Schémas préservant l'asymptotique sur maillages coniques

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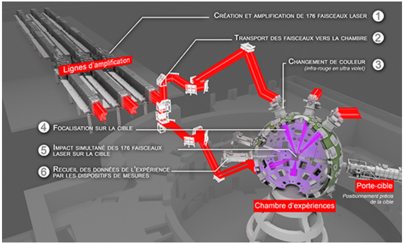
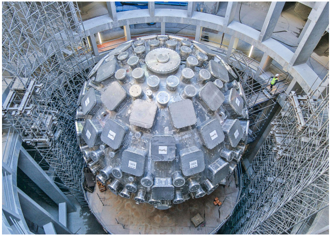
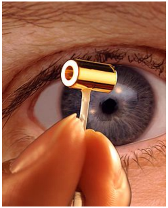
Fusion par confinement inertiel (FCI)







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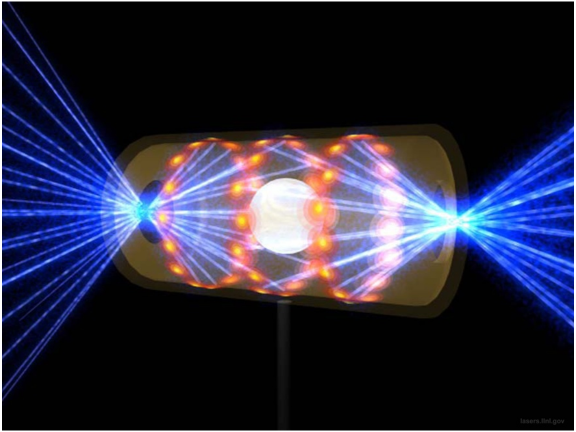
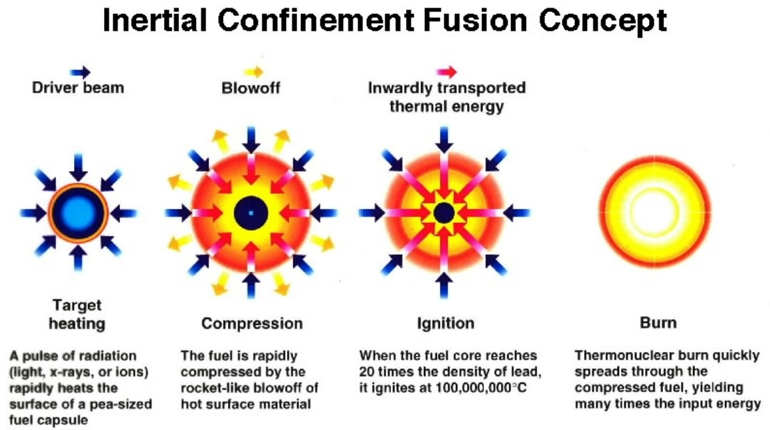


FCI


Inertial Confinement Fusion Concept
Driver beam
Blowoff
Inwardly transported thermal energy
Target heating
Compression
Ignition
Burn
A pulse of radiation (light, x-rays, or ions) rapidly heats the surface of a pea-sized fuel capsule
The fuel is rapidly compressed by the rocket-like blowoff of hot surface material
When the fuel core reaches 20 times the density of lead, it ignites at 100,000,000°C
Thermonuclear burn quickly spreads through the compressed fuel, yielding many times the input energy

Modélisation

$$\begin{cases} \partial_t \rho + \mathrm{div}_x(\rho\vec{v}) = 0, \\ \partial_t(\rho\vec{v}) + \mathrm{div}_x(\rho\vec{v}\otimes\vec{v}) + \nabla_x p = -\vec{\mathscr{S}}_F, \\ \partial_t(\rho E) + \mathrm{div}_x((\rho E + p)\vec{v}) = -\mathscr{S}_E + S, \\ \dfrac{1}{c}\partial_t I + \vec{\Omega}\cdot\nabla_x I = \mathscr{S}, \end{cases}$$

$E = \frac{1}{2}|u|^2 + e,$
E.O.S $p = p(\rho,\theta)$

$I = I(t,x,\Omega)$, $\Omega \in S^2$, intensité radiative.

$$\mathscr{S} = \sigma_a(\rho,\theta)\left(a\theta^4 - I\right) + \sigma_s(\rho,\theta)\left(\fint_{S^2} I d\Omega - I\right), \qquad a = \text{constant}.$$

$$\vec{\mathscr{S}}_F = \int_{S^2} \Omega \mathscr{S} d\Omega, \qquad \mathscr{S}_E = \int_{S^2} \mathscr{S} d\Omega.$$

