Atomic decomposition of a positive compact operator on $L^p$

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We consider of population of individuals, with a feature set $\Omega$. Each individual is susceptible ($S$) or infected ($I$). For disease with no immunity (ex: gonorrhea).

$k(x, y)$: Infection rate from feature $y$ towards feature $x$.
$\gamma(x)$: Recovery rate of feature $x$.

Let $u(t, x)$ be the probability that an individual of feature $x$ is infected at time $t \geq 0$. We consider the following ODE (for $\forall t \geq 0, \forall x \in \Omega$):

$$\partial_t u(t, x) = (1 - u(t, x)) \int_{\Omega} k(x, y) u(t, y) d\mu(y) - \gamma(x) u(t, x).$$
We recall the ODE of evolution of a SIS model:

\[ \partial_t u = (1 - u) T_k(u) - \gamma u \quad (1) \]

\[ u(0, .) = u_0 \]

We call equilibrium of Equation (1) any constant solution \( g \) so that \( 0 \leq g \leq 1 \), that is: \( \gamma g = (1 - g) T_k(g) \).

Let \( g \in L^\infty \) with \( 0 \leq g \leq 1 \). Then \( g \) is an equilibrium if and only if:

\[ T_{(1-g)k/\gamma}(\gamma g) = \gamma g \]

where we denote \( (1 - g)k/\gamma(x, y) = (1 - g(x))k(x, y)/\gamma(y) \) for \( (x, y) \in \Omega^2 \).
Let $M = (M_{i,j})_{1 \leq i, j \leq n}$ be a matrix with nonnegative entries. We define the oriented weighted graph on $\{1, \ldots, n\}$ giving the weight $M_{i,j}$ to the edge $j \to i$. Example:

(a) Matrix

$$
\begin{pmatrix}
1 & 0 \\
1 & 1
\end{pmatrix}
$$

(b) Graph

1 → 2

Figure: Example of a nonnegative matrix and its associated graph

Epidemics interpretation: We split our population into $n$ groups of features. $M_{i,j}$ represents the power of infection from individuals of feature $j$ towards individuals of feature $i$. 

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We call path from $i$ to $j$ on the graph any sequence $(i = x_0, \ldots, x_n = j)$ so that $\forall l, M_{l+1,l} > 0$.

We say that $A \subset \{1, \ldots, n\}$ is invariant if there is no path starting in $A$ that goes outside of $A$. In epidemics terms, any infection starting in $A$ stays within $A$. $A$ is co-invariant if $A^c$ is invariant.

We say that the matrix $M$ is irreducible if its only invariant sets are $\emptyset$ and $\{1, \ldots, n\}$. It is equivalent to say that from all $i, j \in \{1, \ldots, n\}$, there is a path from $i$ to $j$. 

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For any matrix $M \in \mathcal{M}_n(\mathbb{R})$, we denote $\rho(M) = \sup_{\lambda \in \text{Sp}(M)} |\lambda|$ its spectral radius.

**Theorem (Perron-Frobenius theorem ('07 P), ('12 F))**

Let $M \in \mathcal{M}_n(\mathbb{R})$ be a non null matrix with nonnegative entries. Then $\rho(M)$ is a eigenvalue of $M$, related to an eigenvector with nonnegative entries.

We now assume that $M$ is irreducible. We have:

(i) $\rho(M) > 0$.

(ii) $\dim \left( \bigcup_{n \in \mathbb{N}} \ker(M - \rho(M)I_n)^n \right) = 1$ ($\rho(M)$ is simple).

(iii) There exists a unique nonnegative eigenvector, and it has positive entries.
Two possibilities to upgrade Perron-Frobenius theorem:

(i) From finite-dimension to infinite dimension -> From matrices to operators.

(ii) Without irreducibility -> Define irreducible components.

Our goal is to characterize nonnegative eigenfunctions for a positive operator.
Let \((\Omega, \mathcal{F}, \mu)\) be a finite measured space. A kernel is a measurable function \(k : \Omega^2 \rightarrow \mathbb{R}_+\).

When possible, we define, for \(f : \Omega \rightarrow \mathbb{R}\) measurable and \(x \in \Omega\):

\[
T_k(f)(x) = \int_{\Omega} k(x, y) f(y) \, d\mu(y).
\]

\(T_k\) is a positive operator: \(\forall f \geq 0 \text{ a.e.}, T_k(f) \geq 0 \text{ a.e.}\).

