# INPUT-TO-STATE STABILITY OF TIME-DELAY SYSTEMS

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# Time-delay system

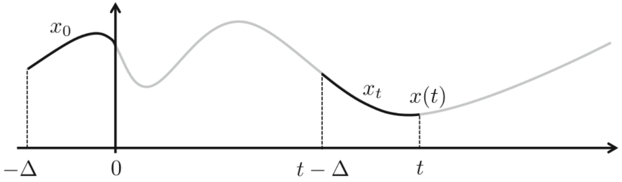


Figure: Time evolution of the solution of a time-delay system.

## Definition 1

**A Nonlinear time-delay system** is a system modeled by functional differential equation of the type:

$$\dot{x}(t) = f(x_t, u(t)), \tag{1}$$

where $u(t) \in \mathbb{R}^m$ is the input, and $x_t : [-\Delta, 0] \to \mathbb{R}^n$ is the solution's history defined by $x_t(s) = x(t+s)$ for all $s \in [-\Delta, 0]$, where $\Delta \geq 0$ denotes the maximum time delay involved (see Figure 1).

Example: $\dot{x}(t) = x(t) + x(t-1) + u(t)$.

In what follows, we assume that the vector field $f$ is Lipschitz on bounded sets and satisfy

$$f(0,0) = 0 \tag{2}$$

to ensure existence and uniqueness of system (1) solution.

$\mathcal{X}^n = \mathcal{C}([-\Delta, 0], \mathbb{R}^n)$.

# Comparision functions


Figure: Class $\mathcal{K}_\infty$ and $\mathcal{KL}$ functions.

# Input-to-State Stability (ISS)

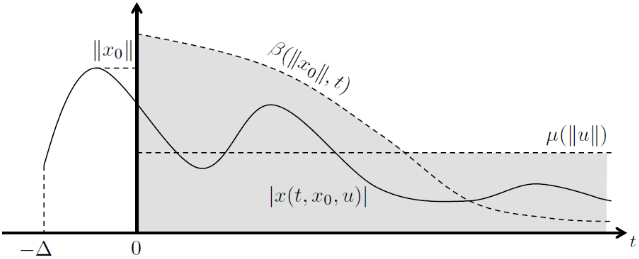


Figure: Schematic representation of the time evolution of an ISS system's solution.

## Definition 2

[Teel, 1998]
System (1) is said to be input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\mu \in \mathcal{K}_\infty$ such that, for all $x_0 \in \mathcal{X}^n$ and all $u \in \mathcal{U}^m$,

$$|x(t, x_0, u)| \leq \beta(||x_0||, t) + \mu(||u_{[0,t]}||), \quad \forall t \geq 0. \tag{3}$$

The function $\mu$ is then called an ISS gain.

In particular, when there exist $k, \lambda \geq 0$, such that $\beta$ is defined as:

$$\beta(s,t) = kse^{-\lambda t},\ \forall s,t \geq 0 \tag{4}$$

then (1) is said to be exponentially input-to-state stable (exp-ISS).

# Lyapunov-Krasovskii-Functional (LKF)

## Definition 3

[Karafyllis and Jiang, 2011, Krasovskii 1963]
A functional $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ is said to be:

1. a Lyapunov-Krasovskii functional candidate (LKF) if it is Lipschitz on bounded sets and there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$,
$$\underline{\alpha}(|\phi(0)|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|). \tag{5}$$
2. a coercive LKF if there exist $\underline{\alpha}, \overline{\alpha} \in \mathcal{K}_\infty$ such that, for all $\phi \in \mathcal{X}^n$,
$$\underline{\alpha}(\|\phi\|) \leq V(\phi) \leq \overline{\alpha}(\|\phi\|). \tag{6}$$

For system (1), an LKF $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ is said to be

1. an ISS LKF with history-wise dissipation if there exist $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that
$$D^+V(\phi, f(\phi, v)) \leq -\alpha(||\phi||) + \gamma(|v|) \tag{7}$$
2. an ISS LKF with LKF-wise dissipation if there exist $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that
$$D^+V(\phi, f(\phi, v)) \leq -\alpha(V(\phi)) + \gamma(|v|) \tag{8}$$
3. an ISS LKF with point-wise dissipation if there exist $\alpha \in \mathcal{K}_\infty$ and $\gamma \in \mathcal{K}_\infty$ such that
$$D^+V(\phi, f(\phi, v)) \leq -\alpha(|\phi(0)|) + \gamma(|v|) \tag{9}$$

where (7)-(9) are all meant to hold for all $\phi \in \mathcal{X}^n$ and all $v \in \mathbb{R}^m$.

# ISS using LKF

We have the following characterization of ISS by using LKF tools (see [Karafyllis et al., 2008, Theorem 3.3] and [Kankanamalage et al., 2017, Theorem 2]).

