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Short- and long-time behavior in (hypo)coercive ODE-systems and Fokker-Planck equations

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Short- and long-time behavior in (hypo)coercive ODE-systems and Fokker-Planck equations

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Goals & strategies

- Given an evolution eq: $\frac{\mathrm{d}}{\mathrm{d}t} f = -L\, f,\ t \geq 0;\quad L$ ... const-in-t operator
- Assume $-L$ is dissipative
- Assume L has a unique steady state: $L\, f_\infty = 0$

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1) optimal long-time decay estimate:
   - exponential decay: $\|f(t) - f_\infty\| \leq c e^{-\mu t} \|f(0) - f_\infty\|$, $t \geq 0$
   - possibly with sharp (= maximum) rate $\mu > 0$
     and minimal $c \geq 1$ [uniform for all $f(0)$]

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2) short-time decay estimate:
   - $\|f(t) - f_\infty\| \leq \left[1 - ct^a + \mathcal{O}(t^{a+1})\right] \|f(0) - f_\infty\|, \quad t \to 0+$
   - relation of a to *hypocoercivity index* of L

- for (nonsymmetric) ODEs $\dot{x} = -\mathbf{C}x$
- for (nonsymmetric) Fokker-Planck equations with linear drift

$\rightarrow$ find their connection

Outline:

1. hypocoercive ODEs
2. long-time decay of Fokker-Planck equations
3. short-time decay of Fokker-Planck equations

Long-time decay for nonsymmetric ODEs

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{C}^n \tag{1}$$

Definition: $\mathbf{C}$ is *coercive* if $x^T \mathbf{C} x \geq \kappa \|x\|^2 \ \forall x$ (for some $\kappa > 0$).

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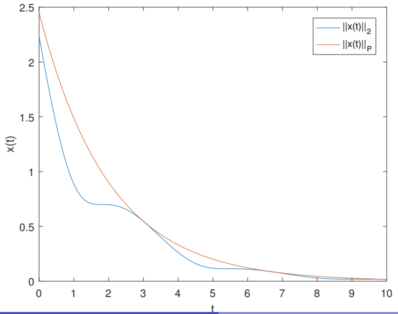
<u>ex:</u> $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$, $\lambda_{\mathbf{C}} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \Rightarrow$ decay rate $= \frac{1}{2}$ for (1).

- $\mathbf{C}$ not coercive $\Rightarrow$ no decay of $\|x(t)\|_2$ by trivial energy method!
- But decay of modified norm $\|x(t)\|_{\mathbf{P}} := \sqrt{x^T\,\mathbf{P}\,x}$; $\mathbf{P} := [2\ \ -1; -1\ 2]$



How to find $\mathbf{P}$ / the Lyapunov functional?

hypocoercive ODEs

$$\dot{x} = -\mathbf{C}x, \qquad t \geq 0,\ x(t) \in \mathbb{C}^n$$

Definition: $\mathbf{C}$ is *hypocoercive* (= positive stable) if $\exists\, \mu > 0$ such that:

$$\Re(\lambda_j) \geq \mu, \qquad j = 1, ..., n.$$

If all eigenvalues of $\mathbf{C}$ are non-defective:

$$\exists c \geq 1: \qquad \|x(t)\|_2 \leq c\|x(0)\|_2 e^{-\mu t}, \quad t \geq 0.$$

- always: $\mu \geq \kappa := \max_x \dfrac{x^T \mathbf{C} x}{\|x\|^2}$ (i.e. spectral gap $\geq$ coercivity)

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Conditions for hypocoercivity:

1. $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2 \in \mathbb{C}^{n \times n}$; $\mathbf{C}_1^* = -\mathbf{C}_1$, $\mathbf{C}_2^* = \mathbf{C}_2 \geq 0$ (w.l.o.g.)
2. No (non-trivial) subspace of $\ker \mathbf{C}_2$ is invariant under $\mathbf{C}_1$

Choice of $\mathbf{P}$ for $\|x\|_{\mathbf{P}}$ / Lyapunov's direct method

Lemma 1

Let $\mathbf{C} \in \mathbb{R}^{n\times n}$ *be positive stable, i.e.* $\mu := \min\{\Re\,\lambda_{\mathbf{C}}\} > 0$.

