Topological invariants of the Kitaev chain

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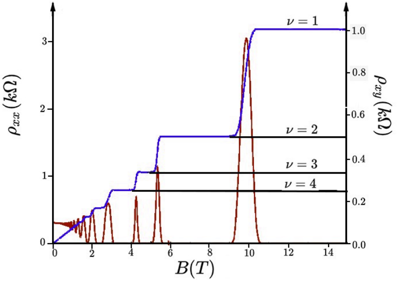

Figure: Quantum Hall Effect[1]

Interpretation: the plateaus correspond to different topological phases of matter .

[1] Marvin E Cage, Kv Klitzing, AM Chang, F Duncan, M Haldane, RB Laughlin, AMM Pruisken, and DJ Thouless. The quantum Hall effect. Springer Science Business Media, 2012.

Bulk-edge correspondence

For some systems one can associate an edge index $I^\sharp$ and a bulk index I, and one has:

$$I = I^\sharp \quad \text{(bulk-edge correspondence)}.$$

Indices are **topological**: continuous with respect to the system.

Quantum Hall Effect: I is related to the Planck constant h, $I^\sharp$ to the electrical resistivity measured.

Model[3]

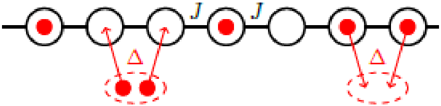

Figure: Kitaev chain model[2]

1D model for fermions with interactions similar to those leading to superconductivity. Hamiltonian H acting on $\mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathbb{C}^2$.

$$(H\psi)_n = A\psi_{n-1} + A^*\psi_{n+1} + V\psi_n, \quad n \in \mathbb{Z}. \tag{1}$$

$$A = \begin{pmatrix} 0 & 1+\delta \\ 1-\delta & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & \mu \\ \mu & 0 \end{pmatrix}, \quad \delta, \mu \in \mathbb{R}.$$

[2] Jean Dalibard. Topologie à une dimension : du modèle SSH aux modes de Majorana. In La matière topologique et son exploration avec les gaz quantiques. Collège de France, May 2018.

[3] A Yu Kitaev. Unpaired Majorana fermions in quantum wires. Physics-Uspekhi, 44(10S):131, 2001.

We transform the operator H acting on $\mathcal{H}$ to a family of operators H_q which are 2×2 matrices.

Bloch wave: $\psi_q = \sum_{n \in \mathbb{Z}} e^{iqn} \delta_n \otimes \begin{pmatrix} \alpha_q \\ \beta_q \end{pmatrix}$. Then $H\psi_q = H_q \psi_q$,

$$H_q = \begin{pmatrix} 0 & 2\cos q - 2i\delta \sin q + \mu \\ 2\cos q + 2i\delta \sin q + \mu & 0 \end{pmatrix}.$$

Theorem $\sigma(H) = \bigcup_q \sigma(H_q)$[4].

[4]T Muthukumar. Bloch-Floquet transform. In ATM Workshop on PDE and Fourier Analysis, Uttarpradesh, 2014.

[5]Michael Reed and Barry Simon. IV: Analysis of Operators, volume 4, chapter XIII. Elsevier, 1978.

$$\sigma(H) = \left\{ \pm\sqrt{(2\cos(q) + \mu)^2 + 4\delta^2 \sin(q)^2}\ ;\ -\pi < q < \pi \right\}.$$

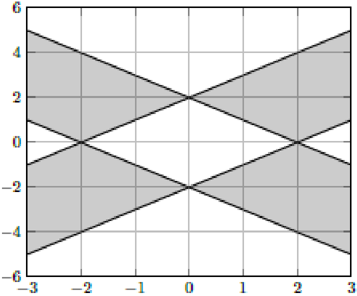

Figure: Spectrum of H with respect to $\mu \in [-3, 3]$.

