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Value Function in deterministic optimal control : sensitivity relations of first and second order

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Outline of the talk

Mayer's Optimal Control Problem

Let $T > 0$. Consider the minimization problem

$$V(t_0, x_0) := \inf \left\{ \phi(x(T)) \ : \ x(\cdot) \in S_{[t_0,T]}(x_0) \right\} \qquad \mathcal{P}(t_0, x_0)$$

where $t_0 \in [0, T]$, $\phi : \mathbb{R}^n \to \mathbb{R}$ and $S_{[t_0,T]}(x_0)$ is the set of all absolutely continuous solutions of the **control system**

$$\begin{cases} \dot{x} = f(x, u(t)), \quad u(t) \in U & \text{a.e. in } [t_0, T] \\ x(t_0) = x_0 \end{cases}$$

where $f : \mathbb{R}^n \times U \to \mathbb{R}^n$ is continuous and U is a complete separable metric space. V is called the **value function**.

Standard Hypothesis: ϕ is locally Lipschitz and

$$\begin{cases} (i) & f(x, U) \text{ is compact for each } x \in \mathbb{R}^n \\ (ii) & f(\cdot, u) \text{ is locally Lipschitz uniformly in } u \in U \\ (iii) & \exists\, \gamma > 0 \text{ so that } \max\{|f(x,u)| : u \in U\} \leq \gamma(1 + |x|)\ \forall x \in \mathbb{R}^n \end{cases}$$

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Dynamic Programming

Value function is nondecreasing along trajectories of control system and is constant along optimal trajectories (for the Mayer problem !)

A trajectory-control pair $(\bar{x}, \bar{u}) : [t_0, T] \to \mathbb{R}^n \times U$ is optimal iff

$$V(t, \bar{x}(t)) = \phi(\bar{x}(T)) \quad \forall\, t \in [t_0, T]$$

Let $K := epi(V) = \{(t, x, z) : z \geq V(t, x)\} \subset [0, T] \times \mathbb{R}^n \times \mathbb{R}$.
If $f(x, U)$ is also convex $\forall\, x \in \mathbb{R}^n$, then K is viable for the system

$$\begin{cases} \dot{t}(s) & = & 1 \\ \dot{x}(s) & \in & f(x(s), U) \quad \text{a.e.} \\ \dot{z}(s) & = & 0 \end{cases}$$

in the sense that for any initial condition $(t_0, x_0, z_0) \in K$, there exists a solution of the above system starting at (t_0, x_0, z_0) and satisfying $(t(s), x(s), z(s)) \in K$ for all $s \in [0, T - t_0]$.

Optimal Trajectories and Optimal Synthesis

A trajectory $\bar{x} : [t_0, T] \to \mathbb{R}^n \times U$ is optimal for the problem $\mathcal{P}(t_0, x_0)$ iff $s \mapsto (t_0 + s, \bar{x}(t_0 + s), V(t_0, x_0))$ is a solution of

$$\begin{cases} \dot{t}(s) = 1 & & t(0) = t_0 \\ \dot{x}(s) \in f(x(s), U) & \text{a.e.} & x(0) = x_0 \\ \dot{z}(s) = 0 & & z(0) = V(t_0, x_0) \\ (t(s), x(s), z(s)) \in K & & \forall\, s \in [0, T - t_0] \end{cases}$$

Let $T_K(t, x, V(t, x))$ denote the contingent cone to K at $(t, x, V(t, x))$. The **optimal synthesis** is given by

$$U(t, x) = \{u \in U \,:\, (1, f(x, u), 0) \in T_K(t, x, V(t, x))\}$$

A trajectory $\bar{x} : [t_0, T] \to \mathbb{R}^n$ is **optimal** for the problem $\mathcal{P}(t_0, x_0)$ if and only if it is a solution to

$$\dot{x}(s) \in f(x(s), U(s, x(s))) \quad x(t_0) = x_0$$

Viability Kernel

If $W : [0,T] \times \mathbb{R}^n \to \mathbb{R}$, $W(T,\cdot) = \phi(\cdot)$ and $epi(W)$ is viable for the system

$$\begin{cases} \dot{t}(s) &=& 1 \\ \dot{x}(s) &\in& f(x(s),U) \quad \text{a.e.} \\ \dot{z}(s) &=& 0 \end{cases}$$

then $W \geq V$.

