Séminaire de Calcul Scientifique du CERMICS

École des Ponts ParisTech

The zero relaxation limit for the Aw-Rascle-Zhang traffic flow model

Nicolas Laurent-Brouty (Inria Sophia-Antipolis)

18 mai 2018

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Nicolas Laurent-Brouty[1,2], Paola Goatin[1]

[1]Université Côte d'Azur, Inria, CNRS, LJAD, France
[2]Ecole des Ponts ParisTech, Champs-sur-Marne, France

May 18th, 2018

nicolas.laurent-brouty@inria.fr

Outline

1. Introduction to conservation laws
2. Wave-Front Tracking approximations
3. Convergence of the WFT approximate solutions
4. Convergence of the relaxed ARZ system towards LWR equation
5. Decay estimates of positive waves

Outline

Introduction to conservation laws

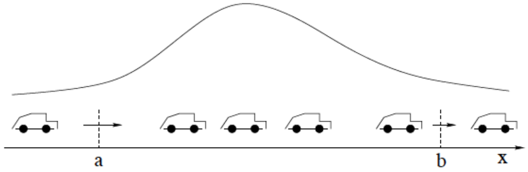


Figure: 1D-representation of a stretch of road

t time

x space variable

$\rho(t,x)$ density of vehicles

$v(t,x)$ velocity of the flow

$$\boxed{f(\rho, v) = \rho v}$$

Conservation of the number of cars:

$$\frac{d}{dt}\int_a^b \rho(y,t)dy = [\text{flux entering at a}] - [\text{flux exiting at b}]$$

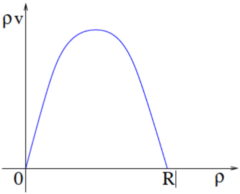
$$\int_a^b \frac{\partial}{\partial t}\rho(y,t)dy = -\int_a^b \frac{\partial}{\partial x}[f(\rho,v)](x,t)dx$$

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial x}f(\rho,v) = 0$$

Application to traffic flow

1. The Lighthill, Whitham, Richards (LWR) model
   Assume $v = v(\rho)$

$$\begin{cases} \partial_t \rho + \partial_x(\rho v(\rho)) = 0 \\ \rho(0,x) = \rho^0(x) \end{cases} \qquad x \in \mathbb{R}, \;\; t > 0,$$



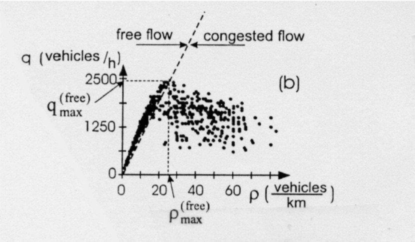


Figure: Fundamental diagram of traffic flow

Application to traffic flow

2. The Payne-Whitham model (PW)
   Define an anticipation factor $A_e(\rho)$ and a response time from drivers δ

$$\begin{cases} \partial_t \rho + \partial_x(\rho v(\rho)) = 0, \\ \partial_t v + v \partial_x v + \frac{1}{\rho} \partial_x (A_e(\rho)) = 0 \end{cases}$$

3. The Aw-Rascle-Zhang (ARZ) model
   Assume a pseudo-pressure, strictly increasing, $p(\rho) > 0$

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t (v + p(\rho)) + v \partial_x (v + p(\rho)) = 0 \end{cases}$$

4. The Aw-Rascle-Zhang model with relaxation

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t (v + p(\rho)) + v \partial_x (v + p(\rho)) = \frac{V_{eq}(\rho) - v}{\delta} \end{cases}$$

The ARZ model with relaxation

The ARZ model with relaxation

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = \frac{V_{\text{eq}}(\rho) - v}{\delta} \end{cases}$$

The model can be put under conservative form:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho(v + p(\rho))) + \partial_x(\rho v(v + p(\rho))) = \rho \frac{V_{\text{eq}}(\rho) - v}{\delta} \end{cases}$$

