Atomic decomposition of a positive compact operator on L^p

Kacem Lefki (LAMA/CERMICS)

Under the supervision of Jean-François Delmas (CERMICS) and Pierre-André Zitt (LAMA)

22 Mars 2023

We consider of population of individuals, with a feature set Ω. Each individual is susceptible (S) or infected (I). For disease with no immunity (ex: gonorrhea).

$k(x, y)$: Infection rate from feature y towards feature x.
$\gamma(x)$: Recovery rate of feature x.

Let $u(t, x)$ be the probability that an individual of feature x is infected at time $t \geqslant 0$. We consider the following ODE (for $\forall t \geqslant 0, \forall x \in \Omega$):

$$\partial_t u(t, x) = (1 - u(t, x)) \int_\Omega k(x, y) u(t, y) \mathrm{d}\mu(y) - \gamma(x) u(t, x).$$

We recall the ODE of evolution of a SIS model:

$$\partial_t u = (1-u)T_k(u) - \gamma u \tag{1}$$
$$u(0,.) = u_0$$

We call equilibrium of Equation (1) any constant solution g so that $0 \leqslant g \leqslant 1$, that is: $\gamma g = (1-g)T_k(g)$.

Let $g \in L^\infty$ with $0 \leqslant g \leqslant 1$. Then g is an equilibrium if and only if:

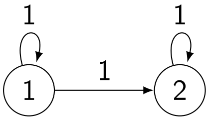
$$T_{(1-g)k/\gamma}(\gamma g) = \gamma g$$

where we denote $(1-g)k/\gamma(x,y) = (1-g(x))k(x,y)/\gamma(y)$ for $(x,y) \in \Omega^2$.

Let $M = (M_{i,j})_{1 \leqslant i,j \leqslant n}$ be a matrix with nonnegative entries. We define the oriented weighted graph on $\{1, \ldots, n\}$ giving the weight $M_{i,j}$ to the edge $j \to i$. Example:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

(a) Matrix



(b) Graph

Figure: Example of a nonnegative matrix and its associated graph

Epidemics interpretation : We split our population into n groups of features. $M_{i,j}$ represents the power of infection from inviduals of feature j towards individuals of feature i.

We call *path* from i to j on the graph any sequence $(i = x_0, \ldots, x_n = j)$ so that $\forall l, M_{l+1,l} > 0$.

We say that $A \subset \{1, \ldots, n\}$ is *invariant* if there is no path starting in A that goes outside of A. In epidemics terms, any infection starting in A stays within A. A is co-invariant if A^c is invariant.

We say that the matrix M is *irreducible* if its only invariant sets are $\emptyset$ and $\{1, \ldots, n\}$. It is equivalent to say that from all $i, j \in \{1, \ldots, n\}$, there is a path from i to j.

For any matrix $M \in \mathcal{M}_n(\mathbb{R})$, we denote $\rho(M) = \sup_{\lambda \in \mathrm{Sp}(M)} |\lambda|$ its spectral radius.

Theorem (Perron-Frobenius theorem ('07 P), ('12 F))

Let $M \in \mathcal{M}_n(\mathbb{R})$ be a non null matrix with nonnegative entries. Then $\rho(M)$ is a eigenvalue of M, related to an eigenvector with nonnegative entries.

We now assume that M is irreducible. We have:

(i) $\rho(M) > 0$.

(ii) $\dim \left(\bigcup_{n \in \mathbb{N}} \ker(M - \rho(M) I_n)^n \right) = 1$ *($\rho(M)$ is simple).*

(iii) *There exists a unique nonnegative eigenvector, and it has positive entries.*

Two possibilities to upgrade Perron-Frobenius theorem:

(i) From finite-dimension to infinite dimension -> From matrices to operators.

(ii) Without irreducibility -> Define irreducible components.

Our goal is to characterize nonnegative eigenfunctions for a positive operator.

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measured space. A kernel is a measurable function $k : \Omega^2 \to \mathbb{R}_+$.
When possible, we define, for $f : \Omega \to \mathbb{R}$ measurable and $x \in \Omega$:

$$T_k(f)(x) = \int_\Omega k(x,y)f(y)\mathrm{d}\mu(y).$$

T_k is a *positive* operator: $\forall f \geqslant 0$ a.e., $T_k(f) \geqslant 0$ a.e..

