A small dive into mathematical finance

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November 4, 2022

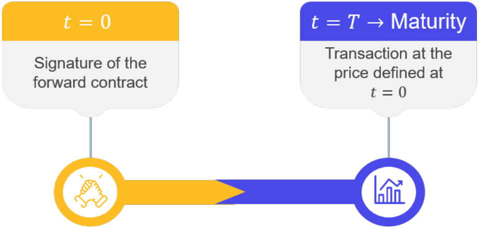
Journées Scientifiques des Jeunes Chercheurs du CERMICS (JSJC)

Definition and genesis

- **Obvious definition :** the field of applied mathematics concerned with the modelling of financial markets.
- Why is it useful to model the **dynamics** of financial markets ? There are essentially two applications :
  - Design of **portfolio allocations** and **investment strategies** with positive expected returns and limited risks.
  - Derivatives pricing and hedging.
- Two **key dates** in mathematical finance's history :
  - **1900** : **Louis Bachelier**'s PhD thesis "Théorie de la spéculation" → first mathematical introduction of the Brownian motion and application to the evolution of stock prices : **mathematical finance is born !**
  - **1973** : pioneering paper of Fischer Black and Myron Scholes → first mathematical model to price and hedge so-called call and put options. **By 1975, almost all traders were using their model to price and hedge option portfolios.**

What are derivatives ?

- **Derivatives** = financial product whose **future payment flows** depend on the value of an underlying asset : a stock, a basket of stocks, interest rates, a commodity, a FX rate, etc. Main types of derivatives :
  - **Futures/Forward contracts :** contract allowing to fix today the price of a future transaction over a given asset.

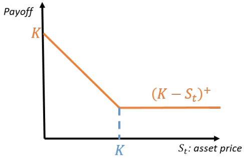


  - **Option contracts :** same as forward contract but the buyer of the contract can decide at the maturity whether he wants or not to perform the transaction.
  - **Swaps :** contract between two parties allowing to exchange a series of payments. Example : exchange fixed interest rate against variable interest rate.

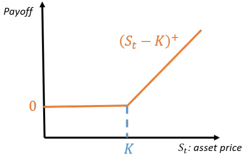
What is the use of derivatives ?

Some examples :

- EDF buys forward contracts on gas during the summer to **secure** the gas supply for the winter and to reduce the uncertainty about their annual energy bill.
- An international company selling stuff in dollars and paying its employees in euros can buy forward contracts or options on the USD/EUR FX rate to **hedge itself** against the future fluctuations of the FX rate.



- Many hedge funds bought sell options (**puts**) to bet on the decrease of GameStop's stock price in January 2021 → **speculative purpose**.



- Listed companies can **retain** key employees by giving them **stock options** (**calls**) → allow them to make a profit if the company's stock price increases.

Focus on option contracts

Option contracts can be bought on two types of market :

- **organized markets** = a centralized and regulated financial market which brings together buyers and sellers in a transparent way.
  - The price is determined by the supply-demand balance.
  - Almost only vanilla options (calls and puts) are traded.
- **over-the-counter (OTC) market** = decentralized market in which market participants trade directly between two parties, without the use of a central exchange or other third party.
  - The price is generally determined by the seller of the option.
  - All kinds of option contracts can be traded.

The main sellers of option contracts on the OTC market are **investment banks** → the **question of the pricing and the hedging** of option contracts is **essential** for them.

Arbitrage opportunity

A central notion in mathematical finance is the **absence of arbitrage opportunities**.

Definition (Arbitrage opportunity)

Investment strategy allowing to make a profit without taking any risk

Example : triangular arbitrage. Suppose :

- USD/EUR rate = 1
- USD/GBP rate = 1.1
- GBP/EUR rate = 0.8

If I have 100€, I can convert them into \$100, then convert them into $100/1.1 \simeq 90.91$£ and finally convert them back to $(100/1.1)/0.8 \simeq 113.64$ €. Assuming no transaction costs, I won $113.64 - 100 = 13.64$€ without taking any risk → there is an arbitrage ! More generally, if I have three currencies A, B and C, the following relation must be verified ($r_{X/Y}$ is the X/Y FX rate) :

$$r_{A/B} = r_{A/C} \times r_{C/B}$$

otherwise there is an arbitrage opportunity.