Consider any lower semicontinuous $W : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ satisfying $W \leq V$ and $W(T,\cdot) = \phi(\cdot)$. Let K be the largest closed subset of $epi(W)$ such that for any initial condition $(t_0, x_0, z_0) \in K$ there exists a viable in K solution of

$$\begin{cases} \dot{t}(s) &=& 1 & & t(0) = t_0 \\ \dot{x}(s) &\in& f(x(s),U) & \text{a.e.} & x(0) = x_0 \\ \dot{z}(s) &=& 0 & & z(0) = V(t_0,x_0) \end{cases}$$

then $K = epi(V)$.

Generalized Differentials

Let $\Omega \subset \mathbb{R}^n$ be open and $g : \Omega \to \mathbb{R}$. For any $x \in \Omega$, the sets

$$\partial^- g(x) = \left\{ p \in \mathbb{R}^n : \liminf_{y \to x} \frac{g(y) - g(x) - \langle p, y - x \rangle}{|\, y - x \,|} \geq 0 \right\}$$

$$\partial^+ g(x) = \left\{ p \in \mathbb{R}^n : \limsup_{y \to x} \frac{g(y) - g(x) - \langle p, y - x \rangle}{|\, y - x \,|} \leq 0 \right\}$$

are the **(Fréchet) subdifferential** and **superdifferential** of g at x, respectively. Denote by ∂g the (Clarke) generalized gradient of g.

$p \in \mathbb{R}^n$ is a **proximal subgradient** of g at $x \in \Omega$ if $\exists\; c,\ \rho \geq 0$

$$g(y) - g(x) - \langle p, y - x \rangle \geq -c|y - x|^2 \quad \forall y \in B(x, \rho).$$

The set of all proximal subgradients of g at x is denoted by $\partial^{-,pr} g(x)$.

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HJB and Characteristics

Define the Hamiltonian $H : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$H(x,p) = \sup_{u \in U} \langle f(x,u), p \rangle \qquad \forall\, (x,p) \in \mathbb{R}^n \times \mathbb{R}^n$$

V is the unique solution, in a suitable sense, of the Hamilton-Jacobi equation

$$\begin{cases} -v_t(t,x) + H(x, -v_x(t,x)) = 0 & \text{in } [0,T] \times \mathbb{R}^n \\ v(T,x) = \phi(x) & x \in \mathbb{R}^n \end{cases}$$

Characteristic system : $p(t) = -v_x(t, x(t))$

$$\begin{cases} \dot{x}(t) = \nabla_p H(x(t), p(t)) & x(T) = x_T \\ -\dot{p}(t) = \nabla_x H(x(t), p(t)) & -p(T) = \nabla\phi(x_T) \end{cases}$$

where $\nabla_x H$ is the gradient of $H(\cdot, p)$, similarly for $\nabla_p H$ whenever $p(\cdot) \neq 0$

HJB and Characteristics

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Characteristic system : $p(t) \neq -v_x(t, x(t))$, in general,

$$\begin{cases} \dot{x}(t) \in \partial_p^- H(x(t), p(t)) & x(T) = x_T \\ -\dot{p}(t) \in \partial_x^- H(x(t), p(t)) & -p(T) \in \partial\phi(x_T) \end{cases}$$

where $\partial_x^- H$ is the subdifferential of $H(\cdot, p)$, similarly for $\partial_p^- H$

Maximum Principle

The maximum principle states that if $(\bar{x}, \bar{u})$ is optimal, then the solution p of the **adjoint system**

$$\begin{cases} -\dot{p}(t) = p(t) f_x(\bar{x}(t), \bar{u}(t)) & \text{a.e.} \\ -p(T) = \nabla \phi(\bar{x}(T)) \end{cases}$$

satisfies the **maximality** condition :

$$\langle p(t), f(\bar{x}(t), \bar{u}(t)) \rangle = \max_{u \in U} \langle p(t), f(\bar{x}(t), u) \rangle$$

(Pontryagin and al., 1956)
If H is differentiable at $(\bar{x}(t), p(t))$, then

$$\nabla_x H(\bar{x}(t), p(t)) = p(t) f_x(\bar{x}(t), \bar{u}(t)),$$
$$\nabla_p H(\bar{x}(t), p(t)) = f(\bar{x}(t), \bar{u}(t))$$

Sensitivity Relations for Smooth Control Systems

Let $\bar{x}(\cdot)$ be an optimal solution for $\mathcal{P}(t_0, x_0)$.