Epidemics interpretation: \(\Omega\) is a set of features, and \(k(x, y)\) the power of infection from individuals of feature \(y\) towards individuals of feature \(x\). The operator \(T_k\) models a "one-step infection".
For any operator $T$ on $L^p$, we denote $\rho(T) = \sup_{\lambda \in \text{Sp}(T)} |\lambda|$. 

**Theorem (Krein-Rutman theorem ('48), Schaefer ('74), de Pagter ('86))**

Let $T$ be a compact positive operator on $L^p$ for $p \in [1, +\infty]$. If $\rho(T) > 0$, then $\rho(T)$ is a eigenvalue of $T$, related to a nonnegative eigenfunction.

If $T$ is irreducible (we will define it later). We have:

(i) $\rho(T) > 0$.

(ii) $\dim \left( \bigcup_{n \in \mathbb{N}} \ker(T - \rho(T) \text{Id})^n \right) = 1$ ($\rho(T)$ is simple).

(iii) There exists a unique nonnegative eigenfunction related to a non-null eigenvalue, and it is positive.
Let $k$ be a kernel and $T$ the associated kernel operator.

**Assumption**

The operator $T$ is a compact operator on $L^p$ for some $p \in (1, +\infty)$.

For $A, B \subset \mathcal{F}$, we define the *infection power* from $B$ towards $A$ as:

$$k(A, B) = \int_{A \times B} k(x, y) \, d\mu(x) \, d\mu(y).$$

Notice that $k(A, B) = 0$ if and only if $k(x, y) = 0$ a.e. on $A \times B$. We say that $A$ is *invariant* if $k(A^c, A) = 0$, and that $A$ is *co-invariant* if $A^c$ is invariant.
Let $A \in \mathcal{F}$ be a measurable set. We define the *future* of $A$, denoted $F(A)$ as the minimal invariant set that contains $A$, and the *past* of $A$, denoted $P(A)$ as the minimal co-invariant set that contains $A$. We denote $A_F = F(A) \setminus A$.

The future of $A$ is the set of all the features that might be infected by an epidemics starting in $A$. The past of $A$ is the set of all the features that might infect $A$ if an epidemics starts there.
For $\Omega' \in \mathcal{F}$ a measurable set with $\mu(\Omega') > 0$, we define the restricted operator $T|_{\Omega'}$ on $L^p(\Omega')$ by $\forall f \in L^p, \forall x \in \Omega'$:

$$T|_{\Omega'}(f)(x) = \int_{\Omega'} k(x, y)f(y)\,d\mu(y).$$

Notice that $A \subset \Omega'$ is a $T|_{\Omega'}$-invariant set if and only if $k(\Omega' \setminus A, A) = 0$. 
We say $T$ is an **irreducible operator** if the only $T$-invariant sets are $\emptyset$ or $\Omega$. We say that $A$ is an **irreducible set** if $T|_A$ is a irreducible operator, that is, the only $T|_A$-invariant sets are $\emptyset$ and $A$.

Let $\mathcal{A}$ be the $\sigma$-field generated by the invariant sets. We call **atom** (in the sense of Schwartz) any minimal element of $\mathcal{A}$ with positive measure. That is, $A$ is an atom if $\mu(A) > 0$, $A \in \mathcal{A}$ and for any set $B \in \mathcal{A}$ with $B \subset A$, we have $B = A$ or $B = \emptyset$.

We say that $A$ is a **convex set** if $F(A) \setminus A$ is an invariant set.
Theorem (Equivalent definitions of atoms)

Let $A \in \mathcal{F}$ be a measurable set with $\mu(A) > 0$. We have equivalence between:

(i) $A$ is an atom.

(ii) $A$ is a minimal convex set.

(iii) $A$ is a maximal irreducible set.
Pros and cons of each definition:

<table>
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<th>Definition</th>
<th>Pros</th>
<th>Cons</th>
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<tr>
<td>Atom</td>
<td>Good for theory</td>
<td>Hard to use in practice</td>
</tr>
<tr>
<td>Min convex</td>
<td>Easy to use</td>
<td>Not general enough</td>
</tr>
<tr>
<td>Max irr</td>
<td>Easy to generalize</td>
<td>Maximal instead of minimal</td>
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For $\lambda \in \mathbb{C}$ and $A \in \mathcal{F}$ a measurable set with $\mu(A) > 0$, we define the *algebraic multiplicity* of $\lambda$ of $T$ restricted to $A$ by:

$$m(\lambda, A, T) \text{ (or } m(\lambda, A)) = \dim \left( \bigcup_{n \in \mathbb{N}} \ker(T|_A - \lambda \text{Id})^n \right).$$

When $m(\lambda, \Omega) = 1$, we say that $\lambda$ is *simple*. The algebraic multiplicity, for a compact operator and for $\lambda \neq 0$, satisfies the following properties:

(i) $m(\lambda, \Omega) < +\infty$.