Eigenvalues:

$$\lambda_1 = v - \rho p'(\rho), \qquad \lambda_2 = v$$

To ensure strict hyperbolicity, we assume:

$$\rho > 0, \qquad p(\rho) \geq 0, \qquad p'(\rho) > 0.$$

Define $w := v + p(\rho)$

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t(\rho w) + \partial_x(\rho v w) = \rho \frac{V_{eq}(\rho) - v}{\delta} \end{cases}$$

convert into Lagrangian coordinates (T, X) with $\tau = \frac{1}{\rho}$.

$$\tilde{p}(\tau) = p\left(\frac{1}{\tau}\right), \qquad \tilde{V}(\tau) = V_{eq}\left(\frac{1}{\tau}\right), \qquad \tilde{p}'(\tau) = -\frac{1}{\tau^2} p'\left(\frac{1}{\tau}\right) < 0,$$

$$\begin{cases} \partial_T \tau - \partial_X v = 0 \\ \partial_T w = \frac{\tilde{V}(\tau) - v}{\delta} \end{cases}$$

with initial data

$$\begin{cases} \tau(0, \cdot) = \tau_0 \\ w(0, \cdot) = w_0 \end{cases}$$

Definition of solutions

$$\begin{cases} U_t + [F(U)]_X = G^\delta(U) \\ U(0,x) = U^0(x) \end{cases} \qquad x \in \mathbb{R}, \ \ t > 0 \qquad (1.1)$$

$$U = \begin{pmatrix} \tau \\ w \end{pmatrix}, \qquad F(U) = \begin{pmatrix} -(w - \tilde{p}(\tau)) \\ 0 \end{pmatrix}, \qquad G^\delta(U) = \begin{pmatrix} 0 \\ \frac{\tilde{V}(\tau)-(w-\tilde{p}(\tau))}{\delta} \end{pmatrix}.$$

Definition (definition of solutions)

Assume $U^0 \in BV(\mathbb{R})$ and $T > 0$. We say that a function $U^\delta : [0,T] \times \mathbb{R} \to \mathbb{R}^2$ is a weak solution to the Cauchy problem (1.1) if the map $t \to U^\delta(t,\cdot) \in L^1_{\text{loc}}(\mathbb{R})$ is continuous, $U^\delta(t=0,\cdot) = U^0(\cdot)$ and if for any $\phi \in C^1_c([0,T] \times \mathbb{R})$

$$\int_{-\infty}^{+\infty} \phi(0,X) U_0^\delta(X) dX + \int_0^T \int_{-\infty}^{+\infty} [\phi_t U^\delta(t,X) + \phi_X F(U^\delta(t,X))] dX dt + \int_0^T \int_{-\infty}^{+\infty} \phi G^\delta(U^\delta(t,X)) dX dt = 0$$

Main results

Theorem

For each relaxation parameter δ, the ARZ model with relaxation admits a weak entropy solution $U^\delta = (\tau^\delta, w^\delta)$.

Theorem

The subsequence of weak entropy solutions $U^\delta = (\tau^\delta, w^\delta)$ of the relaxed ARZ model converges to $\bar{U} = (\bar{\tau}, \bar{w})$ as $\delta \to 0$. Then $\bar{w} = \tilde{V}(\bar{\tau}) + \tilde{p}(\bar{\tau})$ and $\bar{\tau}$ is a weak solution of the scalar Cauchy problem:

$$\begin{cases} \partial_t \tau - \partial_X \tilde{V}(\tau) = 0, \\ \tau(0,\cdot) = \tau_0(\cdot), \end{cases} \qquad X \in \mathbb{R},\ t > 0. \tag{1.2}$$

Outline

What is Wave-Front Tracking?

General idea of wave front tracking for a system of conservation laws:

$$\begin{cases} U_t + [F(U)]_x = 0 \\ U(0,x) = U^0(x) \end{cases} \qquad x \in \mathbb{R}, \ t > 0$$

1. approximate the initial datum U^0 by a piecewise constant function U^0_ϵ such that $\|U^0 - U^0_\epsilon\|_{\mathbf{L}^\infty} \leq \epsilon$
2. for $t = 0^+$ solve for each discontinuity of U^0_ϵ the associated Riemann problem. the solution is piecewise constant.
3. the solution can be propagated along the wavefront until two wave fronts interact.
4. At this time, treat the solution as initial condition and restart the process.