In mathematical finance, the price of a derivative is defined as **price that does not lead to an arbitrage opportunity**

Mathematical setting (1/4)

The market is modelled by :

- a time horizon $T > 0$,
- a probability space $(\Omega, \mathscr{A}, \mathbb{P})$ with a right-continuous filtration $(\mathscr{F}_t)_{0\leq t\leq T}$,
- $K+1$ assets traded continuously from time 0 to time T. Their prices are denoted by $S_t^0, \ldots, S_t^K$ which are stochastic processes (semimartingales). The asset indexed by 0 is a bank account :

$$dS_t^0 = r_t S_t^0 dt$$

where r_t is the short-rate at time t (potentially stochastic).

Definition (Trading strategy)

A trading strategy is a $K+1$-dimensional process $\phi = (\phi_t)_{0\leq t\leq T}$ satisfying some properties of measurability and regularity. The value of the associated portfolio is given by :

$$V_t(\phi) = \langle \phi_t, S_t \rangle = \sum_{k=0}^{K} \phi_t^k S_t^k$$

Mathematical setting (2/4)

Definition (Self-financing trading strategy)

A trading strategy is self-financing if $V_t(\phi) \geq 0$ *for all* t *almost surely and if there is no additional cash inflows or outflows after the initial time, i.e.*

$$dV_t = \sum_{k=0}^{K} \phi_t^k dS_t^k$$

Definition (Arbitrage opportunity)

An arbitrage opportunity is a self-financing trading strategy ϕ *such that :*

1. $V_0(\phi) = 0$
2. $\mathbb{P}(V_T > 0) > 0$

Mathematical setting (3/4)

Theorem (Harrison and Pliska (1983))

The market is free of arbitrage if and only if there exists a martingale measure, i.e. a probability measure $\mathbb{Q}$ *that is equivalent to* $\mathbb{P}$ *and such that the discounted asset prices* $(S_t^k / S_t^0)_{0 \leq t \leq T}$ *are martingales. This martingale measure is often called the* ***risk-neutral*** *probability.*

Definition (Replicable option)

An option is replicable if its payoff H *is a square-integrable and positive random variable and if there exists a self-financing* ϕ *such that* $V_T(\phi) = H$ *a.s.*

Proposition

If the market is free of arbitrage and if an option payoff H *is replicable, then the unique price at time* t *of the option is given by :*

$$\pi(t, S_t) := V_t(\phi) = \mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0} H \mid \mathscr{F}_t\right]$$

where ϕ *is the replicating strategy.*

Mathematical setting (4/4)

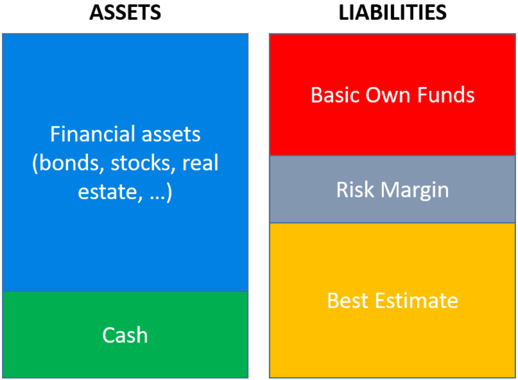
- The last proposition provides both a way to **price** and **hedge** an option contract :
  - the price corresponds to the **initial value $V_0(\phi)$ of the replicating portfolio**
  - the hedge corresponds the **self-financing strategy ϕ**
- This proposition also raises two questions :
  1. How to compute the expectation $\mathbb{E}^{\mathbb{Q}}\left[\frac{S_t^0}{S_T^0} H \mid \mathscr{F}_t\right]$?
  2. How to compute the **self-financing strategy ϕ** ?
- The most widespread approach to answer these questions is to make an assumption about the dynamics of the asset prices $(S_t)_{0 \le t \le T}$ → Example (Black-Scholes) :