If f and ϕ are sufficiently smooth and V is differentiable, then the adjoint state $p(\cdot)$ (of the **maximum principle**) satisfies the partial sensitivity relation

$$-p(t) = V_x(t, \bar{x}(t)) \quad \forall t \in [t_0, T]$$

and the full sensitivity relation

$$(H(\bar{x}(t), p(t)), -p(t)) = \nabla V(t, \bar{x}(t)) \quad \forall t \in [t_0, T)$$

The maximum principle + the last relation imply a necessary and sufficient condition for optimality.

Sensitivity Relations for Smooth Control Systems

Let $\bar{x}(\cdot)$ be an optimal solution for $\mathcal{P}(t_0, x_0)$.

In general, V is merely locally Lipschitz. By Clarke, Vinter 1987 there exists an adjoint state $p(\cdot)$ (co-state from the maximum principle) satisfying

$$-p(t) \in \partial_x V(t, \bar{x}(t)) \quad \text{a.e. in } [t_0, T]$$

and by Vinter 1988 there exists an adjoint state $q(\cdot)$ satisfying

$$(H(\overline{x}(t), q(t)), -q(t)) \in \partial V(t, \overline{x}(t)) \text{ for all } t \in (t_0, T)$$

This relations do not imply sufficient conditions for optimality. They also hold true in the state constrained case.

Sensitivity Relations for Smooth Control Systems

Let $\bar{x}(\cdot)$ be an optimal solution for $\mathcal{P}(t_0, x_0)$.

If $f(\cdot, u)$ and ϕ are differentiable, then the adjoint state $p(\cdot)$ satisfies

$$-p(t) \in \partial_x^+ V(t, \bar{x}(t)) \quad \forall t \in [t_0, T]$$

and

$$(H(\bar{x}(t), p(t)), -p(t)) \in \partial^+ V(t, \bar{x}(t)) \qquad \text{a.e. in } [t_0, T]$$

Subbotina 1989, Cannarsa and HF 1990.

This leads to necessary and sufficient conditions for optimality.

Semiconcave and Semiconvex Functions

Let $\Omega \subset \mathbb{R}^n$, $c \geq 0$, $g : \Omega \to \mathbb{R}$ is $c-$**semiconcave** if

$$g(\lambda x + (1-\lambda)y) \geq \lambda g(x) + (1-\lambda)g(y) - \lambda(1-\lambda)c|x-y|^2$$

for all $x, y \in \Omega$ such that $[x, y] \subset \Omega$ and $\lambda \in [0, 1]$.
g is called $c-$**semiconvex** on Ω if $-g$ is $c-$semiconcave on Ω.
Any locally $C^{1,1}$ function is locally semiconvex.

If f is sufficiently smooth in x and ϕ is C^2, then the value function is locally semiconcave.

Hence it has directional derivatives and if the subdifferential of V is nonempty at some (t, x), then V is differentiable at this point. V is then the unique locally Lipschitz solution of HJB equation in the classical sense.

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Maximum Principle

Theorem

Assume $\forall\, r > 0\ \exists\, c \geq 0$ *such that* $\forall\, p \in S^{n-1}$, $x \mapsto H(\cdot, p)$ *is* $c-$*semiconvex on* $B(0, r)$.
If $\bar{x}(\cdot)$ *is optimal for* $\mathcal{P}(t_0, x_0)$, *then there exists an arc* $p : [t_0, T] \to \mathbb{R}^n$ *which, together with* $\bar{x}(\cdot)$, *satisfies*

$$\begin{cases} \dot{x}(s) & \in & \partial_p^- H(x(s), p(s)), \\ -\dot{p}(s) & \in & \partial_x^- H(x(s), p(s)), \end{cases} \quad \text{for a.e.} \quad s \in [t_0, T]$$

and $-p(T) \in \partial\phi(x(T))$.