WFT approximations

How to treat the relaxation term? $\to$ two step process

1. solve the Cauchy problem associated to the homogeneous system via WFT on a time interval $[t_0, t_0 + \Delta t]$
2. at $t = t_0 + \Delta t$, integrate the source term following $w_t = \frac{\tilde{V}(\tau) - v}{\delta}$

Definition (BV space)

Let Ω an open set. We say that a function $u \in L^1_{loc}(\Omega; \mathbb{R})$ belongs to $BV(\Omega; \mathbb{R})$ if its total variation $TV(u) < \infty$, where for every n-tuple $\{x_1, .., x_n\} \in \Omega$:

$$TV(u) = \sup \sum_{i=1}^{n-1} |u(x_{i+1}) - u(x_i)|$$

Algorithm

Let $T > 0$ and a sequence $\Delta t^\nu > 0$ s.t. $\Delta t^\nu \xrightarrow[\nu\to\infty]{} 0$.

1. Approximate the initial value $U_0 \in BV(\mathbb{R}^+ \times \mathbb{R})$ by a piecewise constant function $U_0^\nu = (\tau_0^\nu, w_0^\nu)$
2. Solve the homogeneous system via WFT and name $U^\nu(t,\cdot)$, $t \in [0, \Delta t^\nu)$ the solution

$$\begin{cases} \partial_t \tau - \partial_X v = 0, \\ \partial_t w = 0, \end{cases}$$

3. At $t = \Delta t^\nu$ integrate the source term $w_t = \frac{\tilde{V}(\tau) - v}{\delta}$, *i.e.* define

$$\tau^\nu(\Delta t^\nu, \cdot) = \tau^\nu(\Delta t^\nu -, \cdot),$$
$$w^\nu(\Delta t^\nu, \cdot) = w^\nu(\Delta t^\nu -, \cdot) + \Delta t^\nu \frac{\tilde{V}(\tau^\nu(\Delta t^\nu, \cdot)) - v(U^\nu(\Delta t^\nu -, \cdot))}{\delta}$$

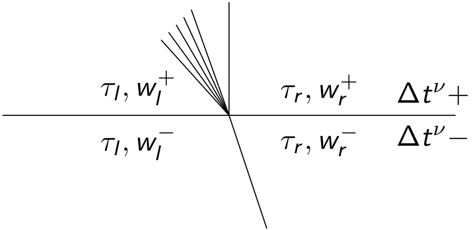


Figure: Notations used in step 3.

4. Treat $U^\nu(\Delta t^\nu, \cdot)$ as a new piecewise constant initial condition and iterate

Existence of an invariant domain

Definition (invariant domain)

$$\mathbf{E} := \left\{ u = (\tau, w) \colon V_{\mathrm{eq}}^{\mathrm{min}} \leq w - \tilde{p}(\tau) < w \leq V_{\mathrm{eq}}^{\mathrm{max}} + \max_{\tau} \tilde{p}(\tau) \right\}$$

Let $M > 0$. $\quad \mathcal{D}(M) := \{ u : \mathbb{R} \to \mathbf{E} \colon TV(w(u)) + TV(v(u)) \leq M \}$

Lemma

For $\Delta t \leq \delta$, the set $\mathbf{E}$ is an invariant domain for the proposed WFT scheme.

Decreasing TV and Lipschitz estimates

Lemma

For $\Delta t \leq \delta$, the total variation of the Riemann invariants of the constructed approximation U^ν is non-increasing in time:

$$TV(w^\nu(t,\cdot)) + TV(v(U^\nu(t,\cdot))) \leq TV(w_0^\nu) + TV(v(U_0^\nu)), \quad \text{for a.e. } t > 0.$$

Lemma

Let $\nu \in \mathbb{N}$ and $U_0^\nu \in \mathcal{D}(M)$. Then $\forall\, a < b, \forall\, 0 \leq s < t$:

$$\int_a^b |\tau^\nu(t,X) - \tau^\nu(s,X)| dX \leq C_M(t-s),$$

$$\int_a^b |w^\nu(t,X) - w^\nu(s,X)| dX \leq (C_M + L_\delta)(t - s + \Delta t).$$