## Theorem 4

*The following properties are equivalent:*

1. (1) *is ISS*
2. (1) *admits a coercive ISS LKF with history-wise dissipation*
3. (1) *admits an ISS LKF with LKF-wise dissipation*

It is not known if ISS can be ensured by pointwise dissipation.
This has been conjectured in [Chaillet et al., 2022, Conjecture 4].

## Conjecture 1

*Assume that there exist an LKF* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$; $\alpha \in \mathcal{K}_\infty$ *and* $\gamma \in \mathcal{K}_\infty$ *such that, for all* $\phi \in \mathcal{X}^n$ *and all* $v \in \mathbb{R}^m$,

$$D^+V(\phi, f(\phi, v)) \leq -\alpha(|\phi(0)|) + \gamma(|v|). \tag{10}$$

*Then the system* (1) *is ISS.*

# Our contribution

## Theorem 5

*Assume that there exist a functional* $V : \mathcal{X}^n \to \mathbb{R}_{\geq 0}$ *which is Lipschitz on bounded sets,* $\bar{\alpha}, \alpha, \gamma \in \mathcal{K}_\infty$ *and* $Q : \mathbb{R}^n \longrightarrow \mathbb{R}_+$ *continuously differentiable, positive definite and radially unbounded such that, for all* $\phi \in \mathcal{X}^n$ *and all* $v \in \mathbb{R}^m$,

$$0 \leq V(\phi) \leq \bar{\alpha}(\|\phi\|) \tag{11}$$

$$D^+ V(\phi, f(\phi, v)) \leq -\alpha(Q(\phi(0))) + \gamma(|v|) \tag{12}$$

*Assume further that there exists a function* $\sigma \in \mathcal{K}_\infty$ *such that, for all* $\phi \in \mathcal{X}^n$ *and all* $v \in \mathbb{R}^m$,

$$\nabla Q(\phi(0)) f(\phi, v) \leq \sigma\left( \max_{\tau \in [-\Delta, 0]} Q(\phi(\tau)) \right) + \gamma(|v|). \tag{13}$$

*Then, under the condition that*

$$\liminf_{r \to \infty} \frac{\alpha(r)}{\sigma(re^{2\Delta})} > 0, \tag{14}$$

*the system* (1) *is ISS.*

# Remarks

1. Condition (13) constitutes a mild requirement.
2. The dissipation rate may not still be a class $\mathcal{K}_\infty$ function because of function $Q$.
3. Theorem 5 extends the result in [Karafyllis et al., 2022, Theorems $3$ & $4$]. (It is enough to consider $\overline{\alpha}, \alpha, \sigma, Q$ as quadratic functions to notice that).

We give here a sufficient condition which allows to build LKF with LKF-wise dissipation from an LKF that dissipates point-wisely.

## Theorem 6

*Assume that $W$ is an LKF such that for all $\phi \in \mathcal{X}^n$ :*

$$W(\phi) = V_1(\phi(0)) + \int_{-\Delta}^{0} V_2(\phi(s))ds \tag{15}$$

$$D^+ W(\phi, f(\phi, v)) \leq -\alpha(Q(\phi(0))) + \gamma(|v|) \tag{16}$$

*where $\alpha \in \mathcal{K}_\infty$ and $Q : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is continuously differentiable, positive definite and radially unbounded. If there exist $\varepsilon > 0$ such that for all $\phi \in \mathcal{X}^n$ :*

$$\alpha(Q(\phi(0))) \geq \varepsilon V_2(\phi(0)), \tag{17}$$

then there exist $c, p > 0$, $\lambda \in \mathcal{K}_\infty$ such that the LKF $V$ defined by

$$V(\phi) = V_1(\phi(0)) + p\int_{-\Delta}^{0} e^{cs} V_2(\phi(s))ds \;\; \forall \phi \in \mathcal{X}^n \tag{18}$$

satisfies

$$D^+V(\phi, f(\phi, v)) \leq -c\lambda(V(\phi)) + \gamma(|v|) \;\; \forall \phi \in \mathcal{X}^n \tag{19}$$

and then allows to ensure ISS of system (1).

1. The trick of adding exponential term in the kernel of integral part of Lyapunov functional as in Theorem 6 does not still allow to build an LKF with LKF-wise dissipation.
2. For instance, the following example fails for this trick:

## Example 7

$$\dot{x}(t) = -x(t) - \frac{x(t)}{1+x(t)^2} + \frac{x(t-1)^4}{1+|x(t)|^3} + \frac{u(t)}{1+x(t)^2}, \tag{20}$$

and the Lyapunov Krasovskii functionals (LKF) $V$, $W$ respectively defined as:

$$V(\phi) := \frac{1}{4}\phi(0)^4 + \int_{-1}^{0} \phi(s)^4 ds, \quad \forall\, \phi \in \mathcal{X}, \tag{21}$$

$$W(\phi) := \frac{1}{4}\phi(0)^4 + k\int_{-1}^{0} e^{cs}\phi(s)^4 ds, \quad \forall\, \phi \in \mathcal{X}, \tag{22}$$

where $k$ and $c$ denote positive constants.

