Séminaire de Mathématiques Appliquées du CERMICS

École des Ponts
ParisTech

Une méthode de domaines fictifs à convergence optimale

Alexei Lozinski (Université de Franche-Comté)

25 mars 2019

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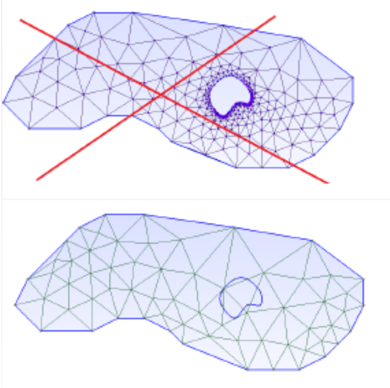
Matching vs. Non-matching grids

Possible uses of non-matching grids



- A simpler treatment of complex geometries, cracks, inclusions, ...
- Inverse problems involving geometrical features of *a priori* unknown shape
- Fluid-Structure interaction, particulate flows, ... (domain changing in time)

Outline

- Classical Fictitious Domain methods for Poisson-Dirichlet problem

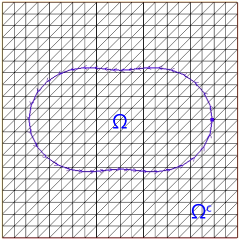
$$
\begin{aligned}
-\Delta u &= f \text{ in } \Omega \\
u &= g \text{ on } \Gamma := \partial\Omega
\end{aligned}
$$

- CutFEM à la Burman & Hansbo
- A “no cut” method
  - A version for Poisson-Dirichlet
  - A version for Poisson-Neumann
- ϕ-FEM : a joint work with Michel Duprez (LJLL)

A classical fictitious domain method

- Extend u to to the whole fictitious domain $\mathcal{O}$ by the solution of the same governing equation

$$
\begin{aligned}
-\Delta u &= f \text{ in } \Omega \\
-\Delta u &= f \text{ in } \Omega^c := \mathcal{O}\backslash\Omega \\
u &= g \text{ on } \Gamma \\
&+ \text{ some b.c. on } \partial\mathcal{O}
\end{aligned}
$$



- The weak form with a Lagrange multiplier λ on Γ:
  Find $u \in H^1(\mathcal{O})$, $\lambda \in H^{-1/2}(\Gamma)$ s.t.

$$
\begin{aligned}
\int_{\mathcal{O}} \nabla u \cdot \nabla v + \int_\Gamma \lambda v &= \int_{\mathcal{O}} fv \quad \forall v \in H^1(\mathcal{O}) \\
\int_\Gamma \mu u &= \int_\Gamma g\mu \quad \forall \mu \in H^{-1/2}(\Gamma)
\end{aligned}
$$

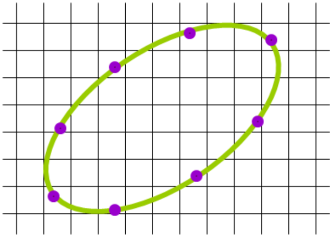
Finite element discretization

- Let

$\mathcal{T}_h$ be a quasi-uniform mesh on $\mathcal{O}$

$V_h = \{$cont. piecewise linear functions on $\mathcal{T}_h\}$

$M_h = \{$piecewise constant on a mesh on $\Gamma\}$

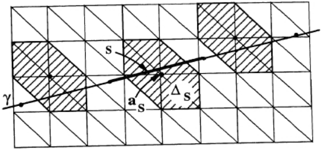


- Find $u_h \in V_h$, $\lambda_h \in M_h$ s.t.

$$\int_{\mathcal{O}} \nabla u_h \cdot \nabla v_h + \int_{\Gamma} \lambda_h v_h = \int_{\mathcal{O}} f v_h \quad \forall v_h \in V_h$$
$$\int_{\Gamma} \mu_h u_h = \int_{\Gamma} g \mu_h \quad \forall \mu_h \in M_h$$

- The mesh on Γ should be sufficiently coarser than $\mathcal{T}_h$ in order to have the inf-sup condition

$$\inf_{v_h \in V_h} \sup_{\mu_h \in M_h} \frac{\int_\Gamma \mu_h v_h}{\|\mu_h\|_{-1/2,\Gamma} |v_h|_{1,\mathcal{O}}} \geqslant \beta > 0$$



- By the theory of saddle-point problem discretization

$$|u - u_h|_{1,\mathcal{O}} \lesssim |u - I_h u|_{1,\mathcal{O}} + \|\lambda - I_h^\Gamma \lambda\|_{-1/2,\Gamma}$$

- By elliptic regularity

$$u \in H^2(\Omega) \cap H^2(\Omega^c) \text{ and } \lambda = \left[\frac{\partial u}{\partial n}\right]_\Gamma \in H^{1/2}(\Gamma)$$

but $u \notin H^2(\mathcal{O})$ (typically $u \in H^{3/2-\varepsilon}(\mathcal{O})$)

The best possible convergence

Even supposing some extra regularity for f one gets

$$|u - u_h|_{1,\mathcal{O}} \lesssim h^{1/2} \|f\|_{1,\mathcal{O}}$$

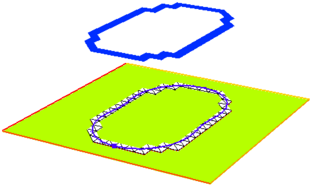
Indeed,

- Approximation of λ is OK: taking I_h^Γ as L^2-projection

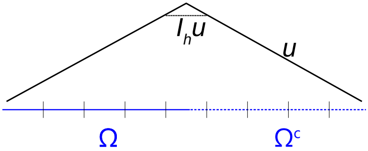
$$\|\lambda - I_h^\Gamma \lambda\|_{-1/2,\Gamma} \lesssim h\|\lambda\|_{1/2,\Gamma} \lesssim h\|f\|_{0,\Omega}$$

- On cut triangles $T \subset \Omega_h^\Gamma$ one cannot expect more than

$$|u - I_h u|_{1,T} \lesssim |u|_{1,\omega_T}$$



Ω_h^Γ



The best possible convergence

Even supposing some extra regularity for f one gets

$$|u - u_h|_{1,\mathcal{O}} \lesssim h^{1/2}\|f\|_{1,\mathcal{O}}$$

Indeed,

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$$\|\lambda - I_h^\Gamma \lambda\|_{-1/2,\Gamma} \lesssim h\|\lambda\|_{1/2,\Gamma} \lesssim h\|f\|_{0,\Omega}$$

