A small dive into mathematical finance

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Definition and genesis

- **Obvious definition**: the field of applied mathematics concerned with the modelling of financial markets.

- Why is it useful to model the **dynamics** of financial markets? There are essentially two applications:
  - Design of **portfolio allocations** and **investment strategies** with positive expected returns and limited risks.
  - Derivatives pricing and hedging.

- Two **key dates** in mathematical finance’s history:
  - **1900**: Louis Bachelier’s PhD thesis "Théorie de la spéculation" → first mathematical introduction of the Brownian motion and application to the evolution of stock prices: **mathematical finance is born!**
  - **1973**: pioneering paper of Fischer Black and Myron Scholes → first mathematical model to price and hedge so-called call and put options. **By 1975, almost all traders were using their model to price and hedge option portfolios.**
What are derivatives?

- **Derivatives** = financial product whose *future payment flows* depend on the value of an underlying asset: a stock, a basket of stocks, interest rates, a commodity, a FX rate, etc. Main types of derivatives:
  - **Futures/Forward contracts**: contract allowing to fix today the price of a future transaction over a given asset.
  - **Option contracts**: same as forward contract but the buyer of the contract can decide at the maturity whether he wants or not to perform the transaction.
  - **Swaps**: contract between two parties allowing to exchange a series of payments. Example: exchange fixed interest rate against variable interest rate.
What is the use of derivatives?

Some examples:

- EDF buys forward contracts on gas during the summer to secure the gas supply for the winter and to reduce the uncertainty about their annual energy bill.

- An international company selling stuff in dollars and paying its employees in euros can buy forward contracts or options on the USD/EUR FX rate to hedge itself against the future fluctuations of the FX rate.

- Many hedge funds bought sell options (puts) to bet on the decrease of GameStop’s stock price in January 2021 → speculative purpose.

- Listed companies can retain key employees by giving them stock options (calls) → allow them to make a profit if the company’s stock price increases.
Focus on option contracts

Option contracts can be bought on two types of market:

- **organized markets** = a centralized and regulated financial market which brings together buyers and sellers in a transparent way.
  - The price is determined by the supply-demand balance.
  - Almost only vanilla options (calls and puts) are traded.

- **over-the-counter (OTC) market** = decentralized market in which market participants trade directly between two parties, without the use of a central exchange or other third party.
  - The price is generally determined by the seller of the option.
  - All kinds of option contracts can be traded.

The main sellers of option contracts on the OTC market are **investment banks** → the question of the pricing and the hedging of option contracts is essential for them.

\[
\begin{align*}
t = 0 & \\
\text{Investment bank sells } 1\text{€ a call of strike } 100\text{€ and maturity 1 year on a stock quoted at } 100\text{€}
\end{align*}
\]

\[
\begin{align*}
t = 1 \text{ year} & \\
\text{The price of the stock is now } 110\text{€; the buyer of the call receives } 110 - 100 = 10\text{€. The bank has lost } 10 - 1 = 9 \text{€}
\end{align*}
\]
Arbitrage opportunity

A central notion in mathematical finance is the **absence of arbitrage opportunities**.

**Definition (Arbitrage opportunity)**

Investment strategy allowing to make a profit without taking any risk

**Example : triangular arbitrage.** Suppose :

- USD/EUR rate = 1
- USD/GBP rate = 1.1
- GBP/EUR rate = 0.8

If I have 100€, I can convert them into $100, then convert them into $100/1.1 \approx 90.91£ and finally convert them back to $(100/1.1)/0.8 \approx 113.64€. Assuming no transaction costs, I won 113.64€ – 100€ = 13.64€ without taking any risk → there is an arbitrage! More generally, if I have three currencies $A$, $B$ and $C$, the following relation must be verified ($r_{X/Y}$ is the $X/Y$ FX rate) :

\[
r_{A/B} = r_{A/C} \times r_{C/B}
\]

otherwise there is an arbitrage opportunity.

In mathematical finance, the price of a derivative is defined as **price that does not lead to an arbitrage opportunity**
The market is modelled by:

- a time horizon $T > 0$,
- a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ with a right-continuous filtration $(\mathcal{F}_t)_{0 \leq t \leq T}$,
- $K+1$ assets traded continuously from time 0 to time $T$. Their prices are denoted by $S_0^t, \ldots, S_K^t$ which are stochastic processes (semimartingales). The asset indexed by 0 is a bank account:

$$dS_0^t = r_t S_0^t \, dt$$

where $r_t$ is the short-rate at time $t$ (potentially stochastic).

**Definition (Trading strategy)**

A trading strategy is a $K+1$-dimensional process $\phi = (\phi_t)_{0 \leq t \leq T}$ satisfying some properties of measurability and regularity. The value of the associated portfolio is given by:

$$V_t(\phi) = \langle \phi_t, S_t \rangle = \sum_{k=0}^{K} \phi_t^k S_t^k$$
Definition (Self-financing trading strategy)

A trading strategy is self-financing if \( V_t(\phi) \geq 0 \) for all \( t \) almost surely and if there is no additional cash inflows or outflows after the initial time, i.e.

\[
dV_t = \sum_{k=0}^{K} \phi_t^k dS_t^k
\]

Definition (Arbitrage opportunity)

An arbitrage opportunity is a self-financing trading strategy \( \phi \) such that:

1. \( V_0(\phi) = 0 \)
2. \( P(V_T > 0) > 0 \)
Theorem (Harrison and Pliska (1983))

The market is free of arbitrage if and only if there exists a martingale measure, i.e. a probability measure \( Q \) that is equivalent to \( P \) and such that the discounted asset prices \( (S_t^k/S_0^k)_{0 \leq t \leq T} \) are martingales. This martingale measure is often called the risk-neutral probability.