1. *If all* $\lambda_{\mathbf{C}}^{\min} \in \{\lambda \in \sigma(\mathbf{C}) \,|\, \Re\lambda = \mu\}$ *are non-defective (i.e. geometric = algebraic multiplicity)*
   $\Rightarrow \quad \exists\, \mathbf{P} \in \mathbb{R}^{n\times n},\ \mathbf{P} > 0 \ : \quad \mathbf{P}\mathbf{C} + \mathbf{C}^\top \mathbf{P} \geq 2\mu \mathbf{P}$.

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2. *If (at least) one* $\lambda_{\mathbf{C}}^{\min}$ *is defective* $\Rightarrow$
   $\forall\,\varepsilon > 0 \quad \exists\,\mathbf{P} = \mathbf{P}(\varepsilon) > 0\ : \quad \mathbf{PC} + \mathbf{C}^\top\mathbf{P} \geq 2(\mu - \varepsilon)\mathbf{P}\,.$

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Proof: $\mathbf{P}$ can be constructed explicitly; e.g. for $\mathbf{C}$ non-defective / diagonalizable:

$$\mathbf{P} := \sum_{j=1}^{n} z_j \otimes \bar{z}_j^\top \ ; \qquad z_j \text{ ... eigenvectors of } \mathbf{C}^\top$$

□

- $\mathbf{P}$ not unique; but the decay rates μ (or $\mu - \varepsilon$) are independent of $\mathbf{P}$.
- For complex $\mathbf{C}$: $\mathbf{P} > 0$ Hermitian with $\mathbf{PC} + \mathbf{C}^* \mathbf{P} \geq 2\mu \mathbf{P}$.

Long-time decay of P–norm

- Sharp decay estimate for $\dot{x} = -\mathbf{C}x$ (non-defective case, $\mathbf{C}$ real):

  Let $\|x\|_P^2 := x^T\, \mathbf{P}\, x$.

$$\frac{d}{dt}\,\|x\|_{\mathbf{P}}^2 \;=\; -x^T\big(\underbrace{\mathbf{P}\mathbf{C}+\mathbf{C}^T\mathbf{P}}_{\geq 2\mu\mathbf{P}}\big)x \leq -2\mu\|x\|_{\mathbf{P}}^2$$

$$\Rightarrow \quad \|x(t)\|_{\mathbf{P}} \;\leq\; \|x(0)\|_{\mathbf{P}}\, e^{-\mu t}, \quad t \geq 0.$$

Long-time decay of $\mathbf{P}$–norm

- Sharp decay estimate for $\dot{x} = -\mathbf{C}x$ (non-defective case, $\mathbf{C}$ real):

  Let $\|x\|_P^2 := x^T \, \mathbf{P} \, x$.

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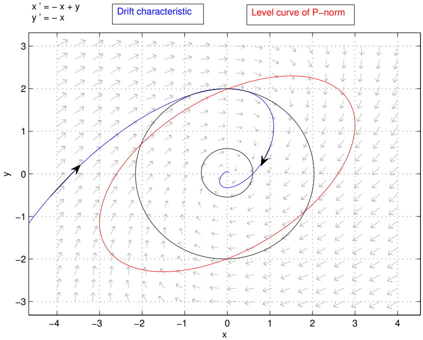
- $\mathbf{P}$–norm can be used for entropy/energy methods of kinetic equations (e.g. relaxation/BGK, Fokker-Planck)

Decay of P–norm (continued)

ex: $\dot{x} = -\mathbf{C}x$ with $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

- At x_2-axis: trajectory $x(t)$ tangent to level curve of $|x|$:



- level curve of "distorted" vector norm $\sqrt{x^T \mathbf{P}\, x}$; $\mathbf{P} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

