Séminaire de Mathématiques Appliquées du CERMICS

École des Ponts
ParisTech

PDMPs with ODE dynamics

Sam Power (Cambridge Centre for Analysis)

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PDMPs with ODE Dynamics

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Overview

1. PDMPs
2. PDMPs for MCMC
3. Construction of Algorithms
4. Remarks, Open Questions, Takeaways

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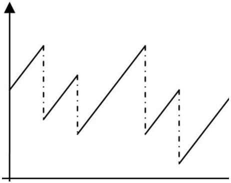

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  - Transition dynamics $Q(z \to dz')$
    - Dictates what happens at events (Markov jump kernel)

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- Typically, jumps fix $x \rightsquigarrow X_t$ has continuous sample paths.
- **Question**:

$$\text{Given target measure } \mu, \text{ vector field } \phi, \tag{1}$$

$$\text{how can I build } (\lambda, Q) \text{ to sample } \mu\,? \tag{2}$$

Aside on Reversibility, Symmetry

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  - To design algorithms, *locality* is the important part
- Symmetry
  - Existing PDMPs are highly symmetric (BPS, ZZ)
  - A priori, not necessary to have symmetry
  - Want to be able to use *all* ODEs!

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  - 'in at z forwards in time = out at z backwards in time'

Choice of Event Rate (1)

- Consider 'probability current'

$$r(z, \tau) \triangleq \underbrace{\langle \nabla H(z), \phi(z, \tau) \rangle}_{\text{Energy Gain}} \quad - \quad \underbrace{\text{div}_z \phi(z, \tau)}_{\text{Compressibility Penalty}} \tag{4}$$

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- We will take $\lambda(z,\tau) = \lambda^0(z,\tau) + \gamma(z)$

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  - $\rightsquigarrow$ Choose $q^{\tau}(z \to dz')$ to be J^{τ}-reversible

Putting together the ingredients

Theorem

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If (ϕ, λ, Q) *is a trajectorially-reversible,* $\tilde{\mu}$*-stationary TA-PDMP, then* $\exists\, \gamma \geqslant 0$ *such that*

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and for $\tau \in \{\pm 1\}$, Q^τ *is* J^τ*-reversible*

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- Comments on proof

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- Key point: Different event types correspond to *decompositions* of r

Split PDMPs (2)

- $z = (z_1, \cdots, z_D)$, $\tau = (\tau_1, \cdots, \tau_D) \in \{\pm 1\}^D$
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$$r(z, \tau) = \sum_{j=1}^{M} r_j(z, \tau) \tag{8}$$

and existence of involutions $\mathcal{F}_j : \{\pm 1\}^D \to \{\pm 1\}^D$ such that

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- Events of type j happen at rate $\lambda_j(z, \tau)$
    - and then jump according to $Q_j^\tau(z \to dz') \cdot \delta(\mathcal{F}_j(\tau), d\tau')$

Making Split-PDMPs work (1)

- Define

$$\lambda_j^0(z,\tau) = (r_j(z,\tau))_+ \tag{10}$$

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Given a fixed splitting, all trajectorially-reversible, $\tilde{\mu}$-stationary Split PDMPs take this form.

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- Choosing ϕ: some room for creativity here.

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- *Curiosity*: Tempering?

Thank you!