Modèle aux moments

Intensité radiative $I = I(t,x,\Omega)$ $t \geq 0$, $x \in \mathbb{R}^3$, $\Omega \in S^2$.

$$\frac{1}{c}\partial_t I + \vec{\Omega}\cdot\nabla_x I = \sigma_a\left(a\theta^4 - I\right) + \sigma_s\left(\langle I\rangle - I\right), \quad \langle I\rangle = \fint_{S^2} I d\Omega.$$

Moments de l'équation :

$$\begin{cases} \partial_t E + \mathrm{div}(F) = 4\pi\sigma_a\left(a\theta^4 - E\right), \\ \partial_t F + c^2\,\mathrm{div}(P) = -c(\sigma_s + \sigma_s)F. \end{cases}$$

$$E = \frac{1}{c}\int_{S^2} I(t,x,\Omega)d\Omega, \;\; F = \int_{S^2} \Omega I(t,x,\Omega)d\Omega, \;\; P = \frac{1}{c}\int_{S^2} \Omega\otimes\Omega I(t,x,\Omega)d\Omega.$$

Approximation affine en Ω : $\quad I(t,x,\Omega) = \frac{c}{4\pi}E(t,x) + \frac{1}{4\pi}\Omega\cdot F(t,x).$

$$P(t,x) = \frac{1}{3}E(t,x)\,\mathrm{Id}.$$

$$\begin{cases} \partial_t E + \mathrm{div}(F) = S, \\ \dfrac{1}{c}\partial_t F + \dfrac{c}{3}\nabla E = -\sigma F. \end{cases}$$

$$A_x = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{c^2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_y = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{c^2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_z = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{c^2}{3} & 0 & 0 & 0 \end{pmatrix},$$

Système symétrisable. $\Sigma = \begin{pmatrix} \frac{c}{\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{\sqrt{3}}{c} & 0 & 0 \\ 0 & 0 & \frac{\sqrt{3}}{c} & 0 \\ 0 & 0 & 0 & \frac{\sqrt{3}}{c} \end{pmatrix}.$

$$\begin{cases} \partial_t E + \mathrm{div}(F) = S, \\ \dfrac{1}{c}\partial_t F + \dfrac{c}{3}\nabla E = -\sigma F. \end{cases}$$

Limite diffusion : $c \approx \sigma \to +\infty$: $F = -\frac{c}{3\sigma}\nabla E$.

$$\partial_t E - \mathrm{div}\left(\frac{c}{3\sigma}\nabla E\right) = S.$$

Premier essai : 1D, schéma décentré amont
$$\begin{cases} \partial_t E + \partial_x F = 0, \\ \partial_t F + \dfrac{c^2}{3}\partial_x E = -\sigma c F. \end{cases}$$

Invariants de Riemann : $U = E + \frac{\sqrt{3}}{c}F$, $V = E - \frac{\sqrt{3}}{c}F$,

$$\begin{cases} \partial_t U + \dfrac{c}{\sqrt{3}}\partial_x U = \sigma \dfrac{c}{2}(V - U), \\ \partial_t V - \dfrac{c}{\sqrt{3}}\partial_x V = \sigma \dfrac{c}{2}(U - V). \end{cases}$$

Schéma décentré amont :

$$\begin{cases} \dfrac{U_j^{n+1} - U_j^n}{\Delta t} + \dfrac{c}{\sqrt{3}}\dfrac{U_j^n - U_{j-1}^n}{\Delta x} = \sigma \dfrac{c}{2}\left(V_j^n - U_j^n\right), \\ \dfrac{V_j^{n+1} - V_j^n}{\Delta t} - \dfrac{c}{\sqrt{3}}\dfrac{V_{j+1}^n - V_j^n}{\Delta x} = \sigma \dfrac{c}{2}\left(U_j^n - V_j^n\right). \end{cases}$$