  $\rightarrow$ uniform decay with sharp rate $\frac{1}{2}$

Hypocoercivity index

Conservative-dissipative system:

$$\dot{x} = -(\mathbf{C}_1 + \mathbf{C}_2)x\,, \qquad \mathbf{C}_1 \in \mathbb{C}^{n\times n}\text{...anti-Hermitian; } \mathbf{C}_2 \geq 0 \text{ Hermit.} \quad (2)$$

Definition 1 (Achleitner-AA-Carlen 2018)

The *hypocoercivity index of* $\mathbf{C} = \mathbf{C}_1 + \mathbf{C}_2$ *is the smallest integer* $m_{HC} \in \mathbb{N}_0$*, such that* $\sum_{j=0}^{m_{HC}} \mathbf{C}_1^j \mathbf{C}_2 (\mathbf{C}_1^*)^j > 0.$

- $\mathbf{C}$ is coercive $\Leftrightarrow$ $\mathbf{C}_2 > 0 \Leftrightarrow m_{HC} = 0$
- $\mathbf{C}$ is hypocoercive $\Leftrightarrow m_{HC} < \infty$
- If $\mathbf{C}$ is hypocoercive: $\frac{n - \operatorname{rank} \mathbf{C}_2}{\operatorname{rank} \mathbf{C}_2} \leq m_{HC} \leq n - \operatorname{rank} \mathbf{C}_2$
- m_{HC} describes the structural complexity of (2).

Hypocoercivity index for $\dot{x} = -(\mathbf{C}_1 + \mathbf{C}_2)x$

ex: $\mathbf{C}_2 = \text{diag}(0, 0, 1, 1)$

(a) $\mathbf{C}_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$, HC index $= 1$ (direct connection)

(b) $\mathbf{C}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, HC index $= 2$ (indirect connection)

Short-time decay / hypocoercivity index for $\dot{x} = -\mathbf{C}x$

Lemma 2 (Achleitner-AA-Carlen 2019)

Let $\mathbf{C}$ *be conservative-dissipative. Then its HC-index is* $m_{HC} \in \mathbb{N}_0$ *iff*

$$\|e^{-\mathbf{C}t}\|_2 = 1 - ct^{2m_{HC}+1} + \mathcal{O}(t^{2m_{HC}+2}), \quad t \to 0+$$

with some $c > 0$.

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Lemma 2 (Achleitner-AA-Carlen 2019)

Let $\mathbf{C}$ be conservative-dissipative. Then its HC-index is $m_{HC} \in \mathbb{N}_0$ iff

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with some $c > 0$.

ex: 2-velocity BGK model, 1D (Goldstein-Taylor model)
for $f(x,t) = \binom{f_+(x,t)}{f_-(x,t)}$ corresponding to $v = \pm 1$:

$$\partial_t f_\pm = \mp \partial_x f_\pm \pm \frac{1}{2}(f_- - f_+) =: -Lf_\pm, \quad t \geq 0, \quad 2\pi\text{–periodic in } x$$

- $\|e^{-Lt}\|_{\mathcal{B}(L^2)}$ decays like $1 - t^3/3 + o(t^3)$ [Miclo-Monmarché '13];

via x-modal decomposition: $\frac{d}{dt}u_k = -\begin{pmatrix} 0 & ik \\ ik & 1 \end{pmatrix} u_k$; $m_{HC} = 1$ for $k \neq 0$

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degenerate Fokker-Planck equations

$$f_t \;=\; \operatorname{div}\Big(\mathbf{D}\nabla f + \mathbf{C}x\, f\Big) =: -Lf\,, \quad x \in \mathbb{R}^d \tag{3}$$

with degenerate $0 \le \mathbf{D} \in \mathbb{R}^{d\times d}$ is degenerate parabolic;
(symmetric part of) L is not coercive.