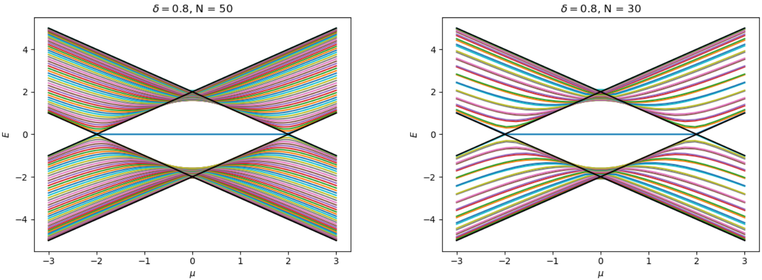

Figure: Spectra of the finite (left) and semi-infinite (right) chain.

Apparition of edge states for Kitaev chain with boundaries.

We consider projectors in a Hilbert space $\mathcal{H}$ of dimension $2n$.
We say that $P(t)$ is **particule-hole symmetric** (PHS), if

$$CP(t)C = 1 - P(-t) = P(-t)^{\perp}. \tag{2}$$

$$\mathcal{P}_C = \{P(t);\ \text{continuous 1-per PHS projectors of rank } n\}.$$

Remarks

- Spectral projectors: $P(q) = \mathbf{1}_{H_q<0}$. For the Kitaev chain, $C = \sigma_z K$.
- P and $P^{\perp} = \mathbf{1} - P$ play very similar roles, the choice $P = \mathbf{1}_{H<0}$ is arbitrary.
- At $t = 0$ and $t = 1/2$, stronger condition: $CPC = 1 - P$.

Index
The index I is defined for $P(t) \in \mathcal{P}_C$ by:

$$I(P) = (\mathrm{pf}(Q(0)), \mathrm{pf}(Q(1/2)) \in \mathbb{Z}_2 \times \mathbb{Z}_2,$$

where $Q(0) = -i(2P(0) - 1)$ and $Q(1/2) = -i(2P(1/2) - 1))$.

Theorem
Let $P_0, P_1 \in \mathcal{P}_C$. We have the equivalence: $I(P_0) = I(P_1)$ if and only if there exists an homotopy $P : [0,1] \to \mathcal{P}_C$ such that $P(0) = P_0, P(1) = P_1$.

Lemma If P satisfies PHS, then $Q = i(\mathbf{1} - 2P)$ is orthogonal and skew-symmetric. Conversely, if $Q \in \mathcal{A}_{2n}(\mathbb{R}) \cap O(2n)$, then $P = \frac{1}{2}(\mathbf{1} + iQ)$ satisfies PHS.

Theorem (diagonalisation of skew-symmetric matrices)[6]
For any $Q \in \mathcal{A}_{2n}(\mathbb{R})$, there exists $S \in SO(2n)$ such that ${}^tSQS = \operatorname{diag}(1, \ldots, 1, \operatorname{pf}(Q)) \otimes J$, where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

[6] Tin-Yau Tam and Mary Clair Thompson. Determinant and pfaffian of sum of skew symmetricmatrices. Linear algebra and its applications, 433(2):412–423, 2010.

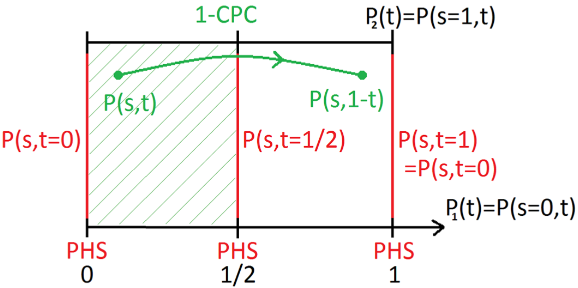

Figure: Homotopy between two families of projectors in $\mathcal{P}_C$.

Theorem (fundamental group of the Grassmannian)[7]
Let $G_n^{\mathbb{C}} = \{$orthogonal projectors of $\mathbb{C}^{2n}$, of rank n$\}$. Then we have $\pi_1(G_n^{\mathbb{C}}) = \{0\}$.

[7] Kenro Furutani. Fredholm–Lagrangian–Grassmannian and the Maslov index. Journal of Geometry and Physics, 51(3):269–331, 2004.