Schéma upwind

$$\begin{cases} \dfrac{U_j^{n+1}-U_j^n}{\Delta t} + \dfrac{c}{\sqrt{3}}\dfrac{U_j^n - U_{j-1}^n}{\Delta x} = \sigma \dfrac{c}{2}\left(V_j^n - U_j^n\right), \\ \dfrac{V_j^{n+1}-V_j^n}{\Delta t} - \dfrac{c}{\sqrt{3}}\dfrac{V_{j+1}^n - V_j^n}{\Delta x} = \sigma \dfrac{c}{2}\left(U_j^n - V_j^n\right). \end{cases}$$

$$\begin{cases} \dfrac{E_j^{n+1}-E_j^n}{\Delta t} + \dfrac{F_{j+1}^n - F_{j-1}^n}{2\Delta x} + \dfrac{c}{2\sqrt{3}\Delta x}\left(2E_j^n - E_{j-1}^n - E_{j+1}^n\right) = 0, \\ \dfrac{F_j^{n+1}-F_j^n}{\Delta t} + \dfrac{c^2}{3}\dfrac{E_{j+1}^n - E_{j-1}^n}{2\Delta x} + \dfrac{c}{2\sqrt{3}\Delta x}\left(2F_j^n - F_{j-1}^n - F_{j+1}^n\right) = -c\sigma F_j^n. \end{cases}$$

Limite diffusion $c \approx \sigma \to +\infty$:

$$F_j^n = -\frac{c}{3\sigma}\frac{E_{j+1}^n - E_{j-1}^n}{2\Delta x}, \quad 2E_j^n - E_{j-1}^n - E_{j+1}^n = 0.$$

Pas consistant avec l'équation de diffusion $\partial_t E - \partial_x\left(\frac{c}{3\sigma}\partial_x E\right) = 0$.

Schémas préservant l'asymptotique

1. S. Jin, D. Levermore 1996.
2. J. Greenberg, A. Y. Leroux 1996.
3. L. Gosse, G. Toscani 2001.
4. C. Berthon, R. Turpault, 2011.
5. C. Buet, B. Després, E. Franck, 2012. ← dimension 2
6. C. Buet, B. Després, 2015.

Schéma de Jin-Levermore

Idée : monter en ordre sur E en utilisant l'asymptotique diffusion :
$\partial_x E = -\frac{3\sigma}{c} F.$

Flux upwind : $U_{j+\frac{1}{2}} = U_j,\ V_{j+\frac{1}{2}} = V_{j+1}.$

$$\begin{cases} E_{j+\frac{1}{2}} + \frac{\sqrt{3}}{c} F_{j+\frac{1}{2}} = E_j + \frac{\sqrt{3}}{c} F_j, \\ E_{j+\frac{1}{2}} - \frac{\sqrt{3}}{c} F_{j+\frac{1}{2}} = E_{j+1} - \frac{\sqrt{3}}{c} F_{j+1}. \end{cases} \rightsquigarrow \begin{cases} E_{j+\frac{1}{2}} + \frac{\Delta x}{2}\frac{3\sigma}{c} F_{j+\frac{1}{2}} + \frac{\sqrt{3}}{c} F_{j+\frac{1}{2}} = E_j + \frac{\sqrt{3}}{c} F_j, \\ E_{j+\frac{1}{2}} - \frac{\Delta x}{2}\frac{3\sigma}{c} F_{j+\frac{1}{2}} - \frac{\sqrt{3}}{c} F_{j+\frac{1}{2}} = E_{j+1} - \frac{\sqrt{3}}{c} F_{j+1}. \end{cases}$$

$$\begin{cases} E_{j+\frac{1}{2}} = \frac{1}{2}\left(E_j + E_{j+1} + \frac{\sqrt{3}}{c} F_j - \frac{\sqrt{3}}{c} F_{j+1} \right), \\ F_{j+\frac{1}{2}} = \frac{1}{1+\sigma\sqrt{3}\frac{\Delta x}{2}} \frac{1}{2}\left(F_j + F_{j+1} + \frac{\sqrt{3}}{c} E_j - \frac{\sqrt{3}}{c} E_{j+1} \right). \end{cases}$$

Schéma de Jin-Levermore

$$M = \frac{1}{1+\sigma\sqrt{3}\frac{\Delta x}{2}}$$

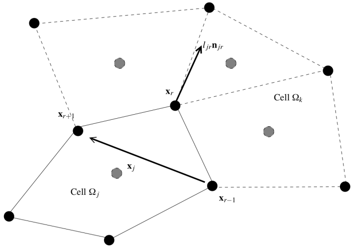
$$\begin{cases} \dfrac{E_j^{n+1}-E_j^n}{\Delta t} + M\dfrac{F_{j+1}^n - F_{j-1}^n}{2\Delta x} + M\dfrac{c}{2\sqrt{3}\Delta x}\left(2E_j^n - E_{j-1}^n - E_{j+1}^n\right) = 0, \\ \dfrac{F_j^{n+1}-F_j^n}{\Delta t} + \dfrac{c^2}{3}\dfrac{E_{j+1}^n - E_{j-1}^n}{2\Delta x} + \dfrac{c}{2\sqrt{3}\Delta x}\left(2F_j^n - F_{j-1}^n - F_{j+1}^n\right) = -c\sigma F_j^n. \end{cases}$$

Limite diffusion $c \approx \sigma \to +\infty$: $M \approx \frac{1}{\sigma\sqrt{3}\frac{\Delta x}{2}}$.