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Definition 2 (Villani 2009)

Consider L on Hilbert space H with $\mathcal{K} = \ker L$; let $\tilde H \hookrightarrow \mathcal{K}^\perp$ (densely)
(e.g. H ... weighted L^2, $\tilde H$ ... weighted H^1).
L is called hypocoercive on $\tilde H$ if $\exists\, \lambda > 0,\ c \ge 1$:

$$\|e^{-Lt} f_0\|_{\tilde H} \le c\, e^{-\lambda t} \|f_0\|_{\tilde H} \qquad \forall\, f_0 \in \tilde H$$

- typically $c > 1$

hypocoercive Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big)$$

can be normalized such that $\mathbf{D} = \mathbf{C}_s$ (from now assumed).
Then $f_\infty(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$; $\mathcal{H} := L^2(f_\infty^{-1})$.

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Condition A for hypocoercivity:

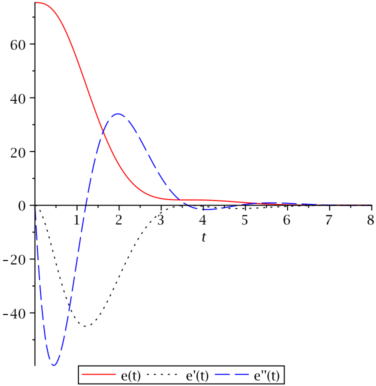
1. No (nontrivial) subspace of $\ker \mathbf{D}$ is invariant under $\mathbf{C}^\top$.
   (equivalent: L is hypoelliptic.)
2. Let $\mathbf{C}_s \in \mathbb{R}^{d\times d} \geq 0$.

   $\Rightarrow$ $\mathbf{C}$ is positive stable (i.e. $\Re\lambda_C > 0$).
   $\exists$ confinement potential; drift towards $x = 0$.

- hypoelliptic + confinement = hypocoercive (for FP eq.)

typical decay of degenerate Fokker-Planck equation

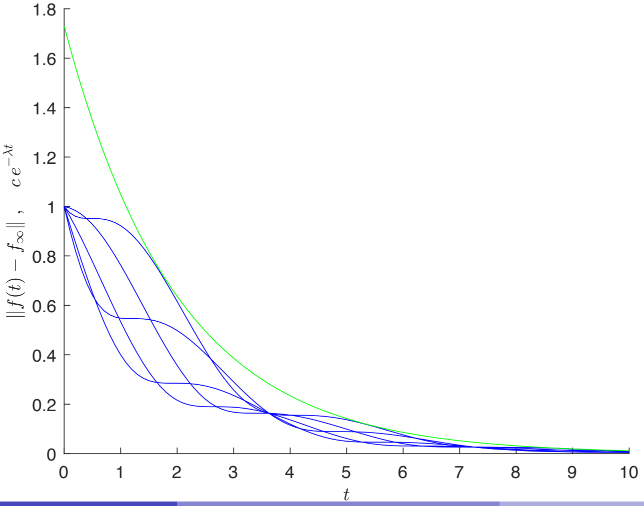
decay of $e(t) := \|f(t) - f_\infty\|^2_{L^2(f_\infty^{-1})}$:



degenerate FP eq. with $\mathbf{D} \geq 0$: $e(t)$ is not convex;
$e'(t) = 0$ for some $f \neq f_\infty$

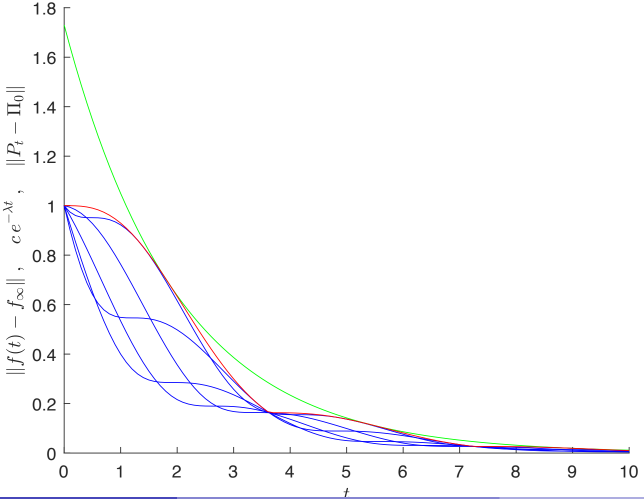
decay estimates for Fokker-Planck equations

