariant the $\dot{H}^1 \times L^2$-norm so that the problem is *energy critical*.

- Nonlinear objects such as $W \in \dot{H}^1$ solution of $\Delta W + W^5 = 0$ (explicit in the radial case) and using Lorentz invariance, $W_c(x - ct)$ where $|c| < 1$.

- A theory has been developed in the critical space related to the notion of nondispersive solution by Kenig, Merle and by Duyckaerts, Kenig, Merle to understand the dynamics. We now have the *full soliton resolution in the radial situation (partial resolution in the nonradial case)* for large data (which gives the first result in the nonintegrable case):

Theorem (Radial soliton resolution, Duyckaerts, Kenig, Merle)

Assume that the solution is global. Then, there exist a solution v_{lin} of the linear wave eq., an integer $J \geq 0$, $\iota_j \in \{\pm 1\}$, and $\lambda_1(t) \ll \lambda_2(t) \ll \ldots \ll \lambda_J(t) \ll t$ as $t \to +\infty$ such that

$$\left|(u(t), \partial_t u(t)) - \left(v_{lin}(t) + \sum_{j=1}^{J} \frac{\iota_j}{\lambda_j^{1/2}(t)} W\left(\frac{x}{\lambda_j(t)}\right), \partial_t v_{lin}(t)\right)\right|_{\dot{H}^1 \times L^2} \underset{t \to +\infty}{\longrightarrow} 0.$$

Theorem (Partial soliton resolution, Duyckaerts, Kenig, Jia, Merle)

Assume that the solution is global. Then, there exist a solution v_{lin} of the linear wave eq., $J \geq 0$, solitary wave Q_{c_i} , $\lambda_i(t), x_i(t)$ and $t_n \to +\infty$ such that

$$\left|(u(t_n), \partial_t u(t_n)) - \left(v_{lin}(t_n) + \sum_{j=1}^{J} \frac{1}{\lambda_j^{1/2}(t_n)} Q_{c_i}\left(\frac{x - x_i(t_n)}{\lambda_j(t_n)}\right), \ldots\right)\right|_{\dot{H}^1 \times L^2} \underset{n \to +\infty}{\longrightarrow} 0.$$

DKM $\quad u$ non dispersif $\iff u \equiv w_l(x - lt)$.

Conclusion

We have illustrated *universality* for some canonical problems. One can see a unity in these problems without apparent links *a priori*:
- similarities in the results,
- similarities in the approaches (monotonicity formulas, few parameters related to special solutions with nonlinear interactions where cancellations play a basic role, etc.).

Nevertheless, the proofs use the specifics of each equation.

Finally, we observe that there are many wide open directions of research related to these approaches...