(ii) $m(\lambda, A) > 0$ if and only if $\lambda$ is an eigenvalue of $T|_A$. 

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Theorem (Schwartz (’61))

Let $\lambda > 0$. Then we have:

$$m(\lambda, \Omega) = \sum_{A \text{ atom}} m(\lambda, A).$$
For any $A \in \mathcal{F}$ with $\mu(A) > 0$, we denote $\rho(A) := \rho(T|A)$. We remind that if $A$ is an atom with $\rho(A) > 0$, by Krein-Rutman theorem we have $m(\rho(A), A) = 1$.

Here are some important consequences of Schwartz's theorem:

**Corollary**

(i) If $\lambda > 0$, there exists a finite amount of atoms with a spectral radius $\geq \lambda$.

(ii) $\rho(T) = \max_{A \text{ atom}} \rho(A)$.

(iii) If $I$ is an invariant set, $\rho(I) = \max_{A \text{ atom}, A \subset I} \rho(A)$. 
Question: How can we "distinguish" an atom using nonnegative eigenfunction "related" to the atom?

Idea: Find a nonnegative eigenfunction \( v \) so that 
\[
\text{supp}(v) := \{ v > 0 \}
\]
is "close enough" to \( A \). For instance 
\( A \cap \text{supp}(v) \neq \emptyset \). As \( A \) is an atom and \( \text{supp}(v) \) an invariant set, we have in this case 
\( F(A) \subset \text{supp}(v) \). Can we have it with 
\( \text{supp}(v) = F(A) \)?

Definition (Distinguished atom)

We say that an atom \( A \) is distinguished if there exists a nonnegative eigenfunction \( v \in L^p \) so that 
\( \text{supp}(v) = F(A) \).
Proposition (Characterization of distinguished atoms)

Let $A$ be an atom. We have the following:

(i) $A$ is distinguished if and only if $\rho(A) > \rho(A_F)$.

(ii) If $A$ is distinguished, there exists a unique nonnegative eigenfunction $\nu \in L^p$ with $\text{supp}(\nu) = F(A)$, and it is related to the eigenvalue $\rho(A)$. 
Idea of proof of the previous proposition:

Let \( \nu \in L^p \) and \( \lambda \in \mathbb{C} \) with \( \text{supp}(\nu) = F(A) \). We have \( T\nu = \lambda \nu \) if and only if:

\[
\lambda \mathbf{1}_A \nu = \mathbf{1}_A T(\mathbf{1}_A \nu) \quad (2)
\]

\[
\lambda \mathbf{1}_{AF} \nu = \mathbf{1}_{AF} T(\mathbf{1}_A \nu) + \mathbf{1}_{AF} T(\mathbf{1}_{AF} \nu) \quad (3)
\]

Then by Equation (2), \( \mathbf{1}_A \nu \) is a nonnegative eigenfunction of \( T|_A \). Therefore by Krein-Rutman theorem, \( \mathbf{1}_A \nu \) is uniquely defined and \( \lambda = \rho(A) \).
Does Equation (3): $\rho(A)1_{AF}v = 1_{AF}T(1_Av) + 1_{AF}T(1_{AF}v)$ have a nonnegative solution $1_{AF}v$?

**Answer:** If and only if $\rho(A_F) < \rho(A)$.

If $\rho(A_F) < \rho(A)$, then we have

$$1_{AF}v = (\rho(A)\text{Id} - T|_{AF})^{-1}(1_{AF}T(1_Av))$$

therefore Equations (2), (3) have a unique solution.
We say that \( \lambda \geq 0 \) is a *distinguished* eigenvalue if there exists a nonnegative eigenfunction \( v \in L^p \) so that \( \lambda v = Tv \).

**Proposition (Jang-Lewis, Victory ('93))**

*A number \( \lambda \geq 0 \) is a distinguished eigenvalue if and only if there exists a distinguished atom \( A \) with \( \rho(A) = \lambda \).*

**Theorem (Characterization of nonnegative eigenfunctions)**

*Let \( \lambda > 0 \) be a distinguished eigenvalue and \( A_1, \ldots, A_n \) be all the distinguished atoms with spectral radius \( \lambda \). Let \( v_i \) be the unique eigenfunction so that \( \text{supp}(v_i) = F(A_i) \). Then \( v \in L^p \) is a nonnegative eigenfunction related to \( \lambda \) if and only if \( v = \sum_{i=1}^{n} c_i v_i \) with \( c_i \geq 0 \).*
We consider the set of feature \((\Omega, \mathcal{F}, \mu)\) with \(\mu\) a finite measure. We recall the ODE of evolution of a SIS model:

\[
\partial_t u = (1 - u) T_k(u) - \gamma u
\]

\[u(0, .) = u_0\]

We call equilibrium of Equation (1) any constant solution \(g\) so that \(0 \leq g \leq 1\), that is: \(\gamma g = (1 - g) T_k(g)\).