Goal 1: best exponential decay $\|f(t)-f_\infty\|_{\mathcal{H}} \le c\, e^{-\lambda t}\|f(0)-f_\infty\|_{\mathcal{H}}$



decay estimates for Fokker-Planck equations

Goal 2: find exact PDE-propagator norm $\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})}$ $\Rightarrow$ Goal 1



propagator norm of (normalized) Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: -Lf, \quad \mathbf{D} = \mathbf{C}_s$$

main Theorem 1 (AA-Signorello-Schmeiser 2019)

Let L satisfy Condition A (i.e. L is hypocoercive). Then

$$\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = \|e^{-\mathbf{C}t}\|_2, \quad t \geq 0$$

Π_0 ... *projection on* $\operatorname{span}[f_\infty]$

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Π_0 ... *projection on* $\operatorname{span}[f_\infty]$

ex: [Gadat-Miclo '13] $f_t = -vf_x + axf_v + (vf)_v + f_{vv}$; $f_\infty(x,v) = c\,e^{-\frac{a}{2}x^2 - \frac{v^2}{2}}$

normalized Fokker-Planck: $\mathbf{C}_a = \begin{pmatrix} 0 & -\sqrt{a} \\ \sqrt{a} & 1 \end{pmatrix}, \quad a > 0$

$$\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = C_a(t)\,\exp\Big(-\frac{1-\sqrt{(1-4a)_+}}{2}t\Big),$$

$C_a(t) = \mathcal{O}(1)$ for $a \ne \frac{1}{4}$, $\quad C_{1/4}(t) = \mathcal{O}(t),\ t \to \infty$

sharp long-time decay of (normal.) Fokker-Planck equation

$$f_t = \mathrm{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: -Lf, \quad \mathbf{D} = \mathbf{C}_s \tag{4}$$

Corollary 1 (of main Theorem)

Let $\mathbf{C} \in \mathbb{R}^{d\times d}$ *be non-defective and satisfy Condition A (i.e.* $\mathbf{C}$ *is hypocoercive). Let* (c_1, μ) *be the optimal constants for* $\dot{x} = -\mathbf{C}x$ *in estimate*

$$\|x(t)\|_2 \le c_1\, e^{-\mu t}\|x_0\|, \quad t \ge 0.$$

Then, they are optimal for (4)*:*

$$\|f(t) - f_\infty\|_{\mathcal{H}} \le c_1\, e^{-\mu t}\|f_0 - f_\infty\|_{\mathcal{H}}, \quad \int_{\mathbb{R}^d} f_0(x)\,dx = 1$$

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$$\|f(t) - f_\infty\|_{\mathcal{H}} \le c_1\, e^{-\mu t}\|f_0 - f_\infty\|_{\mathcal{H}}, \quad \int_{\mathbb{R}^d} f_0(x)\,dx = 1$$

ex: For $d = 2$, $\Re\lambda_1^{\mathbf{C}} = \Re\lambda_2^{\mathbf{C}}$: $\quad c_1 = \sqrt{\operatorname{cond}(\mathbf{P})}$

Rem: For $\mathbf{C}$ defective (in eigenvalues with $\Re\lambda = \mu$): $\quad$ rate $= p(t)e^{-\mu t}$

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short-time decay of Fokker-Planck equation

<u>ex:</u> [Gadat-Miclo '13] $f_t = -vf_x + axf_v + (vf)_v + f_{vv} := -L_a f$

normal. Fokker-Planck: $\mathbf{C}_a = \begin{pmatrix} 0 & -\sqrt{a} \\ \sqrt{a} & 1 \end{pmatrix}$, hypocoercivity index $= 1$

$$\text{for } \ a \geq \frac{1}{4}: \quad \|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = 1 - \frac{a}{6}t^3 + o(t^3), \ t \to 0+$$