Sufficient Conditions for Optimality

Theorem

Let $x(\cdot) \in S_{[t_0,T]}(x_0)$. *If, for almost every* $t \in [t_0, T]$, $\exists\ p(t) \in \mathbb{R}^n$

$$\langle p(t), \dot{x}(t) \rangle = H(x(t), p(t))$$

$$(H(x(t), p(t)), -p(t)) \in \partial^+ V(t, x(t))$$

then x *is optimal for* $\mathcal{P}(t_0, x_0)$.

Regularity Assumptions: $\forall\ r > 0,\ \exists\ c \geq 0,\ \forall\ p \in S^{n-1}$

$$\begin{cases} (i)\ x \mapsto H(x,p) \text{ is c-semiconvex on } B(0,r) \\ (ii)\ \nabla_p H(x,p) \text{ exists and is c-Lipschitz continuous in } x \text{ on } B(0,r) \end{cases}$$

If ϕ is locally semiconcave, then V is also locally semiconcave

Cannarsa and Wolenski, 2011

Sensitivity Relations Involving Superdifferentials

Theorem

Let $\overline{x}(\cdot)$ be optimal for $\mathcal{P}(t_0, x_0)$ and consider any arc $\overline{p}(\cdot)$ such that $(\overline{x}, \overline{p})$ solves the system

$$\begin{cases} -\dot{p}(t) & \in & \partial_x^- H(x(t), p(t)) \\ \dot{x}(t) & \in & \partial_p^- H(x(t), p(t)) \end{cases} \qquad - p(T) \in \partial^+ \phi(\overline{x}(T))$$

Then $\overline{p}(\cdot)$ satisfies the full sensitivity relation

$$(H(\overline{x}(t), \overline{p}(t)), -\overline{p}(t)) \in \partial^+ V(t, \overline{x}(t)) \text{ for all } t \in (t_0, T)$$

and the partial sensitivity relation

$$-\overline{p}(t) \in \partial_x^+ V(t, \overline{x}(t)) \text{ for all } t \in [t_0, T]$$

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Theorem

Assume $\partial_x^- V(t_0, x_0) \neq \emptyset$. Let $\overline{x}(\cdot)$ be optimal for $\mathcal{P}(t_0, x_0)$ and consider any arc $\overline{p}(\cdot)$ such that $(\overline{x}, \overline{p})$ solves the system

$$\begin{cases} \dot{x}(s) \in \partial_p^- H(x(s), p(s)), & x(t_0) = x_0 \\ -\dot{p}(s) \in \partial_x^- H(x(s), p(s)), & -p(t_0) \in \partial_x^- V(t_0, x_0) \end{cases}$$

Then $-\overline{p}(t) \in \partial_x^- V(t, \overline{x}(t))$ for all $t \in [t_0, T]$.

Furthermore, if $\partial^+ \phi(\overline{x}(T)) \neq \emptyset$, then for all $t \in [t_0, T]$, $V(t, \cdot)$ is differentiable at $\overline{x}(t)$ and $\nabla_x V(t, \overline{x}(t)) = -\overline{p}(t)$.
If ϕ is also locally semiconcave, then $V(\cdot, \cdot)$ is differentiable at $(t, \overline{x}(t))$ and $\nabla V(t, \overline{x}(t)) = (H(\overline{x}(t), \overline{p}(t)), -\overline{p}(t))$ $\forall$ $t \in [t_0, T)$.

Second Order Superjets and Subjets

$S(n)$ is the set of symmetric $n \times n$ matrices.
Let $g : \mathbb{R}^n \to [-\infty, +\infty]$ and $x \in dom(g)$.

$(q, Q) \in \mathbb{R}^n \times S(n)$ is a **superjet** of g at x if $\exists\, \delta > 0$, $\forall\, y \in B(x, \delta)$

$$g(y) \leq g(x) + \langle q, y - x \rangle + \frac{1}{2} \langle Q(y - x), y - x \rangle + o(|\, y - x\,|^2)$$

The set of all the superjets of g at x is denoted by $J^{2,+}g(x)$.

$(q, Q) \in \mathbb{R}^n \times S(n)$ is a **subjet** of g at x if $\exists\, \delta > 0$, $\forall\, y \in B(x, \delta)$

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The set of all the subjets of g at x is denoted by $J^{2,-}g(x)$.