# Razumikhin and Halanay approaches

Consider the function $V_0 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$, positive definite and radially unbounded.

1. Nonlinear Razumikhin condition:

$$V_0(\phi(0)) \geq \max\left\{\rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right), \gamma(|v|)\right\} \Rightarrow \nabla V_0(\phi(0)) f(\phi, v) \leq -\alpha(|\phi(0)|) \tag{23}$$

   $\alpha, \rho, \gamma \in \mathcal{K}_\infty$.

2. Linear Razumikhin condition

$$\underline{a}|x|^p \leq V_0(x) \leq \overline{a}|x|^p, \quad (2$$

$$V_0(\phi(0)) \geq \max\left\{\rho_0 \max_{\tau\in[-\Delta,0]} V_0(\phi(\tau)), \gamma(|v|)\right\} \Rightarrow \nabla V_0(\phi(0)) f(\phi, v) \leq -\alpha_0|\phi(0)|^p, \quad (2$$

   $\underline{a}, \overline{a}, \alpha_0, \rho_0, p > 0$, $x \in \mathbb{R}^n$, $\gamma \in \mathcal{K}_\infty$.

**Halanay conditions**

Consider the function $V_0 \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_{\geq 0})$, positive definite and radially unbounded.

1. Nonlinear Halanay's condition:

$$\nabla V_0(\phi(0)) f(\phi, v) \leq -\alpha(V_0(\phi(0))) + \rho\left(\max_{\tau\in[-\Delta,0]} V_0(\phi(\tau))\right) + \gamma(|v|). \tag{26}$$

   $\alpha, \rho, \gamma \in \mathcal{K}_\infty$.

2. Linear Halanay's condition

$$\underline{a}|x|^p \leq V_0(x) \leq \overline{a}|x|^p \tag{27}$$

$$\nabla V_0(\phi(0)) f(\phi, v) \leq -\alpha_0 V_0(\phi(0)) + \rho_0 \max_{\tau\in[-\Delta,0]} V_0(\phi(\tau)) + \gamma(|v|). \tag{28}$$

   $\underline{a}, \overline{a}, \alpha_0, \rho_0, p > 0,\ \gamma \in \mathcal{K}_\infty$.

1. Razumikhin and Halanay results are based on Lyapunov **function** and not functional.
2. In the following we state a result that consider Lyapunov Razumikhin function or Lyapunov Halanay function $V_0$ to build the following Lyapunov-Krasovskii functional $V$

$$V(\phi) := \max_{\tau\in[-\Delta,0]} e^{c\tau} V_0(\phi(\tau)), \quad \forall\, \phi \in \mathcal{X}^n, \tag{29}$$

with $c > 0$.

3. The LKF of the form (29) and details on its derivative are provided in [Karafyllis and Jiang, 2011].

# Nonlinear conditions using

## Theorem 8

1. *If the nonlinear* ***Razumikhin condition*** (23) *is satisfied with the function* $\rho$ *such that*

$$\sup_{s>0} \frac{\rho(s)}{s} < 1, \tag{30}$$

*there exists* $c > 0$ *such that the functional* $V$ *defined by* (29) *is an a coercive ISS LKF with LKF-wise dissipation for system* (1).

2. *If the nonlinear* ***Halanay's condition*** (26) *is satisfied with* $\alpha$ *and* $\rho$ *such that the function*

$$s \mapsto \alpha(s) - e^c \rho(s) \in \mathcal{K}_\infty \tag{31}$$

*for some constant* $c$, *then the functional* $V$ *defined by* (29) *is an a coercive ISS LKF with LKF-wise dissipation for system* (1).

# Remarks

- Condition (30) is restrictive than the real Razumikhin condition on function $\rho$ which is
$$\rho(s) \leq s, \quad \forall\, s > 0.$$
- Condition (31) is restrictive than real Halanay condition on function $\rho$ which is
$$s \mapsto \alpha(s) - \rho(s) \in \mathcal{K}_\infty.$$

# Linear conditions using

## Theorem 9

1. *If the linear* ***Razumikhin condition*** (24)-(25) *is satisfied with*

$$\rho_0 < 1,$$

*there exists* $c > 0$ *such that* $V$ *defined by* (29) *is an a coercive exp-ISS LKF with LKF-wise dissipation for system* (1).

2. *System* (1) *is exp-ISS if* ***linear Halanay's*** *condition is satisfied with*

$$\alpha_0 > \rho_0.$$

This result means that if we can show exp-ISS for a system by Razumikhin's or Halanay's theorem, then we can construct a corecive exp-ISS LKF of the form (29) for the system.