- On triangles $T \subset \Omega_h^\Gamma$ one cannot expect for more than

$$|u - I_h u|_{1,T} \lesssim |u|_{1,\omega_T}$$

- Summing over triangles gives

$$|u - I_h u|_{1,\mathcal{O})} \lesssim h|u|_{2,\mathcal{O}\setminus\Omega_h^\Gamma} + |u|_{1,\Omega_h^\Gamma}$$
$$\lesssim h\|f\|_{L^2(\mathcal{O})} + \sqrt{|\Omega_h^\Gamma|}\|\nabla u\|_{L^\infty(\mathcal{O})} \lesssim h\|f\|_{0,\mathcal{O}} + \sqrt{h}\|f\|_{1,\mathcal{O}}$$

Find $u_h \in V_h$, $\lambda_h \in W_h$ s.t.

$$\int_\Omega \nabla u_h \cdot \nabla v_h + \int_\Gamma \lambda_h v_h = \int_\Omega f v_h \quad \forall v_h \in V_h$$

$$\int_\Gamma \mu_h u_h - \sigma h \sum_{E \in \mathcal{E}_h^\Gamma} \int_\Gamma [\lambda_h][\mu_h] = \int_\Gamma g \mu_h \quad \forall \mu_h \in W_h$$

Here

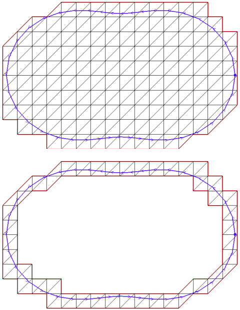
$V_h = \{$cont. piecewise linear functions on $\mathcal{T}_h\}$

$\mathcal{T}_h =$ the original mesh without elements outside Ω

$W_h = \{$piecewise constants on $\mathcal{T}_h^\Gamma\}$

$\mathcal{T}_h^\Gamma =$ restriction of $\mathcal{T}_h$ on Ω_h^Γ

$\mathcal{E}_h^\Gamma = \{$ the edges cut by $\Gamma\}$



One no longer needs λ_h:
Nitsche method + Ghost stabilization

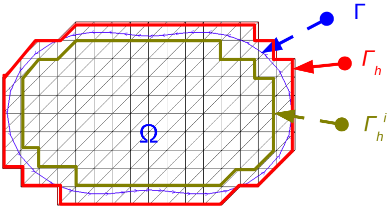
$$\int_\Omega \nabla u_h \cdot \nabla v_h - \int_\Gamma \frac{\partial u_h}{\partial n} v_h \pm \int_\Gamma u_h \frac{\partial v_h}{\partial n} + \frac{\gamma}{h}\int_\Gamma u_h v_h$$
$$+ \sigma h \sum_{E \in \mathcal{E}_h^{\Gamma,i}} \int_E \left[\frac{\partial u_h}{\partial n}\right]\left[\frac{\partial v_h}{\partial n}\right] \qquad (\mathcal{E}_h^{\Gamma,i} = \mathcal{E}_h^\Gamma \cup \Gamma_h^i)$$
$$= \int_{\Omega_h} f v_h + \int_\Gamma g \frac{\partial v_h}{\partial n} + \frac{\gamma}{h}\int_\Gamma g v_h \qquad \forall v_h \in V_h$$

Stabilization for FE of order $k \geq 1$

$$\sigma \sum_{E \in \mathcal{E}_h^{\Gamma,i}} \sum_{j=1}^{k} h^{2j-1} \int_E \left[\frac{\partial^j u_h}{\partial n^j}\right]\left[\frac{\partial^j v_h}{\partial n^j}\right]$$

Notations and results

- $$\begin{aligned} \mathcal{E}_h^{\Gamma,i} &= \mathcal{E}_h^{\Gamma} \cup \{\text{edges on } \Gamma_h^i\} \\ \Gamma_h^i &= \text{the inner boundary of } \Omega_h^{\Gamma} \end{aligned}$$

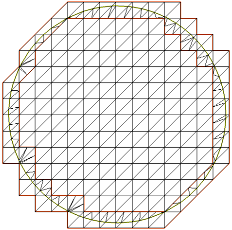


- One gets the optimal convergence

$$|u - u_h|_{1,\Omega} \lesssim h^k |u|_{k+1,\Omega}$$

  - $k = 1$ for the variant with λ_h
  - $k \geq 1$ for the variant à la Nitsche

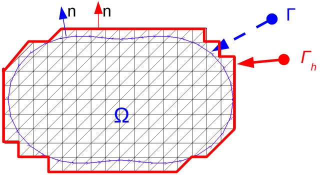
Fictitious Domain vs. CutFEM

	FD	CutFEM
Pro	Ease of implementation: • The standard quadrature for u_h is performed on the whole triangles • Some interpolation in the surface integrals may be needed, but it can be alleviated in the stabilized version (same as CutFEM λ_h)	Optimal convergence
Contra	Slow convergence $\sqrt{h}$	Non standard quadrature on cut triangles 

A "no cut" method for Poisson-Dirichlet

- Assume that our problem can be solved up to Γ_h

$$-\Delta u = f \text{ in } \Omega_h$$
$$u = g \text{ on } \Gamma$$



- Integrate by parts over Ω_h

$$\int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v = \int_{\Omega_h} fv, \quad \forall v \in H^1(\Omega_h)$$

- Weakly impose the b.c. on Γ

$$\int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v + \int_{\Gamma} u \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} uv = \int_{\Omega_h} fv + \int_{\Gamma} g \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} gv$$

- Add a ghost stabilization

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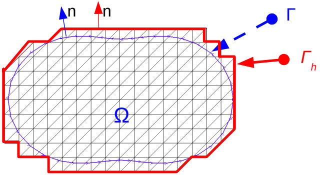

$$-\Delta u = f \text{ in } \Omega_h$$
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$$= \int_{\Omega_h} fv + \int_{\Gamma} g \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} gv$$

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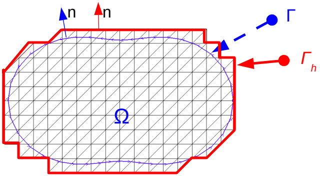

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$$= \int_{\Omega_h} fv + \int_{\Gamma} g \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} gv$$