<u>Conjecture:</u> Decay "power 3 should be seen as an order of the hypocoercivity of the operator L_a ."

short-time decay of Fokker-Planck equation

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GOAL: Make this connection concrete, not just for one example.

short-time decay of Fokker-Planck equation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big), \quad \mathbf{D} = \mathbf{C}_s \tag{5}$$

Definition 3

The *hypocoercivity index of* (5) *is the smallest integer* $m_{HC} \in \mathbb{N}_0$, *such that* $\displaystyle\sum_{j=0}^{m_{HC}} \mathbf{C}_{AH}^j \mathbf{D} (\mathbf{C}_{AH}^*)^j > 0$.

(Also valid for (5) *not normalized, i.e.* $\mathbf{D} \neq \mathbf{C}_s$*.)*

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Corollary 4 (of main Theorem: $\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = \|e^{-\mathbf{C}t}\|_2$)

The HC-index of (5) *is* m_{HC} *iff*

$$\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = 1 - ct^{2m_{HC}+1} + \mathcal{O}(t^{2m_{HC}+2}), \quad t \to 0+$$

with some $c > 0$.

proof: HC-index of (5) = HC-index of ODE ($\dot{x} = -\mathbf{C}x$).

short-time decay of Fokker-Planck: second interpretation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: -Lf, \quad \text{with HC-index } m_{HC} \in \mathbb{N}_0$$

- Then: short-time regularization:

Theorem 5 ([Villani '09] for Hörmander rank; [AA-Erb '14] for HCI)

$$\left\| \nabla \frac{f(t)}{f_\infty} \right\|_{L^2(f_\infty)} \le c\, t^{-(m_{HC}+\frac{1}{2})} \left\| \frac{f_0}{f_\infty} - 1 \right\|_{L^2(f_\infty)}, \quad 0 < t \le \delta \tag{6}$$

short-time decay of Fokker-Planck: second interpretation

$$f_t = \operatorname{div}\Big(\mathbf{D}\,\nabla f + \mathbf{C}\,x\,f\Big) =: -Lf, \quad \text{with HC-index } m_{HC} \in \mathbb{N}_0$$

- Then: short-time regularization:

Theorem 5 ([Villani '09] for Hörmander rank; [AA-Erb '14] for HCI)

$$\left\|\nabla \frac{f(t)}{f_\infty}\right\|_{L^2(f_\infty)} \leq c\, t^{-(m_{HC}+\frac{1}{2})} \left\|\frac{f_0}{f_\infty} - 1\right\|_{L^2(f_\infty)}, \quad 0 < t \leq \delta \qquad (6)$$

- For Fokker-Planck eq. this is equivalent to the short time decay:

Proposition 1 (AA-Schmeiser-Signorello '19)

$$\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = 1 - ct^a + o(t^a), \quad t \to 0+$$

iff regularization (6) *holds with rate* $t^{-a/2}$.

Proof of main result (step 1)

main Theorem 2 (AA-Schmeiser-Signorello 2019)

Let $L = -\operatorname{div}\left(\mathbf{D}\,\nabla\cdot + \mathbf{C}\,x\,\cdot\right)$ *satisfy Condition A (i.e. L is hypocoercive). Then*

$$\|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = \|e^{-\mathbf{C}t}\|_2, \quad t \geq 0$$

Π_0 ... *projection on* $\operatorname{span}[f_\infty]$, $f_\infty = c\, e^{-|x|^2/2}$

- L ... nonsymmetric. Still, $\exists$ a partially orthogonal decomposition:

$$\mathcal{H} := L^2(f_\infty^{-1}) = \bigoplus_{m\in\mathbb{N}_0}^{\perp} V^{(m)}; \quad V^{(m)} = \operatorname{span}[g_\alpha(x) := (-1)^{|\alpha|}\nabla^\alpha f_\infty,\ |\alpha| = m]$$

$$\sigma(L) = \left\{\sum_{j=1}^{d} \alpha_j \lambda_j,\ \alpha \in \mathbb{N}_0^d\right\}; \quad \lambda_j \ldots \text{eigenvalues of } \mathbf{C} \in \mathbb{R}^{d\times d}$$

main proof (step 2): evolution in subspaces $V^{(m)}$

$d_\alpha(t)$ ... coefficient of $g_\alpha(x)$, $\alpha \in \mathbb{N}_0^d$, $x \in \mathbb{R}^d$

ex. $d = 2$:

