A periodic homogenization problem with defects rare at infinity

Rémi Goudey

Ecole des ponts ParisTech and MATHERIALS Team, INRIA Paris

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Introduction

Purpose : To address an homogenization problem for a second order elliptic equation in divergence form when the coefficient is a non-local perturbation of a periodic coefficient :

$$\begin{cases} -\operatorname{div}((a_{per} + \tilde{a})(./\varepsilon)\nabla u^{\varepsilon}) = f & \text{in } \Omega, \\ u^{\varepsilon} = 0 & \text{in } \partial\Omega. \end{cases} \tag{1}$$

Where :

- $\Omega \subset \mathbb{R}^d$ is a bounded domain ($d \geq 1$).
- $f \in L^2(\Omega)$.
- $\varepsilon > 0$ is a small scale parameter.
- $a = a_{per} + \tilde{a}$ is a bounded, elliptic coefficient.
- a_{per} is $\mathbb{Z}^d$-periodic and $\tilde{a}$ represents a non-local perturbation.

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We want :

1. To identify the limit of u^{ε} when the scale parameter $\varepsilon \to 0$ and study the convergence for several topologies $(L^2(\Omega),\ H^1(\Omega),\ldots)$.
2. To make precise the convergence rates.

The periodic problem, when $a = a_{per}$, is well known [1] :

- u^{ε} converges strongly in $L^2(\Omega)$, weakly in $H^1(\Omega)$ to u^* solution to the homogenized equation :

$$\begin{cases} -\operatorname{div}(a^*_{per}\nabla u^*) = f & \text{on } \Omega \\ u^*(x) = 0 & \text{in } \partial\Omega \end{cases} \tag{2}$$

  where (a^*_{per}) is a constant matrix.

- The behavior in $H^1(\Omega)$ is obtained introducing a corrector $w_{per,p}$ defined for all $p \in \mathbb{R}^d$ as the periodic solution (unique up to the addition of a constant) to :

$$-\operatorname{div}(a_{per}(\nabla w_{per,p} + p)) = 0 \quad \text{in } \mathbb{R}^d. \tag{3}$$

[1] [Bensoussan, Lions, Papanicolaou '1978]

This corrector w_{per} allows to both make explicit the homogenized coefficient :

$$(a_{per}^*)_{i,j} = \int_Q e_i^T a_{per}(y) \left(e_j + \nabla w_{per,e_j}\right) dy,$$

and define an approximation

$$u^{\varepsilon,1} = u^*(.) + \varepsilon \sum_{i=1}^{d} \partial_i u^*(.) w_{per,e_i}(./\varepsilon),$$

such that $u^{\varepsilon,1} - u^{\varepsilon}$ **strongly** converges to 0 in $H^1(\Omega)$.

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The perturbed problem

- Our purpose is to extend these results to the setting of the perturbed problem when

$$a = a_{per} + \tilde{a}.$$

- **Main difficulty** : The corrector equation

$$-\operatorname{div}\left((a_{per} + \tilde{a})\left(\nabla w_p + p\right)\right) = 0 \text{ in } \mathbb{R}^d,$$

  cannot be reduced to an equation posed on a bounded domain as is the case in periodic context, which prevents us from using classical techniques (Poincaré inequality).

First extension [2]: The case of local perturbations (i.e $\tilde{a}(x) \to 0$ when $|x| \to \infty$) when $\tilde{a} \in L^r(\mathbb{R}^d)$ for $r \in]1, \infty[$.

- In this case, the homogenized limit is identical to that of the periodic case without defect ($\tilde{a} = 0$) and the existence of a corrector w_p is established. The corrector is of the form :

$$w_p = w_{per,p} + \tilde{w}_p$$

where $w_{per,p}$ is the periodic corrector and $\nabla \tilde{w} \in L^r(\mathbb{R}^d)$,

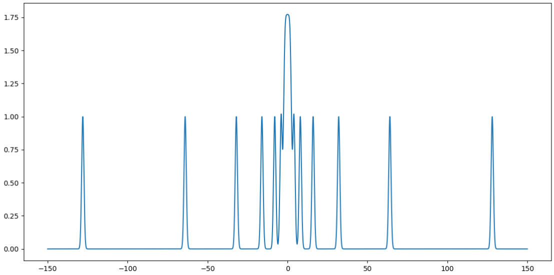
[2] [Blanc, Le Bris, Lions 2012, 2018] & [Blanc, Josien, Le Bris 2020]

The perturbed problem

The case of non-local perturbations

- We consider here a perturbation $\tilde{a}$, that, although it does not vanish, becomes rare at infinity.
- For some fixed $\phi \in \mathcal{D}(\mathbb{R})$, a prototypical one-dimensional example reads as

$$\tilde{a} = \sum_{k \in \mathbb{Z}} \phi(x - \text{sign}(k) 2^{|k|}).$$



Mathematical setting

To formalize the mathematical setting of the defects rare at infinity, we introduce :

$$\mathcal{G} = \{x_k\}_{k \in \mathbb{Z}^d} \subset \mathbb{Z}^d,$$

a infinite discrete set of points, and

$$V_k = \left\{ x \in \mathbb{R}^d \middle| \forall x_q \in \mathcal{G},\ |x - x_k| \leq |x - x_q| \right\},$$

the Voronoi cell containing the point x_k.