Première équation :

$$\frac{E_j^{n+1}-E_j^n}{\Delta t} + \frac{c}{3\sigma\Delta x^2}\left(2E_j^n - E_{j-1}^n - E_{j+1}^n\right) = 0$$

Dimension deux



$$l_{jr} = \frac{1}{2}|x_{r+1} - x_{r-1}|,$$
$$n_{jr} = \frac{1}{2l_{jr}}(x_{r+1} - x_{r-1})^{\perp}.$$
$$\nabla_{x_r}|\Omega_j| = l_{jr}\, n_{jr}.$$

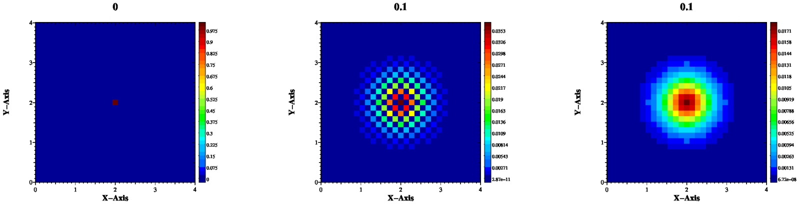
$$\begin{cases} |\Omega_j|\partial_t E_j + \sum_r l_{jr} F_r \cdot n_{jr} = 0, \\ |\Omega_j|\partial_t F_j + \dfrac{c^2}{3}\sum_r l_{jr} E_{jr} n_{jr} = -\sigma c |\Omega_j| F_j \end{cases}$$

$$\begin{cases} E_{jr} = E_j + \dfrac{\sqrt{3}}{c}(F_j - F_r)\cdot n_{jr}, \\ \sum_j l_{jr} n_{jr} E_{jr} = 0. \end{cases}$$

Modification Jin-Levermore : limite diffusion.

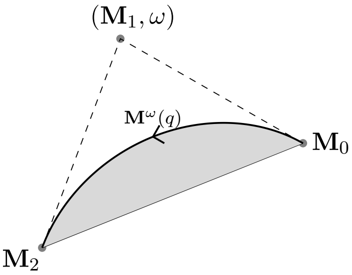
Refs : C. Mazerand, thèse. E. Franck, thèse.

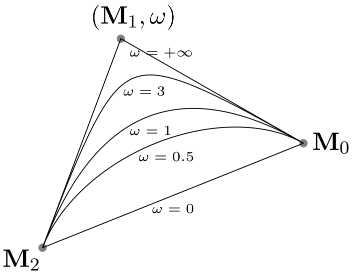
Problème : stencil "en croix" => stabilisation.



Arêtes coniques

$$\mathbf{M}^{\omega}(q) = \begin{pmatrix} x(q) \\ y(q) \end{pmatrix} = \frac{\mathbf{M}_0(1-q)^2 + 2\omega q(1-q)\mathbf{M}_1 + q^2\mathbf{M}_2}{(1-q)^2 + 2\omega q(1-q) + q^2}, \qquad q \in [0,1]$$

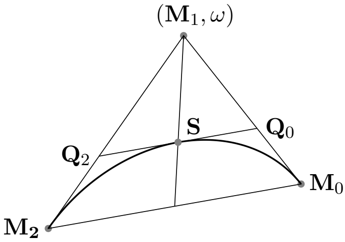




$\omega = 0$: segment $\qquad$ $\omega \in]0,1[$: ellipse $\qquad$ $\omega = 1$: parabole $\qquad$ $\omega > 1$: hyperbole.

Arêtes coniques

Shoulder point : $\mathbf{S} := \mathbf{M}^{\omega}(q = 0.5)$.



