Colloquium du CERMICS

École des Ponts
ParisTech

Recent progress to the theory of stochastic/pathwise solutions

Panagiotis Souganidis (The University of Chicago)

18 mai 2018

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CERMICS, May 2018

work with Pierre-Louis Lions

- plan of the talk
  - stochastic/pathwise viscosity solutions (a very brief review)
  - domain of dependence and speed of propagation
  - homogenization
  - long time behavior a false proof
  - intermittent regularizing effects
  - behavior as $t \to \infty$ in the convex case
  - behavior as $t \to \infty$ work in progress and some open problems
  - some proofs

- pathwise/stochastic viscosity solutions

$$du = H(Du, u, x) \cdot d\omega + F(D^2u, Du, u, x)dt$$

ω continuous (Brownian or, more generally, rough path)

enough to be able to solve the (ode) $dX = -D_xH(P, X) \cdot d\omega \quad dP = D_pH(P, X) \cdot d\omega$

$u \in \mathbb{R}$ F degenerate elliptic

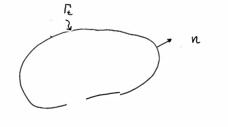
F and H may depend on t

if ω depends on x it must be regular KPZ is outside the scope of the theory

many applications pathwise control, phase transitions, stochastic selection principles, ...

- motion of interfaces

an interface Γ_t evolves with normal velocity $V_n = -\text{mean curvature} + a(x) \cdot dB$



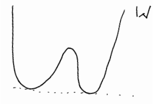
$$\Gamma_t = \{x \in \mathbb{R}^d : u(x,t) = 0\}$$

u solves the level-set pde

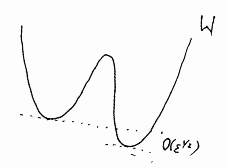
$$du = \text{tr}[(I - \frac{Du \otimes Du}{|Du|^2})D^2u] + a(x)|Du| \cdot dB$$

- phase field theory- asymptotics of reaction diffusion equations perturbed by additive noise

$$u_t^\varepsilon - \Delta u^\varepsilon = \frac{1}{\varepsilon} W'(u^\varepsilon)$$



$u^\varepsilon \underset{\varepsilon \to 0}{\to} \pm 1$ inside and outside a front evolving with n.v. $V_n = -$mean curvature



$$u_t^\varepsilon - \Delta u^\varepsilon = \frac{1}{\varepsilon}(W'(u^\varepsilon) + \varepsilon^{1/2} c)$$

$V_n = -$mean curvature $+ \alpha c$

Kawasaki and Otha conjectured that, if the depths of the wells are perturbed randomly by $\varepsilon^{1/2} dB$, the resulting interface will evolve by $V_n = -$mean curvature $+ \alpha dB$

this is not true — the perturbation is too violent to preserve the stability properties of ± 1

but

if the wells are perturbed by a "mild approximation" $\dot{B}^\varepsilon$ of B, then the interface evolves by

$V_n = -$mean curvature $+ \alpha dB$

Almgren, Yip (convex setting), Funaki (for $d = 2$), Lions and S. general problem

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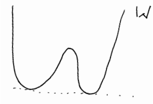

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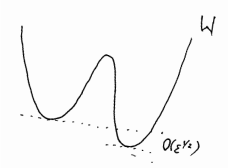

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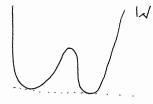

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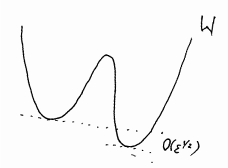

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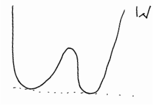

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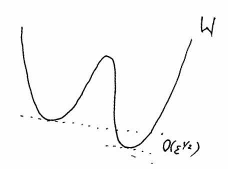

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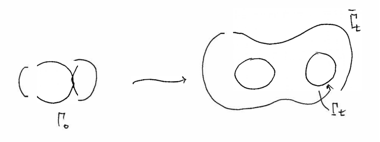
- a stochastic selection principle   work with A. Yip

" unstable configurations"



is there a stochastic mechanism that selects at the limit a unique interface?