When \(g^* \neq \tilde{0}\), it corresponds to an endemic equilibrium.
We denote $k/\gamma : \left\{ \begin{array}{c} \Omega^2 \\
(x, y) \mapsto k(x, y)/\gamma(y) \end{array} \right.$ and $T_{k/\gamma}$ the kernel operator associated to the kernel $k/\gamma$.

**Assumption**

(i) $\gamma$ is bounded

(ii) $T_{k/\gamma}$ is a compact operator on $L^p$ with $p \in (1, +\infty)$.

(iii) $T_{k/\gamma}(L^p) \subset L^\infty$ and $T_{k/\gamma} : (L^p, ||.||_p) \rightarrow (L^\infty, ||.||_\infty)$ is an operator.

**Definition (Basic reproduction number)**

We call basic reproduction number the quantity $R_0 = \rho(T_{k/\gamma})$. 
Theorem (Delmas, Dronnier, Zitt ('21))

Let \( u_0 \) be an initial condition so that \( 0 \leq u_0 \leq 1 \). We have:

(i) Any maximal solution of Equation (1) is global, and verifies
\[
\forall t \in \mathbb{R}_+, \, 0 \leq u(t,.) \leq 1.
\]

(ii) Equation (1) admits a maximal equilibrium \( g^* \). It satisfies
\[
g^* = \lim u(t,.) \text{ where } u \text{ is the solution of Equation (1) with an initial condition } u_0 = 1, \text{ for the a.e. convergence.}
\]

(iii) According to the value of \( R_0 \), we have the different behaviors:

a) If \( R_0 \leq 1 \), then \( g^* = 0 \) and for any initial condition \( u_0 \), we have \( u(t,.) \to 0 \) for the a.e. convergence.

b) If \( R_0 > 1 \), then \( g^* \neq \tilde{0} \). If \( T_k \) is irreducible, then \( g^* > 0 \) and for any initial condition \( u_0 \neq \tilde{0} \), we have \( u(t,.) \to g^* \) for the a.e. convergence.
Question: What would happen if $R_0 > 1$ but with no irreducibility ?

Let $g \in L^\infty$ with $0 \leq g \leq 1$. Then $g$ is an equilibrium if and only if:

$$T_{(1-g)k/\gamma}(\gamma g) = \gamma g$$

where we denote $(1 - g)k/\gamma(x, y) = (1 - g(x))k(x, y)/\gamma(y)$ for $(x, y) \in \Omega^2$.

When $g$ is an equilibrium, then $g < 1$ therefore the operators $T_k$ and $T_{(1-g)k/\gamma}$ have the same invariants, therefore the same atoms. So the study of equilibria becomes a study of nonnegative eigenfunctions.
We say that a family of atoms \( \{A_i\}_{i \in I} \) is an antichain if for \( i \neq j \), we have \( A_i \cap F(A_j) = \emptyset \). Informally, if means that no atom among the family \( \{A_i\}_{i \in I} \) infects another atom of the family.

**Theorem**

(i) Let \( g \) be an equilibrium. There exists a unique antichain \( \{A_1, \ldots, A_n\} \) of atoms with \( R_0(A_i) > 1 \) so that

\[
\text{supp}(g) = \bigcup_{i=1}^{n} F(A_i).
\]

(ii) Let \( \{A_1, \ldots, A_n\} \) be an antichain of atoms with \( R_0(A_i) > 1 \). There exists a unique equilibrium \( g \) so that

\[
\text{supp}(g) = \bigcup_{i=1}^{n} F(A_i).
\]
There are some difficulties and possible generalizations of our results:

(i) Everything also works by replacing $T_k$ by any power compact operator $T$. Except for the application, one can replace "finite measure" by "\(\sigma\)-finite measure".

(ii) If $p = 1$ or $p = +\infty$, there are some issues about the dual of $L^p$.

(iii) General framework of Banach lattices: Irreducibility is easy to generalize on more general spaces that $L^p$, but not atoms and convex sets.
Thank you for your attention !