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A "no cut" method for Poisson-Dirichlet

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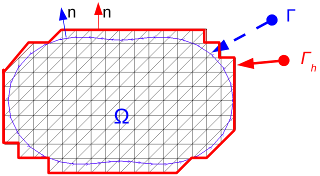


$$-\Delta u = f \text{ in } \Omega_h$$
$$u = g \text{ on } \Gamma$$

- Integrate by parts over Ω_h

$$\int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v = \int_{\Omega_h} fv, \quad \forall v \in H^1(\Omega_h)$$

- Weakly impose the b.c. on Γ

$$\int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v + \int_{\Gamma} u \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} uv = \int_{\Omega_h} fv + \int_{\Gamma} g \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} gv$$

- Add a ghost stabilization

- The FE space

$$V_h = \{v_h \in H^1(\Omega_h) : v_h|_T \in \mathbb{P}_1(T)\, \forall T \in \mathcal{T}_h\}$$

- The bilinear form

$$a_h(u,v) = \int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v + \int_\Gamma u \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_\Gamma uv + \sigma h \sum_{E \in \mathcal{E}_h^{\Gamma,i}} \int_E \left[\frac{\partial u}{\partial n}\right] \left[\frac{\partial v}{\partial n}\right]$$

with $\gamma > 0$ (arbitrary) and $\sigma > 0$ sufficiently big
(both $\gamma > 0$ and $\sigma > 0$ independent of h)

- Search for $u_h \in V_h$ s.t.

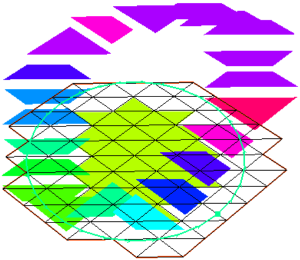
$$a_h(u_h, v_h) = \int_{\Omega_h} f v_h + \int_\Gamma g \frac{\partial v_h}{\partial n} + \frac{\gamma}{h} \int_\Gamma g v_h \quad \forall v_h \in V_h$$

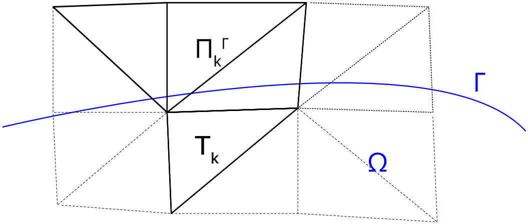
An important lemma for the coerciveness

Recall $\Omega_h^\Gamma = \{T \in \mathcal{T}_h : T \cap \Gamma \neq \varnothing\}$.
$\forall \beta > 0 \quad \exists 0 < \alpha < 1$ s.t. for all $v_h \in V_h$

$$|v_h|^2_{1,\Omega_h^\Gamma} \leqslant \alpha |v_h|^2_{1,\Omega_h} + \beta h \sum_{E \in \mathcal{E}_h^{\Gamma,i}} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E}$$

Proof. One can cover Γ by disjoint connected element patches $\{\Pi_k\}_{k=1,\dots,N_\Pi}$. Each patch consists of a feeding triangle $T_k \subset \Omega$ and the remaining part Π_k^Γ cut by Γ





A patch Π_k

The feeding triangle T_k

The number of triangles in Π_k^Γ is assumed $\leq M$

- Pick a $\beta > 0$ and set

$$\alpha := \max_{\Pi_k, v_h \neq 0} \frac{|v_h|^2_{1,\Pi_k^\Gamma} - \beta h \sum_{E \in \mathcal{E}_h^{\Gamma,i} \cap \Pi_k} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E}}{|v_h|^2_{1,\Pi_k}}$$

with the maximum over all the possible configurations of Π_k and over all the piecewise linear functions on Π_k. The maximum $\alpha \leq 1$ is indeed attained thanks to homogeneity wrt h and v_h.

- Supposing $\alpha = 1$ leads to a contradiction. One would have then for some v_h with $|v_h|_{1,\Pi_k} = 1$

$$|v_h|^2_{1,T_k} + \beta \sum_{E \in \mathcal{E}_h^{\Gamma,i} \cap \Pi_k} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E} = 0$$

  This implies

  - $\nabla v_h = 0$ on T_k
  - $[\nabla v_h] = 0$ on all edges E inside Π_k

  Thus $\nabla v_h = 0$ on Π_k => contradiction.

- We conclude $\exists \alpha < 1$ s.t.

$$|v_h|^2_{1,\Pi_k^\Gamma} \leq \alpha |v_h|^2_{1,\Pi_k} + \beta h \sum_{E \in \mathcal{E}_h^{\Gamma,i} \cap \Pi_k} \left\| \left[\frac{\partial v_h}{\partial n} \right] \right\|^2_{0,E}$$

  Summing this over Π_k yields the announced result

Coerciveness of a_h

Lemma

Provided $\sigma > 0$ is sufficiently big, there exists an h-independent constant $c > 0$ such that $\forall v_h \in V_h$

$$a(v_h, v_h) \geq c|||v_h|||_h^2 \qquad \text{with} \qquad |||v|||_h^2 = |v|_{1,\Omega_h}^2 + \frac{1}{h}\|v\|_{0,\Gamma}^2$$

Proof: Recall, by the definition,

$$a_h(v_h, v_h) = \int_{\Omega_h} |\nabla v_h|^2 - \int_{\Gamma_h} \frac{\partial v_h}{\partial n} v_h + \int_{\Gamma} v_h \frac{\partial v_h}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} v_h^2 + \sigma h \sum_{E \in \mathcal{E}_h^{\Gamma,i}} \int_E \left[\frac{\partial v_h}{\partial n}\right]^2$$

Let B_h be the strip between Γ and Γ_h. By integration by parts

$$-\int_{\Gamma_h} \frac{\partial v_h}{\partial n} v_h + \int_{\Gamma} v_h \frac{\partial v_h}{\partial n} = -\int_{B_h} |\nabla v_h|^2 - \sum_{F \in \mathcal{E}_h^{\Gamma}} \int_{F \cap B_h} v_h \left[\frac{\partial v_h}{\partial n}\right]$$

An H^1 error estimate

Theorem

Suppose $f \in L^2(\Omega_h)$, $g \in H^{3/2}(\Gamma)$, then

$$|u - u_h|_{1,\Omega} + \frac{1}{\sqrt{h}}\|u - u_h\|_{0,\Gamma} \lesssim h\left(\|f\|_{0,\Omega_h} + \|g\|_{3/2,\Gamma}\right)$$