Outline

Convergence of the WFT approximations

Theorem

Let $U_0 = (\tau_0, w_0) \in \mathcal{D}(M)$ for some $M > 0$, and denote by $U^\delta = (\tau^\delta, w^\delta)$ the limit of a subsequence $U^\nu = (\tau^\nu, w^\nu)$ of WFT approximate solutions as $\nu \to \infty$. Then U^δ is the weak entropic solution of

$$\begin{cases} \partial_T \tau - \partial_X v = 0, \\ \partial_T w = \frac{\tilde{V}(\tau) - v}{\delta}, \\ \tau(0, \cdot) = \tau_0, \\ w(0, \cdot) = w_0. \end{cases} \tag{3.1}$$

Sketch of proof

The existence of the limit U^δ and the convergence in $L^1_{loc}([0, +\infty[\times \mathbb{R})$ is guaranteed by Helly's theorem.

Theorem (Helly's theorem)

Consider a sequence of functions U^ν s.t.:

$$TV(U^\nu(t,\cdot)) \leq C, \qquad |U^\nu(t,x)| \leq M \text{ for all } t, x,$$
$$\int_{-\infty}^{\infty} |U^\nu(t,X) - U^\nu(s,X)| dX \leq L(t-s) \text{ for all } t, s \geq 0,$$

Then there exists a subsequence U^μ which converges to some function U in $L^1_{loc}([0, +\infty[\times \mathbb{R})$. The limit satisfies

$$\int_{-\infty}^{\infty} |U(t,X) - U(s,X)| dX \leq L(t-s) \text{ for all } t, s \geq 0,$$

Sketch of proof

$$U_t + [F(U)]_X = G^\delta(U),$$

$$U = \begin{pmatrix} \tau \\ w \end{pmatrix}, \qquad F(U) = \begin{pmatrix} -(w - \tilde{p}(\tau)) \\ 0 \end{pmatrix}, \qquad G^\delta(U) = \begin{pmatrix} 0 \\ \frac{\tilde{V}(\tau) - (w - \tilde{p}(\tau))}{\delta} \end{pmatrix}.$$

Definition (definition of solutions)

Assume $U^0 \in BV(\mathbb{R})$ and $T > 0$. We say that a function $U^\delta : [0, T] \times \mathbb{R} \to \mathbb{R}^2$ is a weak solution to the Cauchy problem (1.1) if the map $t \to U^\delta(t, \cdot) \in L^1_{\text{loc}}(\mathbb{R})$ is continuous, $U^\delta(t = 0, \cdot) = U^0(\cdot)$ and if for any $\phi \in C^1_c([0, T] \times \mathbb{R})$

$$\int_{-\infty}^{+\infty} \phi(0, X) U_0^\delta(X) dX + \int_0^T \int_{-\infty}^{+\infty} [\phi_t U^\delta(t, X) + \phi_X F(U^\delta(t, X))] dX dt + \int_0^T \int_{-\infty}^{+\infty} \phi G(U^\delta(t, X)) dX dt = 0$$

Let $N^\nu \in \mathbb{N}$ such that $T = N^\nu \Delta t^\nu + \beta^\nu, \beta^\nu \in [0, \Delta t^\nu[$. Decompose the integral on intervals $[k\Delta t^\nu, (k+1)\Delta t^\nu]$

$$\int_{k\Delta t^\nu}^{(k+1)\Delta t^\nu} \int_{-\infty}^{+\infty} [\phi_t \tau^\nu - \phi_X (w^\nu - \tilde{p}(\tau^\nu))] dX dt = \int_{-\infty}^{+\infty} \phi\tau^\nu((k+1)\Delta t^\nu -, X) dX - \int_{-\infty}^{+\infty} \phi\tau^\nu(k\Delta t^\nu +, X) dX$$