Epidemics interpretation : Ω is a set of features, and $k(x,y)$ the power of infection from individuals of feature y towards individuals of feature x. The operator T_k models a "one-step infection".

For any operator T on L^p, we denote $\rho(T) = \sup_{\lambda \in \mathrm{Sp}(T)} |\lambda|$.

Theorem (Krein-Rutman theorem ('48), Schaefer ('74), de Pagter ('86))

Let T be a compact positive operator on L^p for $p \in [1, +\infty]$. If $\rho(T) > 0$, then $\rho(T)$ is a eigenvalue of T, related to a nonnegative eigenfunction.

If T is irreducible (we will define it later). We have:

- *(i)* $\rho(T) > 0$.
- *(ii)* $\dim \left(\bigcup_{n \in \mathbb{N}} \ker(T - \rho(T)\,\mathrm{Id})^n \right) = 1$ *($\rho(T)$ is simple).*
- *(iii) There exists a unique nonnegative eigenfunction related to a non-null eigenvalue, and it is positive.*

Let k be a kernel and T the associated kernel operator.

Assumption

The operator T is a compact operator on L^p for some $p \in (1, +\infty)$.

For $A, B \subset \mathcal{F}$, we define the *infection power* from B towards A as:

$$k(A, B) = \int_{A \times B} k(x, y) \mathrm{d}\mu(x) \mathrm{d}\mu(y).$$

Notice that $k(A, B) = 0$ if and only if $k(x, y) = 0$ a.e. on $A \times B$.
We say that A is *invariant* if $k(A^c, A) = 0$, and that A is *co-invariant* if A^c is invariant.

Let $A \in \mathcal{F}$ be a measurable set. We define the *future* of A, denoted $F(A)$ as the minimal invariant set that contains A, and the *past* of A, denoted $P(A)$ as the minimal co-invariant set that contains A. We denote $A_F = F(A) \backslash A$.

The future of A is the set of all the features that might be infected by an epidemics starting in A. The past of A is the set of all the features that might infect A if an epidemics starts there.

For $\Omega' \in \mathcal{F}$ a measurable set with $\mu(\Omega') > 0$, we define the restricted operator $T|_{\Omega'}$ on $L^p(\Omega')$ by $\forall f \in L^p, \forall x \in \Omega'$:

$$T|_{\Omega'}(f)(x) = \int_{\Omega'} k(x,y) f(y) \mathrm{d}\mu(y).$$

Notice that $A \subset \Omega'$ is a $T|_{\Omega'}$-invariant set if and only if $k(\Omega' \backslash A, A) = 0$.

We say T is an *irreducible operator* if the only T-invariant sets are $\emptyset$ or Ω. We say that A is an *irreducible set* if $T|_A$ is a irreducible operator, that is, the only $T|_A$-invariant sets are $\emptyset$ and A.

Let $\mathcal{A}$ be the σ-field generated by the invariant sets. We call *atom* (in the sense of Schwartz) any minimal element of $\mathcal{A}$ with positive measure. That is, A is an atom if $\mu(A) > 0$, $A \in \mathcal{A}$ and for any set $B \in \mathcal{A}$ with $B \subset A$, we have $B = A$ of $B = \emptyset$.

We say that A is a *convex* set if $F(A) \backslash A$ is an invariant set.

Theorem (Equivalent definitions of atoms)

Let $A \in \mathcal{F}$ be a measurable set with $\mu(A) > 0$. We have equivalence between:

(i) A is an atom.

(ii) A is a minimal convex set.

(iii) A is a maximal irreducible set.

Pros and cons of each definition:

	Pros	Cons
Atom	Good for theory	Hard to use in practice
Min convex	Easy to use	Not general enough
Max irr	Easy to generalize	Maximal instead of minimal

For $\lambda \in \mathbb{C}$ and $A \in \mathcal{F}$ a measurable set with $\mu(A) > 0$, we define the *algebraic multiplicity* of λ of T restricted to A by:

$$m(\lambda, A, T)(\text{or } m(\lambda, A)) = \dim\left(\bigcup_{n\in\mathbb{N}} \ker(T|_A - \lambda\,\mathrm{Id})^n\right).$$

When $m(\lambda, \Omega) = 1$, we say that λ is *simple*. The algebraic multiplicity, for a compact operator and for $\lambda \neq 0$, satisfies the following properties:

(i) $m(\lambda, \Omega) < +\infty$.