Observe $u \in H^2(\Omega)$
Let $\tilde{u} \in H^2(\Omega_h)$ be extension of u and $\tilde{f} := -\Delta\tilde{u}$

$$\begin{aligned} a_h(u_h, v_h) &= (f, v_h)_{L^2(\Omega_h)} + \int_\Gamma g\frac{\partial v_h}{\partial n} + \frac{\gamma}{h}\int_\Gamma g v_h \quad \forall v_h \in V_h \\ a_h(\tilde{u}, v_h) &= (\tilde{f}, v_h)_{L^2(\Omega_h)} + \int_\Gamma g\frac{\partial v_h}{\partial n} + \frac{\gamma}{h}\int_\Gamma g v_h \quad \forall v_h \in V_h \end{aligned}$$

Galerkin orthogonality

$$a_h(u_h - \tilde{u}, v_h) = (f - \tilde{f}, v_h)_{0,\Omega_h}, \quad \forall v_h \in V_h$$

Proof continued

Galerkin orthogonality and coercivity of a_h lead to

$$|||u_h - I_h\tilde{u}|||_h \lesssim \sup_{v_h \in V_h} \frac{a_h(u_h - I_h\tilde{u}, v_h)}{|||v_h|||_h}$$

$$= \sup_{v_h \in V_h} \frac{a_h(\tilde{u} - I_h\tilde{u}, v_h) + (f - \tilde{f}, v_h)_{L^2(\Omega_h)}}{|||v_h|||_h}$$

$$\lesssim |e_u|_{1,\Omega_h} + \sqrt{h}\left\|\frac{\partial e_u}{\partial n}\right\|_{0,\Gamma_h} + \frac{1}{\sqrt{h}}\|e_u\|_{0,\Gamma} + \|e_u\|_{0,\Gamma}$$

$$+ \left(h \sum_{E \in \mathcal{E}_h^{\Gamma,i}} \left\|\left[\frac{\partial e_u}{\partial n}\right]\right\|_{0,E}^2\right)^{\frac{1}{2}} + \underbrace{\|f - \tilde{f}\|_{0,\Omega_h^\Gamma} \sup_{v_h \in V_h} \frac{\|v_h\|_{L^2(\Omega_h^\Gamma)}}{|||v_h|||_h}}_{\textit{note that } f = \tilde{f} \textit{ on } \Omega \setminus \Omega_h^\Gamma}$$

with $e_u = \tilde{u} - I_h\tilde{u}$

- All the terms involving e_u are bounded by $h|u|_{2,\Omega_h} \lesssim h|u|_{2,\Omega} \lesssim h\|f\|_{0,\Omega}$

- It remains to bound the term with $f-\tilde{f}$. We have

$$\|f-\tilde{f}\|_{0,\Omega_h^\Gamma} \le \|f\|_{0,\Omega_h} + \|\Delta\tilde{u}\|_{0,\Omega_h} \le C\|f\|_{0,\Omega_h}$$

and by a Poincaré-like inequality in the strip Ω_h^Γ of width $\sim h$

$$\|v_h\|_{0,\Omega_h^\Gamma} \le C\left(\sqrt{h}\|v_h\|_{0,\Gamma} + h|v_h|_{1,\Omega_h^\Gamma}\right) \le Ch|||v_h|||_h$$

An L^2 error estimate (proof by Aubin-Nitsche)

Under the same assumptions as above

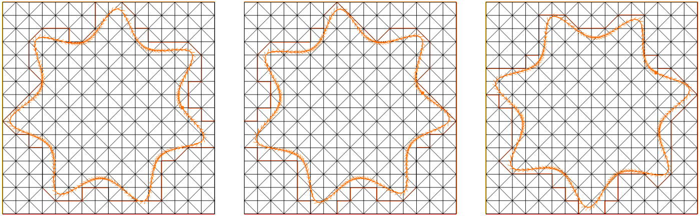
$$\|u-u_h\|_{0,\Omega} \lesssim h^{3/2}\left(\|f\|_{0,\Omega_h} + \|g\|_{3/2,\Gamma}\right)$$

Non-optimality of the L^2 estimate

In fact, the numerical experiments reveal the optimal convergence rate $O(h^2)$, similar to the state of art in the study of the non-symmetric Nitsche method.

Numerical results

Let Ω be a "seven pointed star" embedded in the square $(-0.5, 0.5)^2$.
We rotate this star around $(0, 0)$ by an angle θ_0



$\theta_0 = 0$ $\theta_0 = 0.2$ $\theta_0 = 0.5$

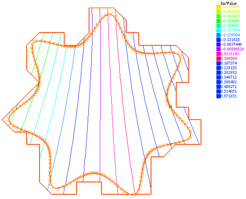
- We solve

$$-\Delta u = 0 \text{ in } \Omega, \quad u = g \text{ on } \Gamma$$

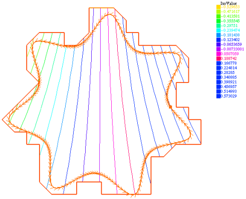
  with a fabricated solution $u = \sin(x)e^y$
- When the domain is rotated, we also rotate the solution

Results on a fixed domain ($\theta_0 = 0$)

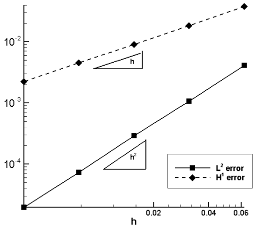
Approximation on the 16×16 mesh



Exact solution



Under the mesh refinement



Implementation in FreeFEM++

- int1d(Th, levelset=phi)(...)

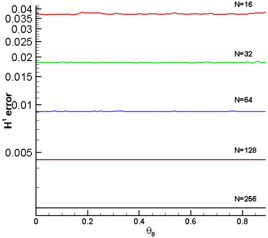
$$\int_{\phi=0} \cdots$$

- int2d(Th, levelset=phi)(...)