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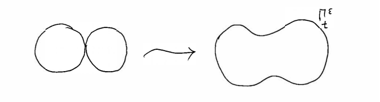
converges a.s. to the maximal solution of the motion without the noise



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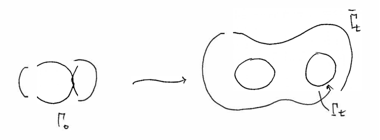

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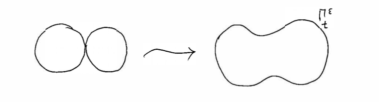

- pathwise stochastic control theory

B_1, B_2 independent Bm with filtrations $(\mathcal{F}_t^{B_1})_{t\geq 0}, (\mathcal{F}_t^{B_2})_{t\geq 0}$

$\mathcal{A}$ the set of admissible ($\mathcal{F}_t^{B_1}$ – progressively measurable controls) controls $(\pi_t)_{t\geq 0}$ with values in A

dynamics
$$\begin{cases} dX_t = b(X_t,\pi_t)dt + \sqrt{2}\sigma_1(X_t,\pi_t)dB_{1,t} + \sigma_2(X_t,\pi_t)\circ dB_{2,t} & (0 \leq s \leq T) \\ X_s = x \end{cases}$$

payoff $\quad J(x,s;\pi) = E_{x,s}[H(X_T)|\mathcal{F}_T^{B_2}]$

value function $\quad u(x,s) = \text{essinf}_{\pi\in\mathcal{A}} J(x,s;\pi)$

pathwise Bellman equation

$$\begin{cases} du + \inf_{\pi\in A}\left[\text{tr}((\sigma_1\sigma_1^\star)(x,\pi)D^2u) + b(x,\pi)\cdot Du\right] + \sigma_2(x)Du\circ dB_2 = 0 \\ u(\cdot,T) = H \end{cases}$$

classical stochastic control problem

payoff $\quad \bar{J}(x,s;\pi) = E_{x,s}[H(X_T)$

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Bellman equation

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- review of pathwise viscosity solutions

$du = H(Du) \cdot d\omega \quad u(\cdot, 0) = u_0$

- "deterministic viscosity solutions" $\omega \in \mathrm{BV} \Rightarrow \exists!$ solution $u \in C_{x,t}$ and comparison

  $\|(u-v)_\pm(\cdot,t)\| \leq \|(u_0 - v_0)_\pm\| \qquad \|Du(\cdot,t)\| \leq \|Du_0\|$

  in general shocks (discontinuities of Du) appear in finite time

is it possible to extend by density to $\omega \in C$? (Itô vs Stratonovich when ω is B motion)

- $du = u_x dB$ ill posed

$$du(x - B(t), t) \underset{\text{Itô's formula and equation}}{=} \left(\frac{1}{2} u_{xx}(x - B(t), t) - u_{xx}(x - B(t), t)\right) dt$$
$$= -\frac{1}{2} u_{xx}(x - B(t), t) dt$$

- $H(p) = |p| \quad u_0(x) = |x|$ density in $\omega \Rightarrow$

$$u(x,t) = \max\left[(|x| + \omega(t))_+, \omega(t) - \min_{0 \leq s \leq t} \omega(s))\right]$$

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"THEOREM" If H is the difference of two convex functions, then $\exists!$ solution with the same properties as in the classical case

- solutions are continuous in H and ω

- solutions to problems with regularized H and ω converge to the same limit

  if $u_{\varepsilon,t} = H_\varepsilon(Du_\varepsilon)\dot{\omega}_\varepsilon$ with $H_\varepsilon, \omega_\varepsilon$ smooth approximations to H and ω, then $\|u_\varepsilon - u\| \underset{\varepsilon\to 0}{\to} 0$