Properties of Superjets

Proposition

Let $g : \mathbb{R}^n \to [-\infty, +\infty]$ be an extended real-valued function and let $x \in dom(g)$. Then the following properties hold:

(i) $J^{2,+}g(x)$ *is a convex subset of* $\mathbb{R}^n \times S(n)$*,*

(ii) *for any* $q \in \mathbb{R}^n$*, the set* $\{Q \in S(n) : \ (q, Q) \in J^{2,+}g(x)\}$ *is a closed convex subset of* $S(n)$*,*

(iii) *if* $g' \leq g$ *and* $g(\widehat{x}) = g'(\widehat{x})$ *for some* $\widehat{x} \in \mathbb{R}^n$*, then* $J^{2,+}g(\widehat{x}) \subset J^{2,+}g'(\widehat{x})$*.*

(iv) *if* $(q, Q) \in J^{2,+}g(x)$*, then* $(q, Q') \in J^{2,+}g(x)$ *for all* $Q' \in S(n)$ *such that* $Q' \geq Q$*. Thus, the set* $J^{2,+}g(x)$ *is either empty or unbounded.*

Matrix Riccati Equation

Assume $H \in C^2(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ and $\partial^+\phi(z) \neq \emptyset$ for all $z \in \mathbb{R}^n$.

Let $\overline{x}$ be an optimal solution of $\mathcal{P}(t_0, x_0)$ and consider a dual arc $\overline{p}$ satisfying $0 \neq \overline{p}(T) \in -\partial^+\phi(\overline{x}(T))$.

From now on set $H_{px}[t] := \nabla^2_{px} H(\overline{x}(t), \overline{p}(t))$, and let $H_{xp}[t], H_{pp}[t], H_{xx}[t]$ be defined analogously.

Riccati Equation : $R(T) = -\nabla^2\phi(\bar{x}(T))$

$$\dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0$$

If $V(t, \cdot)$ is C^2 in a neighborhood of $\bar{x}(t)$ for all $t \in [t_0, T]$, then

$$(\nabla_x V(t, \bar{x}(t)), \nabla^2_{xx} V(t, \bar{x}(t)) = (-\bar{p}(t), -R(t))$$

This is a second order sensitivity relation.

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If $V(t, \cdot)$ is C^2 in a neighborhood of $\bar{x}(t)$ for all $t \in [t_0, T]$, then

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This is a second order sensitivity relation.

Sensitivity Relations Involving Superjets

Theorem

Let $(q,Q) \in J^{2,+}\phi(\overline{x}(T))$, $q \neq 0$ and $\overline{p}(\cdot)$ be the dual arc such that $\overline{p}(T) = -q$. Consider the solution $R(\cdot)$ of

$$\begin{cases} \dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0 \\ R(T) = -Q, \end{cases}$$

defined on $[a,T]$ for some $a \in [t_0, T)$. Then

$$(-\overline{p}(t), -R(t)) \in J_x^{2,+}V(t, \overline{x}(t)) \qquad \textit{for all } t \in [a,T].$$

Proof is an adaptation of the one in Caroff and HF, TAMS 1996

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Sensitivity Relations Involving Subjets

Theorem

Let $H \in C^{2,1}_{loc}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$ *and for some* $R_0 \in S(n)$

$$(-\overline{p}(t_0), -R_0) \in J_x^{2,-} V(t_0, x_0).$$

If the solution $R(\cdot)$ *of the Riccati equation*

$$\begin{cases} \dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0 \\ R(t_0) = R_0 \end{cases}$$

is well defined on $[t_0, a]$ *for some* $a \in (t_0, T]$, *then*

$$(-\overline{p}(t), -R(t)) \in J_x^{2,-} V(t, \overline{x}(t)) \quad \text{for all } t \in [t_0, a].$$

Forward Propagation of Twice Differentiability

Theorem

If $V(t_0,\cdot)$ is twice differentiable at x_0 and the solution $R(\cdot)$ of

$$\begin{cases} \dot{R}(t) + H_{px}[t]R(t) + R(t)H_{xp}[t] + R(t)H_{pp}[t]R(t) + H_{xx}[t] = 0 \\ R(t_0) = -\nabla_{xx}V(t_0,x_0) \end{cases}$$

is well defined on $[t_0,a]$, then $V(t,\cdot)$ is twice differentiable at $\overline{x}(t)$ for any $t \in [t_0,a]$ and $R(t) = -\nabla_{xx}V(t,\overline{x}(t))$.