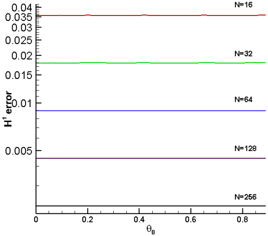
$$\int_{\phi<0} \cdots$$

Error vs. mesh refinement and domain placement (H^1)

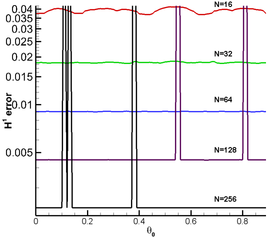
(No-Cut method $\gamma = 1, \ \sigma = 0.01$)



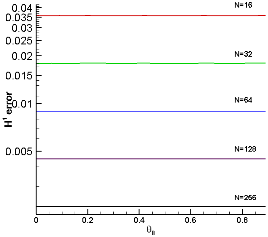
(CutFEM–Nitsche sym. $\gamma = 5, \ \sigma = 0.1$)



(CutFEM with λ_h $\sigma = 0.01$)

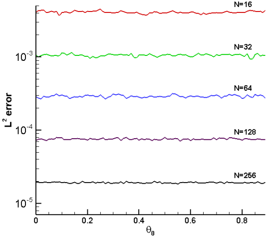


(CutFEM–Nitsche asym. $\gamma = 1, \ \sigma = 0.01$)

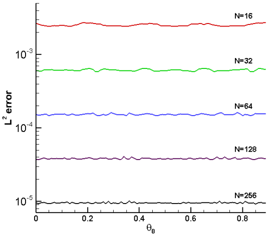


Error vs. mesh refinement and domain placement (L^2)

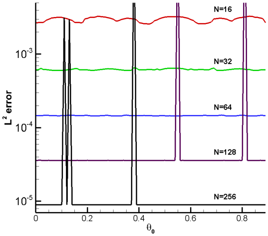
(No-Cut method $\gamma = 1$, $\sigma = 0.01$)



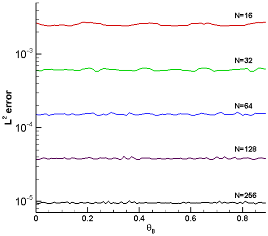
(CutFEM–Nitsche sym. $\gamma = 5$, $\sigma = 0.1$)



(CutFEM with λ_h $\sigma = 0.01$)

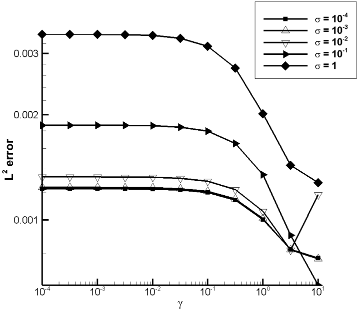


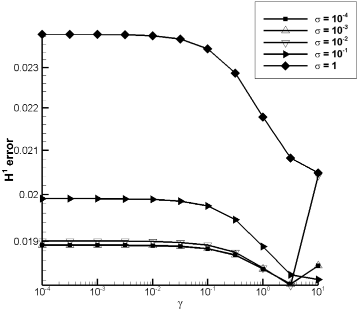
(CutFEM–Nitsche asym. $\gamma = 1$, $\sigma = 0.01$)



Dependence on stabilization parameters

The same problem on the 32×32 mesh.
Non rotated Ω ($\theta_0 = 0$)





The relative errors in $L^2(\Omega)$ and $H^1(\Omega)$ norms as functions of parameters

- γ for the Nitsche stabilization (on horizontal axis)
- σ for the ghost penalty (separate curves)

What about Neumann boundary conditions?

$$-\Delta u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = g \text{ on } \Gamma$$

First idea

Formally

$$-\Delta u = f \text{ in } \Omega_h, \qquad \frac{\partial u}{\partial n} = g \text{ on } \Gamma.$$

Integration by parts over Ω_h and weak b.c. on Γ

$$\int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v + \int_{\Gamma} \frac{\partial u}{\partial n} v = \int_{\Omega_h} fv + \int_{\Gamma} gv$$

Add ghost penalty ...

What about Neumann boundary conditions?

$$-\Delta u = f \text{ in } \Omega, \qquad \frac{\partial u}{\partial n} = g \text{ on } \Gamma$$

First idea

$$\int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v + \int_{\Gamma} \frac{\partial u}{\partial n} v = \int_{\Omega_h} fv + \int_{\Gamma} gv$$

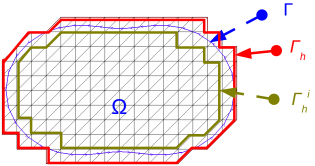
This does not work

- It seems impossible to establish the coercivity without controlling $\|u\|_{0,\Gamma}$.
- ∇u_h is piecewise constant if u_h is a $\mathbb{P}_1$ FE: impossible to impose $\nabla u_h \cdot n \approx g$ on an arbitrary curve Γ

A "no cut" method for Poisson-Neumann

- Assume that our problem can be solved up to Γ_h

$$-\Delta u = f \text{ in } \Omega_h, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma$$



Moreover introduce $y = -\nabla u$ on Ω_h^Γ (between Γ_h and Γ_h^i)

- Integrate by parts over Ω_h and impose $-y \cdot n = g$ on Γ

$$\int_{\Omega_h} \nabla u \cdot \nabla v + \int_{\Gamma_h} y \cdot nv - \int_\Gamma y \cdot nv = \int_{\Omega_h} fv + \int_\Gamma gv, \ \forall v \in H^1(\Omega_h)$$

- Impose $y = -\nabla u$ and $\operatorname{div} y = f$ on Ω_h^Γ

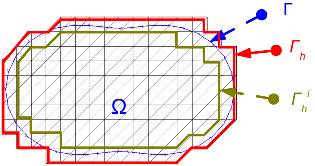
$$\int_{\Omega_h} \nabla u \cdot \nabla v + \int_{\Gamma_h} y \cdot nv - \int_\Gamma y \cdot nv + \gamma \int_{\Omega_h^\Gamma} \operatorname{div} y \operatorname{div} z$$

$$+\sigma \int_{\Omega_h^\Gamma} (y + \nabla u) \cdot (z + \nabla v)$$

$$= \int_{\Omega_h} fv + \int_\Gamma gv + \gamma \int_{\Omega_h^\Gamma} f \operatorname{div} z, \qquad \forall v \in H^1(\Omega_h), z \in H^1(\Omega_h^\Gamma)$$

- Assume that our problem can be solved up to Γ_h

$$-\Delta u = f \text{ in } \Omega_h, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma$$