- $\|u(\cdot,t)\|$ and $\operatorname{osc} u(\cdot,t)$ decrease in t

  the contraction property

- $H(0) = 0 \;\Rightarrow\; \max u(\cdot,t)$ and $\min u(\cdot,t)$ decrease in t

  formally $d[\max u(\cdot,t)] \leq 0$ and $d[\max u(\cdot,t)] \geq 0$

- domain of dependence and finite speed of propagation   joint work with Gassiat and Gess

is there a domain of dependence for $du = H(Du, x) \cdot d\omega$?

$$u_1(\cdot, 0) = u_2(\cdot, 0) \text{ in } B(0, R_0) \quad \Rightarrow \quad u_1(\cdot, t) = u_2(\cdot, t) \text{ in } B(0, R(t))?$$

- a partial result

$H(p, x) = H_1(p) - H_2(p) \quad H_1, H_2$ convex

$$u(\cdot, 0) \equiv A \text{ in } B(0, R) \quad \Rightarrow \quad u(\cdot, t) \equiv A \text{ in } B(0, R(t))$$

$$R(t) := R - L(\max_{s\in[0,T]} \omega(s) - \min_{s\in[0,T]} \omega(s))$$

- a negative result   Gassiat

$du = (|u_x| - |u_y|) \cdot d\omega \qquad u(x, y, 0) = |x - y| + \Theta(x, y) \qquad \Theta \geq 1 \text{ if } x, y \geq R$

$u(0, 0, T) \geq 0 \quad \text{if} \quad \|\omega\|_{TV_{[0,T]}} > R$

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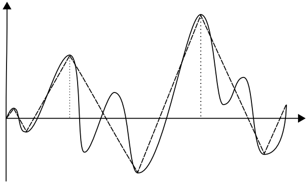
- finite speed of propagation

$du = H(Du, x) \cdot d\omega \quad H$ convex in $p \quad \omega \in C_0([0, T])$

$\rho_H(\xi, T) := \sup \left\{ R \geq 0 : \ u^1(\cdot, 0) = u^2(\cdot, 0) \text{ in } B_R(0) \text{ and } \ u^1(0, T) \neq u^2(0, T) \right\}$

skeleton $\quad R_{0,T}(\omega)$



- positive results
  - $\rho_{H,T}(\omega) \leq L \, \|R_{0,T}(\omega)\|_{TV([0,T])}$
  - $B \quad$ Brownian motion $\quad \Rightarrow \quad \|R_{0,T}(B)\|_{TV([0,T])} < \infty$ a.s.
  - $H(p) = |p| \quad \Rightarrow \quad \rho_H(\xi, T) \geq \|R_{0,T}(\xi)\|_{TV([0,T])}$

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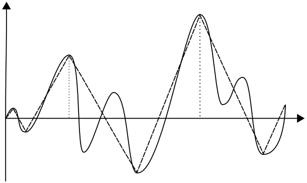

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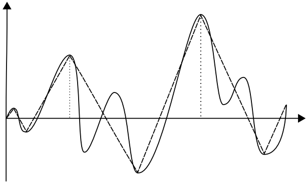

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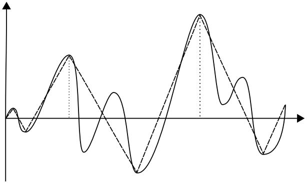

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- homogenization

Ben Seeger

- $u_t^\varepsilon = H(Du^\varepsilon, \frac{x}{\varepsilon})\dot{B}^\varepsilon$ H convex coercive periodic $B^\varepsilon \to B$ in distribution

there exists $\overline{H}$ convex st $\begin{cases} u^\varepsilon \to \overline{u} \\ d\overline{u} = \overline{H}(D\overline{u}) \circ dB \end{cases}$

- $u_t + H(Du, x) = f(x)\dot{\omega}$ ω piecewise constant with slope ± 1

$u^\varepsilon = u(\frac{x}{\varepsilon}, \frac{t}{\varepsilon})$ $\omega^\varepsilon(t) = \varepsilon^{1/2}\omega(\frac{t}{\varepsilon}) \to B$ Brownian motion