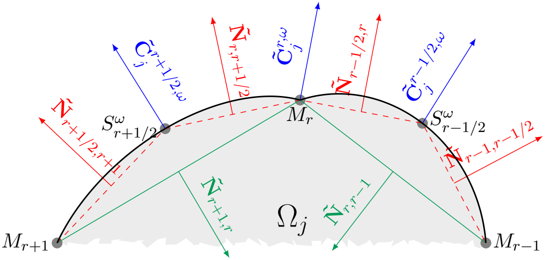
$$\mathbf{S} = \frac{1}{2}(\mathbf{Q_0} + \mathbf{Q_2}), \qquad \mathbf{Q_0} = \frac{1}{1+\omega}(\omega\mathbf{M_1} + \mathbf{M_0}), \qquad \mathbf{Q_2} = \frac{1}{1+\omega}(\omega\mathbf{M_1} + \mathbf{M_2}).$$

B. Boutin, E. Deriaz, P. Hoch, ESAIM Proc 2011, M. Li, X.-S. Gao, S.-C. Chou, Visual Comput., 2006.

Arêtes coniques

Aire

$$|\Omega_j| = \sum_e A(\mathbf{O}, \mathbf{M}_0^e, \mathbf{M}_2^e) + f(\omega) A(\mathbf{M}_0^e, \mathbf{M}_1^e, \mathbf{M}_2^e) = \frac{1}{2} \sum_{dof} \widetilde{\mathbf{C}}_j^{dof,\omega} \cdot \mathbf{OM}_{dof}.$$



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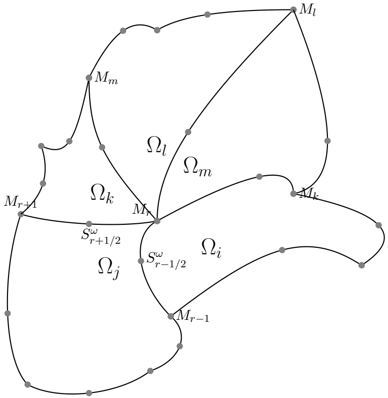
Arêtes coniques



Dof aux sommets $\mathbf{M}_r$ ET aux shoulder points $\mathbf{S}_{r+1/2}$.

Schéma GLACE

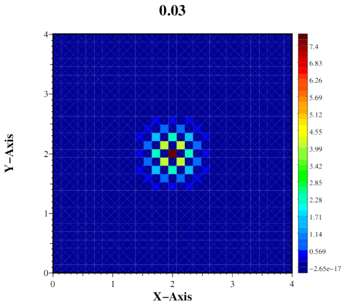
$$\begin{cases} E_i^{n+1} - E_i^n + \dfrac{\Delta t}{|\Omega_i|} \displaystyle\sum_{r/\boldsymbol{M}_r \in \Omega_i} \boldsymbol{C}_i^r \cdot \boldsymbol{F}_r^{n+1} = 0, \\ \boldsymbol{F}_i^{n+1} - \boldsymbol{F}_i^n + \dfrac{c^2 \Delta t}{3|\Omega_i|} \displaystyle\sum_{r/\boldsymbol{M}_r \in \Omega_i} \boldsymbol{C}_i^r E_{ir}^{n+1} = -c\sigma_i \Delta t \boldsymbol{F}_i^{n+1}, \end{cases}$$

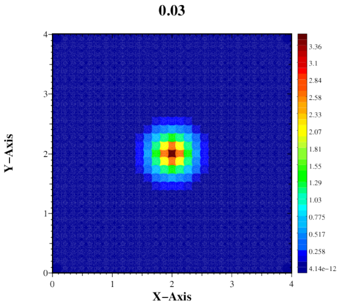
$$\begin{cases} E_{ir}^{n+1} = E_i^{n+1} + \dfrac{\sqrt{3}}{c}\left(\boldsymbol{F}_i^{n+1} - \boldsymbol{F}_r^{n+1}\right) \cdot \boldsymbol{n}_{ir} - c\sigma_r \boldsymbol{F}_r^{n+1} \cdot (\boldsymbol{M}_r - \boldsymbol{x}_i), \\ \displaystyle\sum_{i/\boldsymbol{M}_r \in \Omega_i} (\alpha_{ir} + c\sigma_r \beta_{ir}) \boldsymbol{F}_r^{n+1} = \sum_{i/\boldsymbol{M}_r \in \Omega_i} \boldsymbol{C}_i^r E_i^{n+1} + \sum_{i/\boldsymbol{M}_r \in \Omega_i} \boldsymbol{C}_i^r \otimes \boldsymbol{n}_{ir} \boldsymbol{F}_i^{n+1}. \end{cases}$$

$$\alpha_{ir} = \boldsymbol{C}_i^r \otimes \boldsymbol{n}_{ir} \text{ et } \beta_{ir} = \boldsymbol{C}_i^r \otimes (\boldsymbol{M}_r - \boldsymbol{x}_i) \text{ avec } \boldsymbol{n}_{ir} = \frac{\boldsymbol{C}_i^r}{\|\boldsymbol{C}_i^r\|}.$$