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

  where $(W_t)_{t \ge 0}$ is a Brownian motion. The expectation is then computed using a **closed-form formula** if possible or using **Monte-Carlo simulations**. The hedge is generally obtained as the derivative of the option price with respect to the underlying asset price ($\partial_S \pi(t, S_t)$).

A word about mathematical finance in the insurance industry

- Solvency II (SII) is the current regulatory framework for insurance and reinsurance companies in Europe that came into force in 2016.
- It introduces two major innovations in the actuarial landscape : **Market-Consistent** valuation of the balance-sheet and **Risk-based capital requirements**.
- To understand these notions, we present below the simplified balance sheet of an insurer :



The balance sheet valuation

- **Valuation of assets :** easy !
- **Valuation of liabilities :** that is where the problems begin...
- The **Best Estimate** is the expected value of all future discounted cash-flows of the insurer :

$$BEL = \mathbb{E}^{\mathbb{Q}}\left[\sum_{t=1}^{T} e^{-\int_0^t r_s ds} F_t\right]$$

  where $\mathbb{Q}$ is the risk-neutral probability. The use of this probability measure is imposed by the regulation : this is the so-called **market-consistent** valuation. Note that the cash-flows F_t of the insurer typically include :
  - contracts exits,
  - payment of the minimum guaranteed rate with profit-sharing participation,
  - payment of pensions and insurance claims ,
  - collect of premia from clients.

  In practice, there is **no closed-form formula** !
- The **Risk Margin** is a margin added to the Best Estimate to take into account the fact that most of the insurance liabilities are not replicable → theoretically we could not use the risk-neutral probability $\mathbb{Q}$.
- The **Basic Own Funds** BOF are then computed as the difference the value of the assets A minus the Best Estimate BE and the Risk Margin RM :

$$BOF = A - BE - RM.$$

  If $BOF < 0$, the company is **bankrupt** !

Solvency Capital Requirement

- At this stage, we have only described how to value the balance sheet at $t = 0$ but it does not say anything about the risk of **future insolvency**! Indeed, the assets and liabilities are going to change in the future and as a consequence it is possible that the BOF become negative.
- This leads us to the second major innovation of Solvency II : the **Solvency Capital Requirement** (SCR). The SCR is the minimum value of BOF needed to keep the risk of bankruptcy on a year horizon below 0.5%. Mathematically, it is defined as the solution of :

$$\mathbb{P}(BOF_1 = A_1 - BE_1 - RM_1 < 0 \mid BOF_0 = SCR) = 0.5\%.$$

where $\mathbb{P}$ is the real-world probability. In practice, the SCR is approximated by the 99.5% Value-at-Risk of the insurer portfolio loss at a one-year risk horizon :

$$SCR = VaR_{99.5\%}\left(BOF_0 - e^{-\int_0^1 r_s ds} BOF_1\right).$$

- Hence, estimating the SCR is **very challenging** because of :
  - the interactions between the assets and the liabilities
  - the necessity to project the assets and liabilities using the real-world probability and to price them using the risk-neutral probability with no-closed form formula.

Bibliography

Harrison, J. M., & Pliska, S. R. (1981). Martingales and stochastic integrals in the theory of continuous trading. Stochastic processes and their applications, 11(3), 215-260.

Harrison, J. M., & Pliska, S. R. (1983). A stochastic calculus model of continuous trading : complete markets. Stochastic processes and their applications, 15(3), 313-316.

Lamberton, D. & Lapeyre, B. (1997). Introduction au calcul stochastique appliqué à la finance.

Tankov, P. Mathématiques financieres. Cours du master M2MO.