$u_t^\varepsilon + H(Du^\varepsilon, \frac{x}{\varepsilon}) = \varepsilon^{1/2} f(\frac{x}{\varepsilon})\dot{\omega}^\varepsilon$

there exists $\overline{H}$ st $\begin{cases} u^\varepsilon \to \overline{u} \\ \overline{u}_t + \overline{H}(D\overline{u}) = 0 \end{cases}$ BUT

$\overline{H}$ does not come from the periodic homogenization

$\varepsilon^{1/2}\omega(\frac{t}{\varepsilon})$ creates a stationary ergodic environment

- $u_t^\varepsilon + |Du^\varepsilon| = \varepsilon^{1/2} f(\frac{x}{\varepsilon})\dot{\omega}^\varepsilon$

$\lim_{\varepsilon \to 0} u^\varepsilon(x,t) = \overline{c}\, t$ $\overline{c} := \lim_{T\to\infty} \frac{1}{T} \inf\left\{ \int_0^T f(Y(s))\dot{B}(s) \ : |\dot{Y}| \le 1 \right\}$

- homogenization Ben Seeger

  - $u_t^\varepsilon = H(Du^\varepsilon, \frac{x}{\varepsilon})\dot{B}^\varepsilon$ H convex coercive periodic $B^\varepsilon \to B$ in distribution

    there exists $\overline{H}$ convex st $\begin{cases} u^\varepsilon \to \overline{u} \\ d\overline{u} = \overline{H}(D\overline{u}) \circ dB \end{cases}$

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- long time behavior

$$du = H(Du) \cdot d\omega \quad u(\cdot,0) = u_0 \qquad \begin{cases} H \text{ continuous, } u_0 \text{ continuous and ``periodic'' in } x \\ \omega \text{ continuous and } \omega(0) = 0 \\ H(0) = 0 \quad \Rightarrow \text{ constants are solutions} \end{cases}$$

$u(\cdot,t) \underset{t\to\infty}{\to} u^\infty$ with u^∞ constant in x and depending only on u_0 and ω ?

- $H(p) = v \cdot p \quad v \in \mathbb{R}^d$

  $u(x,t) = u_0(x + v\omega(t)) \not\to$ constant

- $H(p) = |p|^2 \quad \dot\omega(t) \geq 0$ and $\omega(t) \underset{t\to\infty}{\to} \infty$

  $u(x,t) = \inf_y [u_0(y) + \frac{1}{4\omega(t)}|x-y|^2] \to \inf u_0$

"nonlinearity" and monotonicity of $\omega \Rightarrow$ limit

what happens if ω oscillates ?

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- $t \to \infty$ a false proof!

$d = 1$ $\omega =$ Brownian motion u_0 periodic

$$du = H(u_x) \underset{(S)}{\circ} dB = H(u_x) \underset{(I)}{\cdot} dB + \frac{1}{2} H'(u_x)^2 u_{xx} dt$$

$$d(\int_0^1 u dx) = (\int_0^1 H(u_x) dx) dB \qquad \int_0^1 \phi(u_x) u_{xx} = 0$$

$M_t = \int_0^1 u(x,t) dx$ is a bounded martingale $\Rightarrow$

$$M_t \underset{t \to \infty}{\to} M_\infty \text{ and } \int_0^\infty (\int_0^1 H(u_x) dx)^2 dt < \infty \text{ a.s.}$$

if $H(z) > 0$ for $z \neq 0$, then $u(x,t,\omega) \underset{t \to \infty}{\to} M_\infty$ a.s.

argument incorrect due to shocks!

what if H is more regular, for example, convex?

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what if H is more regular, for example, convex?