Limite diffusion

$\sigma = c = 10^4$, donnée initiale $E(0,x) = \delta_0(x)$. $E(t,x) = \frac{6\sigma}{\pi c t} e^{-\frac{3\sigma}{2ct}|x|^2}$





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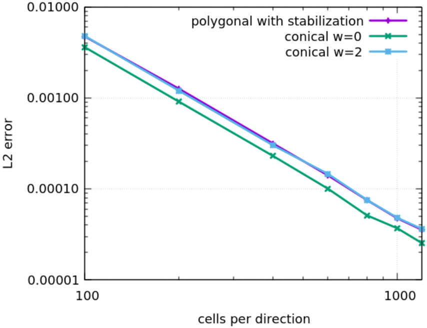
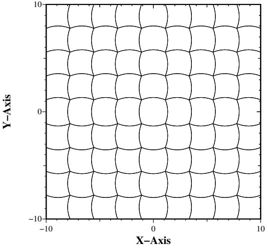




Régime transport

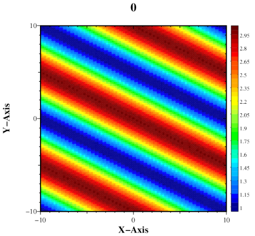
Conditions périodiques aux bord => Fourier : solution analytique.

Si $4\pi\sqrt{j^2+k^2} \leq \sigma L\sqrt{3}$, $E(x,y,t) = \cos\left(\frac{2j\pi x}{L} + \frac{2k\pi y}{L}\right) e^{-\sigma ct}\left[\alpha\left(\sigma c\frac{\sinh(\gamma t)}{2\gamma} + \cosh(\gamma t)\right) + \beta\frac{\sinh(\gamma t)}{\gamma}\right]$,

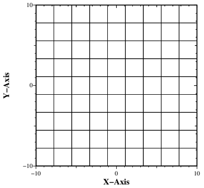
Si $4\pi\sqrt{j^2+k^2} > \sigma L\sqrt{3}$, $E(x,y,t) = \cos\left(\frac{2j\pi x}{L} + \frac{2k\pi y}{L}\right) e^{-\sigma ct}\left[\alpha\left(\sigma c\frac{\sin(\gamma t)}{2\gamma} + \cos(\gamma t)\right) + \beta\frac{\sin(\gamma t)}{\gamma}\right]$,

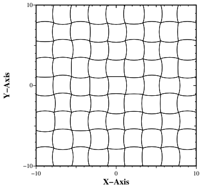
$$\gamma = \frac{1}{2}\sqrt{\left|\sigma^2 c^2 - 16\frac{\pi^2 c^2}{3L^2}\left(j^2+k^2\right)\right|}.$$

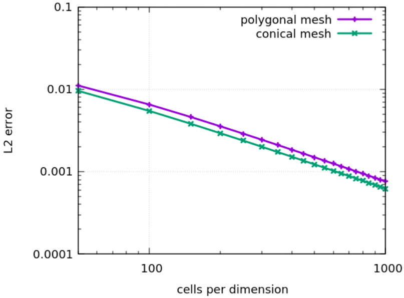
$$\sigma = c = 1,\ j = 1,\ k = 2$$



Régime transport

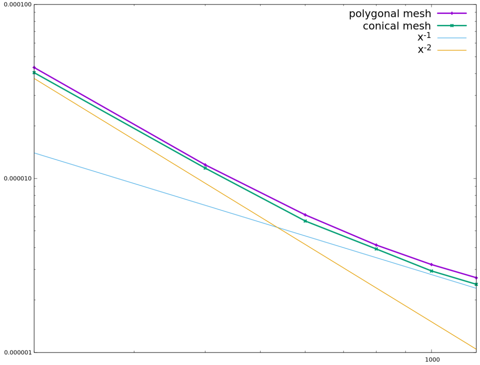






Régime diffusion

$\sigma = c = 10^4$, solution analytique



dirac.gif