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- Impose $y = -\nabla u$ and $\operatorname{div} y = f$ on Ω_h^Γ

$$\int_{\Omega_h} \nabla u \cdot \nabla v + \int_{\Gamma_h} y \cdot nv - \int_{\Gamma} y \cdot nv + \gamma \int_{\Omega_h^\Gamma} \operatorname{div} y \operatorname{div} z$$

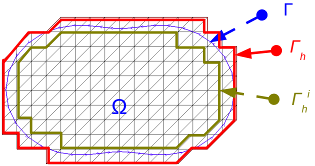
$$+\sigma \int_{\Omega_h^\Gamma} (y + \nabla u) \cdot (z + \nabla v)$$

$$= \int_{\Omega_h} fv + \int_{\Gamma} gv + \gamma \int_{\Omega_h^\Gamma} f \operatorname{div} z, \qquad \forall v \in H^1(\Omega_h), z \in H^1(\Omega_h^\Gamma)$$

A "no cut" method for Poisson-Neumann

- Assume that our problem can be solved up to Γ_h

$$-\Delta u = f \text{ in } \Omega_h, \quad \frac{\partial u}{\partial n} = g \text{ on } \Gamma$$



Moreover introduce $y = -\nabla u$ on Ω_h^Γ (between Γ_h and Γ_h^i)

- Integrate by parts over Ω_h and impose $-y \cdot n = g$ on Γ

$$\int_{\Omega_h} \nabla u \cdot \nabla v + \int_{\Gamma_h} y \cdot nv - \int_\Gamma y \cdot nv = \int_{\Omega_h} fv + \int_\Gamma gv, \ \forall v \in H^1(\Omega_h)$$

- Impose $y = -\nabla u$ and $\operatorname{div} y = f$ on Ω_h^Γ

$$\int_{\Omega_h} \nabla u \cdot \nabla v + \int_{\Gamma_h} y \cdot nv - \int_\Gamma y \cdot nv + \gamma \int_{\Omega_h^\Gamma} \operatorname{div} y \operatorname{div} z$$

$$+\sigma \int_{\Omega_h^\Gamma} (y + \nabla u) \cdot (z + \nabla v)$$

$$= \int_{\Omega_h} fv + \int_\Gamma gv + \gamma \int_{\Omega_h^\Gamma} f \operatorname{div} z, \qquad \forall v \in H^1(\Omega_h), z \in H^1(\Omega_h^\Gamma)$$

- The FE space

$$V_h = \{v_h \in H^1(\Omega_h) : v_h|_T \in \mathbb{P}_1(T)\,\forall T \in \mathcal{T}_h,\ \int_{\Omega_h} v_h = 0\}$$
$$Z_h = \{z_h \in H^1(\Omega_h^\Gamma)^2 : z_h|_T \in \mathbb{P}_1(T)^2\,\forall T \in \mathcal{T}_h^\Gamma\}$$

- The bilinear form

$$a_h^N(u,y;v,z) = \int_{\Omega_h} \nabla u\cdot\nabla v + \int_{\Gamma_h} y\cdot nv - \int_\Gamma y\cdot nv + \gamma\int_{\Omega_h^\Gamma} \operatorname{div} y \operatorname{div} z + \sigma\int_{\Omega_h^\Gamma}(y+\nabla u)\cdot(z+\nabla v) + \sigma h\int_{\Gamma_h^i}[\nabla u]\cdot[\nabla v]$$

$\Leftarrow$ ghost penalty

- Search for $u_h \in V_h$, $y_h \in Z_h$ such that $\forall (v_h, z_h) \in V_h \times Z_h$

$$a_h^N(u_h, y_h; v_h, z_h) = \int_{\Omega_h} f v_h + \int_\Gamma g v_h + \gamma\int_{\Omega_h^\Gamma} f \operatorname{div} z_h$$

Analysis of the coerciveness

- The bilinear form can be rewritten by the divergence Theorem

$$a_h^N(u,y;v,z) = \int_{\Omega_h} \nabla u \cdot \nabla v + \int_{B_h} (v \operatorname{div} y + y \cdot \nabla v) + \gamma \int_{\Omega_h^\Gamma} \operatorname{div} y \operatorname{div} z$$
$$+ \sigma \int_{\Omega_h^\Gamma} (y + \nabla u) \cdot (z + \nabla v) + \sigma h \int_{\Gamma_h^i} [\nabla u] \cdot [\nabla v]$$

- The term $\int_{B_h} v \operatorname{div} y$ can be controlled thanks to the "div-div"' stabilization. The following lemma allows us to control $\int_{B_h} y \cdot \nabla v$

Lemma

There exist $0 < \alpha < 1$ and $\beta > 0$ depending only on the mesh regularity such that $\forall v_h \in V_h, z_h \in Z_h$

$$-\int_{B_h} z_h \cdot \nabla v_h \leqslant \alpha |v_h|^2_{1,\Omega_h} + \beta \|z_h + \nabla v_h\|^2_{0,\Omega_h^\Gamma} + \beta h \|[\nabla v_h]\|^2_{0,\Gamma_h^i}$$

Coercivity and error estimates

Coercivity of the bilinear form

Provided γ, σ are sufficiently big, there exists an h-independent constant $c > 0$ such that $\forall v_h \in V_h, z_h \in Z_h$

$$a_h^N(v_h, z_h; v_h, z_h) \geq c|||v_h, z_h|||_h^2$$

with
$|||v, z|||_h^2 = |v|_{1,\Omega_h}^2 + \|\operatorname{div} z\|_{0,\Omega_h^\Gamma}^2 + \|z + \nabla v\|_{0,\Omega_h^\Gamma}^2 + h\|[\nabla v]\|_{0,\Gamma_h^i}$

Error estimates in H^1 and L^2

Suppose $f \in H^1(\Omega_h)$, $g \in H^{3/2}(\Gamma)$. Provided γ, σ are sufficiently big,

$$|u - u_h|_{1,\Omega} \lesssim h(\|f\|_{1,\Omega_h} + \|g\|_{3/2,\Gamma})$$
$$\|u - u_h\|_{0,\Omega} \lesssim h^{3/2}(\|f\|_{1,\Omega_h} + \|g\|_{3/2,\Gamma})$$