- intermittent regularity

Theorem: for H convex and all continuous ω,

$$-\frac{c\,I_d}{\omega(t)-\min\limits_{0\le s\le t}\omega(s)} \le (D^2H)^{1/2}(Du)\cdot D^2u\cdot (D^2H)^{1/2}(Du) \le \frac{c\,I_d}{\max\limits_{0\le s\le t}\omega(s)-\omega(t)}$$

- $|(D^2H)^{1/2}(Du)\cdot D^2u\cdot (D^2H)^{1/2}(Du)| \le c\ \max\Big(\frac{1}{\max\limits_{0\le s\le t}\omega(s)-\omega(t)}, \frac{1}{\omega(t)-\min\limits_{0\le s\le t}\omega(s)}\Big)$

- Gassiat and Gess studied the case $H(p)=|p|^2$ using the explicit Lax-Oleinik formula

- the result as one sided bound is new even in the classical case (ω monotone)

- $D^2H>0$ and $\min\limits_{0\le s\le t}\omega(s)<\omega(t)<\max\limits_{0\le s\le t}\omega(s) \ \Rightarrow\ u(\cdot,t)\in C^{1,1}(\mathbb{R}^d)$

- the long time behavior in the convex case

H convex $\qquad H(0)=0 \qquad DH(p)\cdot p>0 \quad p\neq 0$

Theorem: for all ω st $\exists t_n \to \infty$ such that

either $\omega(t_n) - \min_{0\le s\le t_n} \omega(s) \underset{n}{\to} \infty$ or $\max_{0\le s\le t_n} \omega(s) - \omega(t_n) \underset{n}{\to} \infty$

$$u(\cdot,t) \underset{t\to\infty}{\to} u_\infty$$

- true for Brownian motion
- what is the law of u_∞? nontrivial — only partial results available

idea of proof:

$(D^2H)^{1/2}(Dv)\cdot D^2v\cdot(D^2H)^{1/2}(Dv)\le 0 \;\Rightarrow\; \mathrm{div}DH(Dv)=\mathrm{tr}D^2H(Dv)D^2v\le 0 \underset{\text{periodicity}}{\Rightarrow}$

$0=\int_Q \mathrm{tr}[D^2H(Dv)D^2v]dx\le 0 \;\Rightarrow\; \mathrm{div}DH(Dv)=D^2H(Dv)D^2v=0$

$\underset{\text{multiply by } v \text{ and integrate over } Q}{\Rightarrow} \int_Q DH(Dv)\cdot Dv=0 \underset{\text{assumption on } H}{\Rightarrow} Dv=0$

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- asymptotic behavior ($t \to \infty$) work in progress and open problems
  - H not convex or concave? x-dependent problems?
  - $du = \Sigma_{i=1}^{K} H_i(Du_i) \cdot \dot{\omega}_i \quad \omega_i$ "independent"

    $\dot{\omega}_2 = -\dot{\omega}_1 = \dot{\omega} \qquad du = (H_1 - H_2)(Du)\dot{\omega}$

    $H_1 = H_2 = H \qquad \dot{\omega} = \dot{\omega}_1 + \dot{\omega}_2 \qquad du = H(Du)(\omega_1 + \omega_2)$
  - systematic approach to ergodicity (Lions course 2015) when ω Brownian
  - $$\begin{cases} du = H(Du) \circ dB \\ \max_x u(x,t,\omega) \text{ independent of } t \quad \text{a.s} \\ \min_x u(x,t,\omega) \text{ independent of } t \quad \text{a.s} \end{cases} \Rightarrow \quad u \equiv c(\omega) \text{ ??}$$

- the proof of the estimate
  - divide $(0,\infty)$ into intervals where $\dot{\omega}$ is either positive or negative
  - in each such interval $u(x,t) = v(x,\omega(t))$ where v solves a $v_t = \pm H(Dv)$
  - enough to propagate an (appropriate) upper and lower bound for v on each such interval
  - result follows by iteration
  - important step is to establish an actual decay on each interval

- sketch of the proof in a special case

$$u_t = |Du|^2 \text{ in } \mathbb{R}^d \times (0,\infty) \quad u(\cdot,0) = u_0 \quad u(x,t) = \sup_{y\in\mathbb{R}^d}[u_0(y) - \frac{|x-y|^2}{4t}]$$

$$\text{“}(D^2u)_t = 2DuD(D^2u) + 2|D^2u|^2\text{”} \qquad u_{xx,t} = 2u_x u_{xxx} + 2(u_{xx})^2$$