Then

$$\int_0^T \int_{-\infty}^{+\infty} [\phi_t \tau^\nu - \phi_X (w^\nu - \tilde{p}(\tau^\nu))] dX dt = \sum_{k=0}^{N^\nu - 1} \int_{k\Delta t^\nu}^{(k+1)\Delta t^\nu} \cdots + \int_{N^\nu \Delta t^\nu}^{T} \cdots$$

$$= \int_{-\infty}^{+\infty} \phi(N^\nu \Delta t^\nu, X) \tau^\nu (N^\nu \Delta t^\nu -, X) dX - \int_{-\infty}^{+\infty} \phi(0, X) \tau^\nu (0+, X) dX + \int_{N^\nu \Delta t^\nu}^{T} \int_{-\infty}^{+\infty} [\phi_t \tau^\nu - \phi_X (w^\nu - \tilde{p}(\tau^\nu))] dX dt.$$

Thus

$$\int_{-\infty}^{+\infty} \phi(0, X) \tau_0(X) dX + \int_0^T \int_{-\infty}^{+\infty} [\phi_t \tau^\nu - \phi_X (w^\nu - \tilde{p}(\tau^\nu))] dX dt \to 0$$

Now let us consider w^ν

$$\int_{k\Delta t^\nu}^{(k+1)\Delta t^\nu}\int_{-\infty}^{+\infty} \phi_t w^\nu dXdt = \int_{-\infty}^{+\infty} [\phi w^\nu((k+1)\Delta t-, X) - \phi w^\nu(k\Delta t^\nu +, X)]dX$$
$$= \int_{-\infty}^{+\infty} \phi w^\nu((k+1)\Delta t^\nu -, X)dX$$
$$- \int_{-\infty}^{+\infty} \phi\Big[w^\nu(k\Delta t^\nu -, X) + \Delta t^\nu \frac{\tilde{V}(\tau^\nu) - v^\nu}{\delta}(k\Delta t^\nu -, X)\Big]dX.$$

where we use that

$$w^\nu(k\Delta t^\nu +, X) = w^\nu(k\Delta t^\nu -, X) + \Delta t^\nu \frac{\tilde{V}(\tau^\nu) - v^\nu}{\delta}(k\Delta t^\nu -, X)$$

Finally use that $\tau^\nu \to \tau^\delta$ and $v^\nu \to v^\delta$ in L^1_{loc} and conclude with the Dominated Convergence Theorem.

For the entropy solution aspect, fix a smooth convex entropy η with associated entropy-flux q s.t. :

$$\begin{aligned} &\nabla\eta^T(z)DF(z) = \nabla^T q(z), \\ &\nabla\eta^T(z)G(z) \leq 0, \end{aligned} \qquad \text{for } z \in \mathbb{R}^2_+.$$

The same way, we show that

$$\int_0^T \int_{-\infty}^{+\infty} \Big[\eta(U^\delta(t,X))\phi_t(t,X) + q(U^\delta(t,X))\phi_x(t,X)$$
$$+ \nabla\eta^T(U^\delta(t,X))G(t,X,U^\delta(t,X))\phi(t,X)\Big] dXdt + \int_{-\infty}^{+\infty} \phi(0,X)\eta(U_0^\delta(X))dX \geq 0.$$

Outline

General idea

Numerically, one can observe that the solutions to the relaxed-ARZ system

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0 \\ \partial_t (v + p(\rho)) + v\partial_x (v + p(\rho)) = \frac{V_{\text{eq}}(\rho) - v}{\delta} \\ \rho(0,x) = \rho^0(x) \\ v(0,x) = v^0(x) \end{cases} \qquad x \in \mathbb{R},\ \ t > 0,$$

converges to a solution of LWR when $\delta \to 0$

$$\begin{cases} \partial_t \rho + \partial_x(\rho V_{\text{eq}}(\rho)) = 0 \\ \rho(0,x) = \rho^0(x) \end{cases} \qquad x \in \mathbb{R},\ \ t > 0,$$

"The system forces the velocity towards the equilibrium speed"

zero-relaxation limit

We recall that the sequence of WFT approximation satisfies:

$$\int_a^b |\tau^\nu(t,X) - \tau^\nu(s,X)| dX \le C_M(t-s),$$
$$\int_a^b |w^\nu(t,X) - w^\nu(s,X)| dX \le (C_M + L_\delta)(t - s + \Delta t).$$

To pass to the limit as $\delta \to 0$, we need a stronger estimate on L_δ.