A sketch of the proof of the H^1 estimate

- Let $\tilde{u} \in H^3(\Omega_h)$ be an extension of u from Ω

$$\|\tilde{u}\|_{3,\Omega_h} \le C\|u\|_{3,\Omega} \le C(\|f\|_{1,\Omega} + \|g\|_{3/2,\Gamma})$$

  and $y = -\nabla \tilde{u}$ on Ω_h^Γ.
- By Galerkin orthogonality and coercivity

$$\frac{1}{c}|||u_h - I_h\tilde{u}, y_h - I_h y|||_h \le \sup_{(v_h,z_h)\in V_h\times Z_h} \frac{a_h^R(u_h - I_h\tilde{u}, y_h - I_h y; v_h, z_h)}{|||v_h, z_h|||_h}$$
$$= \sup_{(v_h,z_h)\in V_h\times Z_h} \frac{a_h^R(e_u, e_y; v_h, z_h) + (f - \tilde{f}, v_h)_{L^2(\Omega_h)} + \gamma(f - \tilde{f}, \operatorname{div} z_h)_{L^2(\Omega_h^\Gamma)}}{|||v_h, z_h|||_h}$$

  with $e_u = \tilde{u} - I_h\tilde{u}$, $e_y = y - I_h y$.
- The interpolation estimate for y:

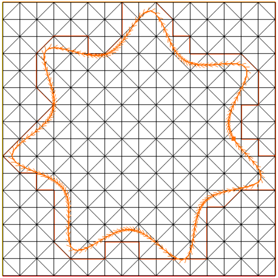
$$|e_y|_{1,\Omega_h^\Gamma} \le Ch|\tilde{y}|_{1,\Omega_h^\Gamma} \le Ch|u|_{3,\Omega}$$

- We no longer have $\|v_h\|_{0,\Omega_h^\Gamma} \le Ch|||v_h, z_h|||_h$. We rely instead on

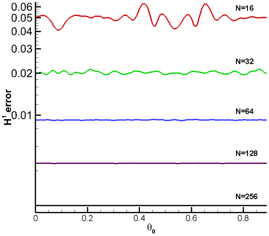
$$\|f - \tilde{f}\|_{0,\Omega_h} = \|f - \tilde{f}\|_{0,\Omega_h\setminus\Omega} \le Ch|f - \tilde{f}|_{1,\Omega_h} \le Ch(|f|_{1,\Omega_h} + \|\tilde{u}\|_{3,\Omega_h})$$

Numerical results for the Neumann problem

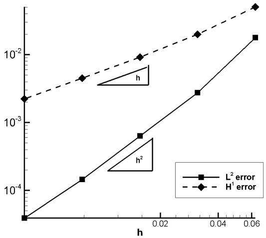
The domain

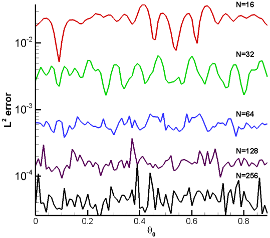


Rotating Ω



Mesh refinement





Extension to P_k FE: Poisson-Dirichlet

- The FE space

$$V_h = \{v_h \in H^1(\Omega_h) : v_h|_T \in \mathbb{P}_k(T)\, \forall T \in \mathcal{T}_h\}$$

- The bilinear form

$$a_h(u,v) = \int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v + \int_\Gamma u \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_\Gamma uv$$
$$+ \sigma h^2 \sum_{T \subset \Omega_h^\Gamma} \int_T (\Delta u)(\Delta v) + \sigma h \sum_{E \in \mathcal{E}_h^{\Gamma,i}} \int_E \left[\frac{\partial u}{\partial n}\right] \left[\frac{\partial v}{\partial n}\right]$$

- Search for $u_h \in V_h$ s.t. for all $v_h \in V_h$

$$a_h(u_h, v_h) = \int_{\Omega_h} f v_h + \int_\Gamma g \frac{\partial v_h}{\partial n} + \frac{\gamma}{h} \int_\Gamma g v_h - \sigma h^2 \sum_{T \subset \Omega_h^\Gamma} \int_T f \Delta v$$

Extension to P_k FE: Poisson-Dirichlet

- The FE space

$$V_h = \{v_h \in H^1(\Omega_h) : v_h|_T \in \mathbb{P}_k(T) \, \forall T \in \mathcal{T}_h\}$$

- The bilinear form

$$a_h(u,v) = \int_{\Omega_h} \nabla u \cdot \nabla v - \int_{\Gamma_h} \frac{\partial u}{\partial n} v + \int_{\Gamma} u \frac{\partial v}{\partial n} + \frac{\gamma}{h} \int_{\Gamma} uv$$
$$+ \sigma h^2 \sum_{T \subset \Omega_h^\Gamma} \int_T (\Delta u)(\Delta v) + \sigma h \sum_{E \in \mathcal{E}_h^{\Gamma,i}} \int_E \left[\frac{\partial u}{\partial n}\right] \left[\frac{\partial v}{\partial n}\right]$$

- To prove the coerciveness, integrate by parts on B_h

$$-\int_{\Gamma_h} \frac{\partial v_h}{\partial n} v_h + \int_{\Gamma} v_h \frac{\partial v_h}{\partial n} = -\int_{B_h} |\nabla v_h|^2 - \sum_{T \subset \Omega_h^\Gamma} \int_{T \cap B_h} v_h \Delta v_h - \sum_{E \in \mathcal{E}_h^\Gamma} \int_{E \cap B_h} \cdots$$

$$a(v_h, v_h) \geqslant (1-\alpha)|v_h|^2_{1,\Omega_h} - h \, ||| \, v_h \, |||_h \, \|\Delta v_h\|_{0,\Omega_h^\Gamma} + \sigma h^2 \|\Delta v_h\|^2_{0,\Omega_h^\Gamma}$$
$$+ \cdots \geqslant c \, ||| \, v_h |||^2_h$$

Other possible extensions

- P_k FE for Poisson-Neumann
- Robin boundary conditions
- General eliptic equations

$$-\operatorname{div}(a(x)\nabla u) + b \cdot \nabla u + cu = f$$

  OK provided
  - a is not strongly oscillating on length h
  - usual positivity assumptions on a, b, c
- Stokes equations: should be OK since we have velocity-pressure inf-sup (for example $P_2 - P_1$ FE pair) for free.
- Back to simple model equation $-\Delta u = f$:
  Computing numerically the integrals over Γ is still challenging even for a "no-cut" method
  - The piecewise-affine approximation of Γ is yet to be investigated theoretically
  - "Boundary value correction" = a Taylor expansion, could be introduced for higher order elements, cf. Burman, Hansbo, Larson (2017)