- equation preserves concavity $\quad D^2u_0 \le 0 \;\Rightarrow\; D^2u(\cdot,t) \le 0.$
- equation preserves semiconvexity $\quad D^2u_0 \ge -CI_d \;\Rightarrow\; D^2u(\cdot,t) \ge -CI_d$
- equation regularizes $\quad D^2u(\cdot,t) \ge -\frac{2I_d}{t}$

not good enough !! for the iteration we need to improve last two

Theorem: $\quad D^2u_0 \ge -CI_d \;\Rightarrow\; D^2u(\cdot,t) \ge -\frac{2C}{1+Ct}I_d$

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$$u_t = H(Du) \text{ in } \mathbb{R}^d \times (0,T) \quad u(\cdot,0) = u_0 \qquad F(p) = (D^2H(p))^{1/2}$$

- claim 1 G symmetric matrix

$$G(Du_0)D^2u_0G(Du_0) \geq -C_0 I_d \Rightarrow G(Du(\cdot,t))D^2u(\cdot,t)G(Du(\cdot,t)) \geq -C_0 I_d$$

- claim 2 $G = F$ and any $u_0 \Rightarrow$

$$G(Du(\cdot,t))D^2u(\cdot,t)G(Du(\cdot,t)) \geq -\frac{C}{t} I_d$$

- claim 3 $G = F$ and C_0 for $u_0 \Rightarrow$

$$G(Du(\cdot,t))D^2u(\cdot,t)G(Du(\cdot,t)) \geq -\frac{C_0}{1 + C_0 t} I_d$$

- $u(x,t) = \sup_{y\in\mathbb{R}^d} \left[u_0(y) - tH^\star(\frac{x-y}{t})\right] = u_0(\bar{y}) - tH^\star(\frac{x-\bar{y}}{t})$

$$u(x \pm h\eta, t) = \sup_{y\in\mathbb{R}^d} \left[u_0(y) - tH^\star(\frac{x \pm h\eta - y}{t})\right] \geq u_0(\bar{y}) - tH^\star(\frac{x \pm h\eta - \bar{y}}{t})$$

$$\Rightarrow \quad < D^2u(x,t)\eta, \eta > \geq -\frac{1}{t} < D^2H^\star(\frac{x-y}{t})\eta, \eta >$$

$\eta = F(DH^\star(\frac{x-y}{t}))\xi \;\; \Rightarrow \;\;$ claim 2

- $u(x,t) = \sup_{y \in \mathbb{R}^d} \left[u_0(x-y) - tH^\star(\frac{y}{t}) \right] = u_0(x-\bar{y}) - tH^\star(\frac{\bar{y}}{t})$

$$u(x \pm h\eta, t) = \sup_{y \in \mathbb{R}^d} \left[u_0(x \pm h\eta - y) - tH^\star(\frac{y}{t}) \right] \geq u_0(x \pm h\eta - \bar{y}) - tH^\star(\frac{\bar{y}}{t})$$

$$\Rightarrow \quad < D^2 u(x,t)\eta, \eta > \geq < D^2 u_0 \eta, \eta >$$

$\eta = G(Du)\xi \quad \Rightarrow \quad$ claim 1

- $u(x,t) = \sup_{y,z\in\mathbb{R}^d:\ y+z=x} \left[u_0(y) - tH^\star(\frac{z}{t})\right] = u_0(\bar{y}) - tH^\star(\frac{\bar{z}}{t})$

$$u(x \pm h\eta, t) = \sup_{y,z\in\mathbb{R}^d: y+z=x\pm h\eta} \left[u_0(y \pm \theta h\eta) - tH^\star(\frac{z \pm (1-\theta)h\eta}{t})\right] \geq$$

$$u_0(\bar{y} \pm \theta h\eta) - tH^\star(\frac{\bar{z} \pm (1-\theta)h\eta}{t})$$

$$\Rightarrow \quad < D^2u(x,t)\eta, \eta > \geq -\left[\theta^2 C_0 + (1-\theta)^2 \frac{1}{t}\right] < \eta, \eta >$$

minimize over θ and $\eta = F(Du(x,t))\xi \quad \Rightarrow$ claim 3