Lemma

$$L_\delta \le \frac{2}{\delta} e^{-\frac{s}{\delta}} \int_a^b |\tilde{V}(\tau_0(X)) - v_0(X)| dX.$$

zero-relaxation limit

Theorem

Let $U_0 = (\tau_0, w_0) \in \mathcal{D}(M)$ for some $M > 0$, and denote by $\bar{U} = (\bar{\tau}, \bar{w})$ the limit of a subsequence of weak entropy solutions $U^\delta = (\tau^\delta, w^\delta)$ of (3.1), as $\delta \to 0$. Then $\bar{w} = \tilde{V}(\bar{\tau}) + \tilde{p}(\bar{\tau})$ and $\bar{\tau}$ is a weak solution of the scalar Cauchy problem:

$$\begin{cases} \partial_t \tau - \partial_X \tilde{V}(\tau) = 0, \\ \tau(0, \cdot) = \tau_0(\cdot), \end{cases} \qquad X \in \mathbb{R}, \;\; t > 0.$$

Sketch of proof

Difficulty: prove convergence of the sequence U^δ on the set $[0,\infty[\times\mathbb{R}$ in $0+$, since

$$L_\delta \leq \frac{2}{\delta} e^{-\frac{s}{\delta}} \int_a^b |\tilde{V}(\tau_0(X)) - v_0(X)| dX.$$

Idea: consider sets of the type $[1/n,\infty[\times[-n,n]$, where you have:

$$L_\delta \leq \frac{C}{\delta}(b-a)e^{-\frac{1}{n\delta}} \to 0$$

Then on each set isolate a converging subsequence thanks to Helly's theorem, and construct a converging sequence in $L^1_{loc}([0,+\infty[\times\mathbb{R})$ with a diagonalization process.

Then pass to the limit in the weak solution definition, to obtain $\bar{w}(t,\cdot) = \tilde{V}(\bar{\tau}(t,\cdot)) + \tilde{p}(\bar{\tau}(t,\cdot))$ and then $\bar{\tau}_t - \tilde{V}(\bar{\tau})_x = 0$.

Outline

Oleinik estimates - the entropy condition

Proposition (Kruzkov entropy condition)

The unique solution u to the scalar conservation law must satisfy, for any $k \in \mathbb{R}$:

$$\int_{\mathbb{R}^+} \int_{\mathbb{R}} (|u-k|\phi_t + sgn(u-k)(f(u)-f(k))\phi_x)dxdt + \int_{\mathbb{R}} |u_0 - k|\phi(0,x)dx \geq 0$$

Proposition

A weak solution u of the scalar conservation law is entropic if and only if there exists a constant $C > 0$ such that:

$$u(x+z,t) - u(x,t) \leq \frac{C}{t} z \quad \forall t > 0, \forall x \in \mathbb{R}, \forall z > 0.$$

Decay estimates of positive waves

Proposition

Assume that $\exists c_0 > 0$ *such that* $\forall u, u' \in \mathbf{E}$*:*

$$\begin{aligned} |\lambda_1(u) - \lambda_1(u')| &= |\tilde{p}'(\tau) - \tilde{p}'(\tau')| \leq c_0, \\ |\lambda_1(u) - \lambda_2(u')| &= |\tilde{p}'(\tau)| \geq 2c_0. \end{aligned} \tag{5.1}$$

Then, there exists a constant $C > 0$ *such that, for any interval* $]a, b[$, *for any time horizon* $T > 0$, *and every initial condition, the measure* $\mu_T^{1+}(]a, b[)$ *of positive 1-waves contained in the solution of* (3.1) *satisfies*

$$\mu_T^{1+}(]a, b[) \leq C \frac{b-a}{T} e^{C\frac{T}{\delta}(TV(w_0)+TV(v_0))} + C\frac{T}{\delta}\left(TV(w_0) + TV(v_0)\right). \tag{5.2}$$

Remark

Unfortunately, the estimate is not sufficient to recover Oleinik's estimates and proove that the sequence converges towards the entropy solution of LWR.

Thank you for your attention

- Paola Goatin and Nicolas Laurent-Brouty. "The zero relaxation limit for the Aw-Rascle-Zhang traffic flow model". working paper or preprint. Apr. 2018. URL: https://hal.inria.fr/hal-01760930.