- We want to solve

$$-\Delta u = f \text{ in } \Omega, \qquad u = 0 \text{ on } \quad \Gamma$$

with Ω given by the levelset

$$\Omega = \{\phi < 0\}$$

- Introduce the mesh $\mathcal{T}_h$ as above (interior and cut triangles), the FE space

$$V_h = \{v_h \in H^1(\Omega_h) : v_h|_T \in \mathbb{P}_k(T)\ \forall T \in \mathcal{T}_h\}, \qquad \Omega_h \supset \Omega$$

and approximate $\phi \approx \phi_h \in V_h$, $u \approx \phi_h w_h$ with $w_h \in V_h$

$$\int_{\Omega_h} \nabla(\phi_h w_h) \cdot \nabla(\phi_h v_h) - \int_{\Gamma_h} \frac{\partial}{\partial n}(\phi_h w_h)\phi_h v_h + G_h(w_h, v_h) = \int_{\Omega_h} f\phi_h v$$

where G_h stands for the the ghost penalty

$$G_h(w_h, v_h) = \sigma h \sum_{E \in \mathcal{F}_\Gamma} \int \left[\frac{\partial}{\partial n}(\phi_h w_h)\right] \left[\frac{\partial}{\partial n}(\phi_h v_h)\right] + \sigma h^2 \sum_{T \subset \mathcal{T}_h^\Gamma} \int_T \Delta(\phi_h w_h)\Delta(\phi_h v_h)$$

Theorem

Provided u and ϕ smooth enough and σ big enough

$$|u - u_h|_{1,\Omega\cap\Omega_h} \leq Ch^k \|f\|_{k,\Omega\cup\Omega_h}$$

$$\|u - u_h\|_{0,\Omega} \leq Ch^{k+1/2} \|f\|_{k,\Omega_h}$$

Main ingredients for the proof

- A Hardy inequality: for any $u \in H^{k+1}(\mathcal{O})$ vanishing on Γ,

$$\|u/\phi\|_{k,\mathcal{O}} \leq C\|u\|_{k+1,\mathcal{O}}$$

- Interpolation: $\tilde{u}$ an extension of u and $w = \tilde{u}/\phi$

$$\begin{aligned} |\phi w - \phi_h I_h w|_{1,\Omega_h} &\leq |(\phi - \phi_h)w|_{1,\Omega_h} + |\phi_h(w - I_h w)|_{1,\Omega_h} \\ &\leq Ch^k \|\phi\|_{W^{k+1,\infty}(\Omega_h)} \|w\|_{k+1,\Omega_h} \\ &\leq Ch^k \|\phi\|_{W^{k+1,\infty}(\Omega_h)} \|\tilde{u}\|_{k+2,\Omega_h} \end{aligned}$$

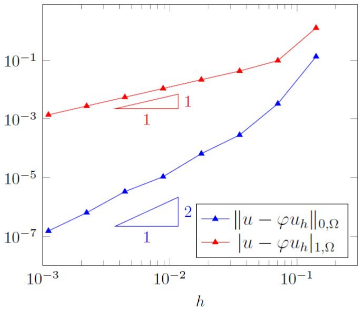
Numerical results for ϕ-FEM with P_1 FE

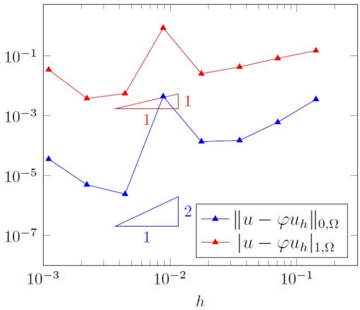
- Let $\Omega := \{\phi < 0\}$ be a circle inside $(0,1)^2$

$$\phi = (x - 1/2)^2 + (y - 1/2)^2 - 1/8$$

- Solve the homogeneous Poisson-Dirichlet problem on Ω with

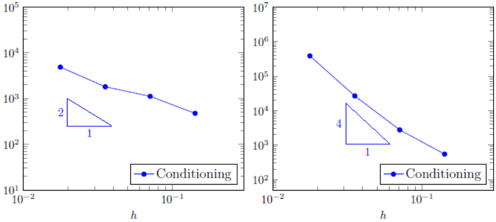
$$u = \phi \times \exp(x) \times \sin(2\pi y)$$



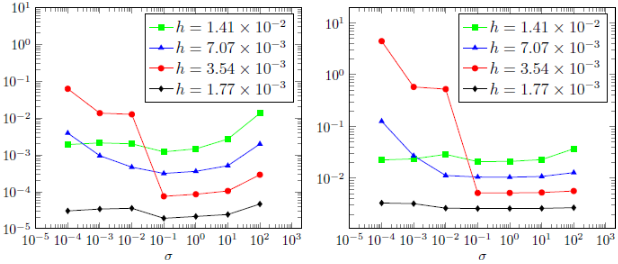


Errors with ghost penalty $\sigma = 20$ (left) and without ghost penalty $\sigma = 0$ (right)

Numerical results for ϕ-FEM with P_1 FE

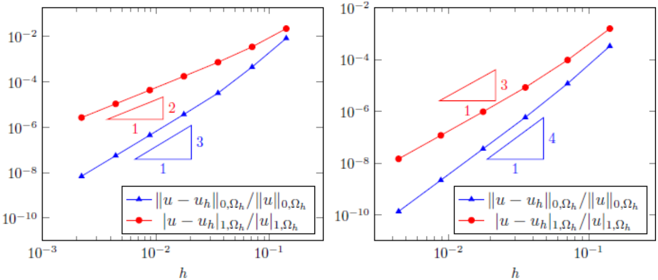


Matrix condition numbers with ghost penalty $\sigma = 20$ (left) and without ghost penalty $\sigma = 0$ (right)



Influence of the ghost penalty parameter σ

Numerical results for ϕ-FEM with P_2 and P_3 FE



Finite elements of degree $k = 2$ (left) and $k = 3$ (right)

Open questions and perspectives

- Stationary Stokes equations
  - A "no-cut" method giving a good approximation for the force?
- Unsteady (Navier-)Stokes equations
  - How to treat the derivative wrt time at a point x that was covered by solid at time t_{n-1} and is covered by fluid at time t_n?
- A better "no-cut" method for Neumann bc? Can one construct one so that to avoid higher regularity assumption?
- A ϕ-FEM for Neumann bc?