(ii) $m(\lambda, A) > 0$ if and only if λ is an eigenvalue of $T|_A$.

Theorem (Schwartz ('61))

Let $\lambda > 0$. Then we have:

$$m(\lambda, \Omega) = \sum_{A \text{ atom}} m(\lambda, A).$$

For any $A \in \mathcal{F}$ with $\mu(A) > 0$, we denote $\rho(A) := \rho(T|_A)$. We remind that if A is an atom with $\rho(A) > 0$, by Krein-Rutman theorem we have $m(\rho(A), A) = 1$.

Here are some important consequences of Schwartz's theorem:

Corollary

(i) If $\lambda > 0$, there exists a finite amount of atoms with a spectral radius $\geqslant \lambda$.

(ii) $\rho(T) = \max_{A \text{ atom}} \rho(A)$.

(iii) If I is an invariant set, $\rho(I) = \max_{A \text{ atom}, A \subset I} \rho(A)$.

Question: How can we "distinguish" an atom using nonnegative eigenfunction "related" to the atom ?

Idea: Find a nonnegative eigenfunction v so that $\text{supp}(v) := \{v > 0\}$ is "close enough" to A. For instance $A \cap \text{supp}(v) \neq \emptyset$. As A is an atom and $\text{supp}(v)$ an invariant set, we have in this case $F(A) \subset \text{supp}(v)$. Can we have it with $\text{supp}(v) = F(A)$?

Definition (Distinguished atom)

We say that an atom A is distinguished if there exists a nonnegative eigenfunction $v \in L^p$ so that $\text{supp}(v) = F(A)$.

Proposition (Characterization of distinguished atoms)

Let A be an atom. We have the following:

(i) A is distinguished if and only if $\rho(A) > \rho(A_F)$.

(i) If A is distinguished, there exists a unique nonnegative eigenfunction $v \in L^p$ with $\mathrm{supp}(v) = F(A)$, and it is related to the eigenvalue $\rho(A)$.

Idea of proof of the previous proposition:

Let $v \in L^p$ and $\lambda \in \mathbb{C}$ with $\text{supp}(v) = F(A)$. We have $Tv = \lambda v$ if and only if:

$$\lambda \mathbb{1}_A v = \mathbb{1}_A T(\mathbb{1}_A v) \quad (2)$$

$$\lambda \mathbb{1}_{A_F} v = \mathbb{1}_{A_F} T(\mathbb{1}_A v) + \mathbb{1}_{A_F} T(\mathbb{1}_{A_F} v) \quad (3)$$

Then by Equation (2), $\mathbb{1}_A v$ is a nonnegative eigenfunction of $T|_A$. Therefore by Krein-Rutman theorem, $\mathbb{1}_A v$ is uniquely defined and $\lambda = \rho(A)$.

Does Equation (3): $\rho(A)\mathbb{1}_{A_F} v = \mathbb{1}_{A_F} T(\mathbb{1}_A v) + \mathbb{1}_{A_F} T(\mathbb{1}_{A_F} v)$ have a nonnegative solution $\mathbb{1}_{A_F} v$?

Answer: If and only if $\rho(A_F) < \rho(A)$.

If $\rho(A_F) < \rho(A)$, then we have
$\mathbb{1}_{A_F} v = (\rho(A)\,\mathrm{Id} - T|_{A_F})^{-1}(\mathbb{1}_{A_F} T(\mathbb{1}_A v))$ therefore Equations (2), (3) have a unique solution.

We say that $\lambda \geqslant 0$ is a *distinguished* eigenvalue if there exists a nonnegative eigenfunction $v \in L^p$ so that $\lambda v = Tv$.

Proposition (Jang-Lewis, Victory ('93))

A number $\lambda \geqslant 0$ is a distinguished eigenvalue if and only if there exists a distinguished atom A with $\rho(A) = \lambda$.

Theorem (Characterization of nonnegative eigenfunctions)

Let $\lambda > 0$ be a distinguished eigenvalue and $A_1, \ldots, A_n$ be all the distinguished atoms with spectral radius λ. Let v_i be the unique eigenfunction so that $\operatorname{supp}(v_i) = F(A_i)$. Then $v \in L^p$ is a nonnegative eigenfunction related to λ if and only if $v = \sum_{i=1}^{n} c_i v_i$ with $c_i \geqslant 0$.