Definition (Replicable option)

An option is replicable if its payoff \( H \) is a square-integrable and positive random variable and if there exists a self-financing \( \phi \) such that \( V_T(\phi) = H \) a.s.

Proposition

If the market is free of arbitrage and if an option payoff \( H \) is replicable, then the unique price at time \( t \) of the option is given by:

\[
\pi(t, S_t) := V_t(\phi) = \mathbb{E}^Q \left[ \frac{S^0_t}{S^0_T} H \mid \mathcal{F}_t \right]
\]

where \( \phi \) is the replicating strategy.
The last proposition provides both a way to **price** and **hedge** an option contract:

- the price corresponds to the **initial value** $V_0(\phi)$ of the replicating portfolio
- the hedge corresponds the **self-financing strategy** $\phi$

This proposition also raises two questions:

1. How to compute the expectation $\mathbb{E}^Q \left[ \frac{S_t^0}{S_T^0} H \mid \mathcal{F}_t \right]$?
2. How to compute the **self-financing strategy** $\phi$?

The most widespread approach to answer these questions is to make an assumption about the dynamics of the asset prices $(S_t)_{0 \leq t \leq T} \rightarrow$ Example (Black-Scholes):

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $(W_t)_{t \geq 0}$ is a Brownian motion. The expectation is then computed using a **closed-form formula** if possible or using **Monte-Carlo simulations**. The hedge is generally obtained as the derivative of the option price with respect to the underlying asset price ($\partial_S \pi(t, S_t)$).
A word about mathematical finance in the insurance industry

• Solvency II (SII) is the current regulatory framework for insurance and reinsurance companies in Europe that came into force in 2016.
• It introduces two major innovations in the actuarial landscape: **Market-Consistent** valuation of the balance-sheet and **Risk-based capital requirements**.
• To understand these notions, we present below the simplified balance sheet of an insurer:

```
<table>
<thead>
<tr>
<th>ASSETS</th>
<th>LIABILITIES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Financial assets (bonds, stocks, real estate, ...)</td>
<td>Basic Own Funds</td>
</tr>
<tr>
<td>Cash</td>
<td>Risk Margin</td>
</tr>
<tr>
<td></td>
<td>Best Estimate</td>
</tr>
</tbody>
</table>
```
The balance sheet valuation

- **Valuation of assets**: easy!
- **Valuation of liabilities**: that is where the problems begin...
- The **Best Estimate** is the expected value of all future discounted cash-flows of the insurer:

\[
BEL = \mathbb{E}^Q \left[ \sum_{t=1}^{T} e^{-\int_0^t r_s ds} F_t \right]
\]

where \( Q \) is the risk-neutral probability. The use of this probability measure is imposed by the regulation: this is the so-called **market-consistent** valuation.

Note that the cash-flows \( F_t \) of the insurer typically include:
- contracts exits,
- payment of the minimum guaranteed rate with profit-sharing participation,
- payment of pensions and insurance claims,
- collect of premia from clients.

In practice, there is **no closed-form formula**!

- The **Risk Margin** is a margin added to the Best Estimate to take into account the fact that most of the insurance liabilities are not replicable → theoretically we could not use the risk-neutral probability \( Q \).
- The **Basic Own Funds** \( BOF \) are then computed as the difference the value of the assets \( A \) minus the Best Estimate \( BE \) and the Risk Margin \( RM \):

\[
BOF = A - BE - RM.
\]

If \( BOF < 0 \), the company is **bankrupt**!
Solvency Capital Requirement

- At this stage, we have only described how to value the balance sheet at \( t = 0 \) but it does not say anything about the risk of future insolvency! Indeed, the assets and liabilities are going to change in the future and as a consequence it is possible that the BOF become negative.

- This leads us to the second major innovation of Solvency II: the Solvency Capital Requirement (SCR). The SCR is the minimum value of BOF needed to keep the risk of bankruptcy on a year horizon below 0.5%. Mathematically, it is defined as the solution of:

\[
\mathbb{P}(BOF_1 = A_1 - BE_1 - RM_1 < 0 \mid BOF_0 = SCR) = 0.5%.
\]

where \( \mathbb{P} \) is the real-world probability. In practice, the SCR is approximated by the 99.5% Value-at-Risk of the insurer portfolio loss at a one-year risk horizon:

\[
SCR = Var_{99.5\%}\left(BOF_0 - e^{-\int_0^1 r_s ds} BOF_1\right).
\]

- Hence, estimating the SCR is very challenging because of:
  - the interactions between the assets and the liabilities
  - the necessity to project the assets and liabilities using the real-world probability and to price them using the risk-neutral probability with no-closed form formula.


Tankov, P. Mathématiques financières. Cours du master M2MO.