- $m = 1$: $\frac{\mathrm{d}}{\mathrm{d}t}\binom{d_{(1,0)}}{d_{(0,1)}} = -\mathbf{C}\binom{d_{(1,0)}}{d_{(0,1)}}$

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- $m = 2$: $\begin{pmatrix} d_{(2,0)} \\ d_{(1,1)} \\ d_{(0,2)} \end{pmatrix}$ ... impractical !

better: $D^{(2)}(t) := \begin{pmatrix} d_{(2,0)} & d_{(1,1)}/2 \\ d_{(1,1)}/2 & d_{(0,2)} \end{pmatrix}(t) \in \mathbb{R}^{2\times 2}$

$$\frac{\mathrm{d}}{\mathrm{d}t} D^{(2)} = -\left(\mathbf{C} D^{(2)} + D^{(2)} \mathbf{C}^T\right)$$

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$$\frac{\mathrm{d}}{\mathrm{d}t}D^{(m)}(t) = -m\,\mathrm{Sym}\Big(\underbrace{\mathbf{C} \odot D^{(m)}(t)}_{\text{mult. on 1st index}}\Big) \quad \ldots \quad \text{tensored drift ODE}$$

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$\Rightarrow$ FP = 2nd quantization of ODE in Bosonic Fock space of $\mathbb{R}^2$

evolution in subspaces $V^{(m)}$

• ingredient for evolution equation in $V^{(m)}$:
rank-1 decomposition of order-m tensors:

$$D^{(m)} = \sum_{k=1}^{s} \mu_k v_k^{\otimes m}, \quad \mu_k \in \mathbb{R},\ v_k \in \mathbb{R}^d \tag{7}$$

Lemma 3

Let (7) be the decomposition of $D^{(m)}(0)$. Then, the evolution in $V^{(m)}$ is given by

$$D^{(m)}(t) = \sum_{k=1}^{s} \mu_k [v_k(t)]^{\otimes m}, \quad \dot{v}_k = -\mathbf{C} v_k \ .$$

main proof (step 3): decay in subspaces $V^{(m)}$

Lemma 4

Let $h(t) := \|e^{-\mathbf{C}t}\|_2$*, in particular* $h(t) \leq 1$.

$$\Rightarrow \quad \|D^{(m)}(t)\|_F \leq h(t)^m \, \|D^{(m)}(0)\|_F, \quad t \geq 0, \;\; m \in \mathbb{N}$$

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Lemma 4

Let $h(t) := \|e^{-\mathbf{C}t}\|_2$, in particular $h(t) \leq 1$.

$$\Rightarrow \quad \|D^{(m)}(t)\|_F \leq h(t)^m \, \|D^{(m)}(0)\|_F, \quad t \geq 0, \;\; m \in \mathbb{N}$$

- partial Parseval's identity:

$$\|f(t) - f_\infty\|_{\mathcal{H}}^2 = \sum_{m \in \mathbb{N}} m! \, \|D^{(m)}(t)\|_F^2$$

$$\Rightarrow \quad \|e^{-Lt} - \Pi_0\|_{\mathcal{B}(\mathcal{H})} = h(t), \quad t \geq 0$$

- I.e., decay behavior determined only by 1st subspace!

Conclusion

- Hypocoercivity index characterizes the short-time decay of ODEs ($\dot{x} = -\mathbf{C}x$) and Fokker-Planck equations: $f_t = \mathrm{div}(\mathbf{C}[\nabla f + xf])$; as well as the regularization rate in Fokker-Planck equations.
- Optimal decay estimates of (drift) ODEs carry over to Fokker-Planck equations.

Conclusion

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