If ϕ is locally semiconcave, then the interval $[t_0,a]$ can be taken equal to $[t_0,T]$.

A similar result holds true also backward in time.

Avoiding Conjugate Times

Assume $\phi \in C^2(\mathbb{R}^n)$ and consider the Riccati equation

$$\begin{cases} \dot{R} + H_{px}[t]R + RH_{xp}[t] + RH_{pp}[t]R + H_{xx}[t] = 0, \\ R(T) = -\nabla^2\phi(\bar{x}(T)). \end{cases}$$

If for some $t_c \in [t_0, T]$, $R(\cdot)$ is well defined on $(t_c, T]$ and $\lim_{t \searrow t_c} \| R(t) \| = +\infty$, then t_c is the conjugate time for $\bar{x}(T)$.

Theorem

Let $\overline{x}$ be optimal for $\mathcal{P}(t_0, x_0)$ with $\nabla\phi(\overline{x}(T)) \neq 0$. If $\partial_x^{-,pr} V(t_0, x_0) \neq \emptyset$, then $R(\cdot)$ is well defined on $[t_0, T]$ and $V(t, \cdot)$ is of class C^2 in a neighborhood of $\overline{x}(t)$ for all $t \in [t_0, T]$.

$\partial_x^{-,pr} V(t, x) \neq \emptyset$ on a dense subset of $x \in \mathbb{R}^n$.

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Mayer's Problem under State Constraints

Consider the minimization problem

$$V(t_0, x_0) := \min \left\{ \phi(x(T)) \ : \ x(\cdot) \in S_{[t_0,T]}(x_0), \ x([t_0, T]) \subset K \right\}$$

where $K \subset \mathbb{R}^n$ is nonempty and closed.

Inward Pointing Condition (IPC):

$$\mathrm{co}\, f(y, U) \cap \mathrm{int}\, C_K(y) \neq \emptyset \quad \forall\, y \in \partial K$$

where $C_K(y)$ denotes the Clarke tangent cone to K at y.

If ϕ is locally Lipschitz, then V is locally Lipschitz on $[0, T] \times K$.

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Generalized Gradients on Closed Sets

Define partial generalized gradient of V

$$\partial_x V(t,y) \ := \ \mathrm{co}\,\mathrm{Limsup}_{y' \to_{\mathrm{IntK}} y} \{\nabla_x V(t,y')\}$$

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$$\partial V(t,y) \ := \ \mathrm{co}\,\mathrm{Limsup}_{(t',y') \to_{[0,T]\times \mathrm{IntK}} (t,y)} \{\nabla V(t',y')\,\}$$

In the next result $N_K(y)$ denotes Clarke's normal cone to K at y and $\widehat{V} : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is given by

$$\widehat{V}(y) = \begin{cases} -V(t_0,y) & \text{if } y \in K \\ +\infty & \text{otherwise.} \end{cases}$$

Necessary Optimality Conditions

Theorem (Bettiol, Frankowska, Vinter, AMO 2015)

Let $\bar{x}(.)$ be optimal for the initial condition (t_0, x_0). Then there exist an arc $p(\cdot)$, a finite positive Borel measure $\mu(\cdot)$ on $[t_0, T]$ and a Borel measurable selection $\nu(t) \in N_K(\bar{x}(t)) \cap B$ μ-a.e. in $[t_0, T]$ such that for $q(s) := p(s) + \int_{[t_0,s]} \nu(\tau) d\mu(\tau)$

$$(-\dot{p}(t), \dot{\bar{x}}(t)) \in \partial H(\bar{x}(t), q(t)) \quad \text{a.e. in } [t_0, T]$$

$-q(T) \in \partial\phi(\bar{x}(T))$, $p(t_0) \in \partial\widehat{V}(\bar{x}(t_0))$ and the following sensitivity relations hold true for a.e. $t \in [t_0, T]$:

$$-q(t) \in \partial_x V(t, \bar{x}(t))$$

$$(H(\bar{x}(t), q(t)), -q(t)) \in \partial V(t, \bar{x}(t))$$


Merci
de votre écoute