# Application

Consider the following mathematical model of a chemical reactor with an exothermic chemical reaction taking place in it and a cooling jacket with negligible axial heat conduction of the cooling medium:

$$\partial_t u(t,z) + c\partial_z u(t,z) = -\xi u(t,z) + \xi x(t), \quad t \geq 0,\ z \in [0,1] \tag{32}$$

$$u(t,0) = 0, \quad t \geq 0 \tag{33}$$

$$\dot{x}(t) = g(x(t)) - (\mu+1)x(t) + \mu \int_0^1 u(t,z)dz, \quad t \geq 0 \tag{34}$$

where $c, \xi, \mu > 0$ and $g : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz, non-decreasing function with $g(0) = 0$ and $g(x) = -a$ for all $x \leq -b$ for some constants $a, b > 0$.

In Chapter 8 of the book [Karafyllis and Krstic, 2019], it is shown that the system (32),(33),(34) is globally exponentially stable in the state norm $|x(t)| + \|u[t]\|_\infty$ provided that

$$\sup_{x \in \mathbb{R}} (g'(x)) < 1 + \mu e^{-\xi c^{-1}}. \tag{35}$$

We consider a pertubed version of (32),(33),(34), namely system (32),(33) with

$$\dot{x}(t) = g(x(t)) - (\mu+1)x(t) + \mu \int_0^1 u(t,z)dz + v(t), \quad t \geq 0 \text{ a.e} \tag{36}$$

where $v(t) \in \mathbb{R}$ is an external disturbance.
By using Theorem 9 we prove that

## Proposition 1

1. *The exponential stability of* (32),(33),(36) *is proved in the stronger state norm* $|x(t)| + \|u[t]\|_\infty + \|\partial_z u[t]\|_\infty$, *provided that*

$$G := \sup_{x \neq 0} \left( x^{-1} g(x) \right) < 1 + c\xi^{-1}\mu \left( 1 - e^{-\xi c^{-1}} \right), \tag{37}$$

2. *inequality* (37) *is less demanding than inequality* (35) *(meanining that the stability analysis in [Karafyllis and Krstic, 2019] is extended),*
3. *an explicit ISS Lyapunov functional is constructed and then provide explicit estimate on the state norm of the system.*

# Estimate on the state norm

Every solution of (32), (33), (36) with $v \in L^{\infty}_{loc}(\mathbb{R}_{\geq 0})$ satisfies the following estimate for all $t \geq 0$ :

$$\max\left(|x(t)|, c\xi^{-1}\|\partial_z u[t]\|_{\infty}, \frac{1}{1-e^{-\xi c^{-1}}}\|u\|_{\infty}\right) \leq c\xi^{-1} e^{(k+\xi)c^{-1}}\|\partial_z u[0]\|_{\infty} e^{-\min\left(k,\frac{f(k)}{2}\right)t} + \frac{e^{kc^{-1}}}{\sqrt{f(k)\min(2k,f(k))}} \sup_{0\leq s\leq t} |v(s)|.$$

where $f(k) = \mu + 1 - G - \frac{\xi\mu}{\xi-k}\left(1 - \frac{c}{\xi-k}\left(1 - e^{-(\xi-k)c^{-1}}\right)\right)$ and $G := \sup\limits_{x\neq 0}\left(x^{-1}g(x)\right)$ .

Chaillet, A., Karafyllis, I., Pepe, P., and Wang, Y. (2022).
The ISS framework for time-delay systems: a survey.

Halanay, A. (1966).
*Differential equations: Stability, oscillations, time lags*, volume 23.
Academic press.

Kankanamalage, H. G., Lin, Y., and Wang, Y. (2017).
On Lyapunov-Krasovskii characterizations of input-to-output stability.
*IFAC-PapersOnLine*, 50(1):14362–14367.
20th IFAC World Congress.

Karafyllis, I. and Jiang, Z.-P. (2011).
*Stability and Stabilization of Nonlinear Systems*.

Karafyllis, I. and Krstic, M. (2019).
*Input-to-state stability for PDEs*.
Springer.

Karafyllis, I., Pepe, P., and Jiang, Z. (2008).
Input-to-output stability for systems described by retarded functional differential equations.

*European Journal of Control*, 14(6):539–555.

Karafyllis, I., Pepe, P., Wang, Y., and Chaillet, A. (2022).
Growth conditions for global exponential stability and exp-ISS of time-delay systems under point-wise dissipation.
*HAL*.

Pepe, P. (2021).
A nonlinear version of Halanay's inequality for the uniform convergence to the origin.
*Mathematical Control and Related Fields*.

Teel, A. (1998).
Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem.
*IEEE Transactions on Automatic Control*, 43(7):960–964.