We consider the set of feature $(\Omega, \mathcal{F}, \mu)$ with μ a finite measure. We recall the ODE of evolution of a SIS model:

$$\partial_t u = (1-u)T_k(u) - \gamma u$$
$$u(0,.) = u_0$$

We call equilibrium of Equation (1) any constant solution g so that $0 \leqslant g \leqslant 1$, that is: $\gamma g = (1-g)T_k(g)$.

When $g^* \neq \tilde{0}$, it corresponds to an endemic equilibrium.

We denote $k/\gamma : \begin{cases} \Omega^2 & \to & \mathbb{R}_+ \\ (x,y) & \mapsto & k(x,y)/\gamma(y) \end{cases}$ and $T_{k/\gamma}$ the kernel operator associated to the kernel k/γ.

Assumption

(i) γ *is bounded*

(ii) $T_{k/\gamma}$ *is a compact operator on* L^p *with* $p \in (1, +\infty)$.

(iii) $T_{k/\gamma}(L^p) \subset L^\infty$ *and* $T_{k/\gamma} : (L^p, ||.||_p) \to (L^\infty, ||.||_\infty)$ *is an operator.*

Definition (Basic reproduction number)

We call basic reproduction number the quantity $R_0 = \rho(T_{k/\gamma})$.

Theorem (Delmas, Dronnier, Zitt ('21))

Let u_0 be an initial condition so that $0 \leqslant u_0 \leqslant 1$. We have:

(i) *Any maximal solution of Equation* (1) *is global, and verifies* $\forall t \in \mathbb{R}_+, 0 \leqslant u(t,.) \leqslant 1$.

(ii) *Equation* (1) *admits a maximal equilibrium g^*. It satisfies* $g^* = \lim u(t,.)$ *where u is the solution of Equation* (1) *with an initial condition $u_0 = 1$, for the a.e. convergence.*

(iii) *According to the value of R_0, we have the different behaviors:*

a *If $R_0 \leqslant 1$, then $g^* = 0$ and for any initial condition u_0, we have $u(t,.) \to 0$ for the a.e. convergence.*

b *If $R_0 > 1$, then $g^* \neq \tilde{0}$. If T_k is irreducible, then $g^* > 0$ and for any initial condition $u_0 \neq \tilde{0}$, we have $u(t,.) \to g^*$ for the a.e. convergence.*

Question: What would happen if $R_0 > 1$ but with no irreduciblity ?

Let $g \in L^\infty$ with $0 \leqslant g \leqslant 1$. Then g is an equilibrium if and only if:

$$T_{(1-g)k/\gamma}(\gamma g) = \gamma g$$

where we denote $(1-g)k/\gamma(x,y) = (1-g(x))k(x,y)/\gamma(y)$ for $(x,y) \in \Omega^2$.

When g is an equilibrium, then $g < 1$ therefore the operators T_k and $T_{(1-g)k/\gamma}$ have the same invariants, therefore the same atoms. So the study of equilibria becomes a study of nonnegative eigenfunctions.

We say that a family of atoms $\{A_i\}_{i\in I}$ is an *antichain* if for $i \neq j$, we have $A_i \cap F(A_j) = \emptyset$. Informally, if means that no atom among the family $\{A_i\}_{i\in I}$ infects another atom of the family.

Theorem

(i) Let g be an equilibrium. There exists a unique antichain $\{A_1, \ldots, A_n\}$ of atoms with $R_0(A_i) > 1$ so that

$$\mathrm{supp}(g) = \bigcup_{i=1}^{n} F(A_i).$$

(ii) Let $\{A_1, \ldots, A_n\}$ be an antichain of atoms with $R_0(A_i) > 1$. There exists a unique equilibrium g so that

$$\mathrm{supp}(g) = \bigcup_{i=1}^{n} F(A_i).$$

There are some difficulties and possible generalizations of our results:

(i) Everything also works by remplacing T_k by any power compact operator T. Except for the application, one can replace "finite measure" by "σ-finite measure".

(ii) If $p = 1$ or $p = +\infty$, there are some issues about the dual of L^p.

(iii) General framework of Banach lattices: Irreducibility is easy to generalize on more general spaces that L^p, but not atoms and convex sets.

Thank you for your attention !