- Each point x_k models the presence of a defect in the periodic background.
- To ensure the defects are sufficiently rare at infinity, we need the points are increasingly distant from one another when far from the origin.

Mathematical setting

The set $\mathcal{G}$ is required to satisfy the following three conditions :

$$\forall x_k \in \mathcal{G}, \quad |V_k| < \infty, \tag{H1}$$

$$\exists C_1 > 0,\ C_2 > 0,\ \forall x_k \in \mathcal{G}, \quad C_1 \leq \frac{1 + |x_k|}{D\left(x_k, \mathcal{G} \setminus \{x_k\}\right)} \leq C_2, \tag{H2}$$

$$\exists C_3 > 0, \forall x_k \in \mathcal{G}, \quad \frac{Diam\left(V_k\right)}{D\left(x_k, \mathcal{G} \setminus \{x_k\}\right)} \leq C_3, \tag{H3}$$

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Remarks

- (H2) is the most significant assumption, it ensures the distance between a point x_k and the others to **scale exponentially** far from the origin.
- (H1) and (H3) are only technical assumptions that limit the size of the Voronoi cells and ensure the worst case scenario, where the set $\mathcal{G}$ contains as many points as possible while satisfying (H2).

Example of admissible set of points

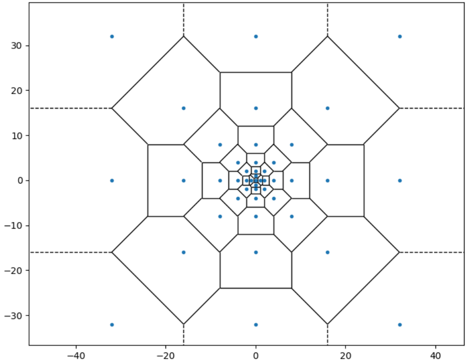

Figure: Example of points that satisfy our assumptions along with their associated Voronoi diagram for $d = 2$. Here the coordinates of the points are powers of 2.

The space of perturbations

In order to describe the perturbations we define :

$$\mathcal{B}^2(\mathbb{R}^d) = \left\{ f \in L^2_{unif}(\mathbb{R}^d) \;\middle|\; \exists f_\infty \in L^2(\mathbb{R}^d), \lim_{|x_k| \to \infty} \| f - \tau_{-x_k} f_\infty \|_{L^2(V_k)} = 0 \right\},$$

equipped with the norm

$$\|f\|_{\mathcal{B}^2(\mathbb{R}^d)} = \|f\|_{L^2_{unif}(\mathbb{R}^d)} + \|f_\infty\|_{L^2(\mathbb{R}^d)} + \sup_{x_k \in \mathcal{G}} \|f - \tau_{-x_k} f_\infty\|_{L^2(V_k)}.$$

Where :

- $L^2_{unif}(\mathbb{R}^d) = \left\{ f \in L^2_{loc}(\mathbb{R}^d), \sup_{x \in \mathbb{R}^d} \|f\|_{L^2(B_1(x))} < \infty \right\},$
- $\|f\|_{L^2_{unif}(\mathbb{R}^d)} = \sup_{x \in \mathbb{R}^d} \|f\|_{L^2(\mathrm{B}_1(x))}$
- $\tau_x f = f(. + x)$

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Remarks

- A function in $\mathcal{B}^2(\mathbb{R}^d)$ behaves, locally at the vicinity of each point x_k, as a fixed L^2-function truncated over the domain V_k.
- $\left(\mathcal{B}^2(\mathbb{R}^d), \|.\|_{\mathcal{B}^2(\mathbb{R}^d)}\right)$ is a Banach space.

A property of $\mathcal{B}^2(\mathbb{R}^d)$

Proposition (Average)

Let $u \in \mathcal{B}^2(\mathbb{R}^d)$, then :

$$\langle |u| \rangle = \lim_{R\to\infty} \frac{1}{|\mathrm{B}_R|} \int_{\mathrm{B}_R} |u(x)| dx = 0$$

- Consequence of the geometric distribution of the points x_k (Assumption (H2)).
- On average, the perturbations belonging to $\mathcal{B}^2(\mathbb{R}^d)$ do not impact the periodic background.
- If $\tilde{a} \in \mathcal{B}^2(\mathbb{R}^d)$, the homogenized limit of u^ε is therefore expected to be the same as in the periodic case without defect (i.e. when $\tilde{a} = 0$).

Assumptions

In our work, we consider a matrix-valued coefficient of the form

$$a = a_{per} + \tilde{a},$$

such that

- $a_{per} \in L^2_{per}(\mathbb{R}^d)^{d\times d}$, $\tilde{a} \in \mathcal{B}^2(\mathbb{R}^d)^{d\times d}$,
- $\exists \lambda > 0$, $\forall x, \xi \in \mathbb{R}^d$, $\lambda|\xi|^2 \leq \langle a(x)\xi, \xi\rangle, \quad \lambda|\xi|^2 \leq \langle a_{per}(x)\xi, \xi\rangle$,
- $a_{per}, \ \tilde{a}, \ \tilde{a}_\infty \in \left(L^\infty(\mathbb{R}^d) \cap \mathcal{C}^{0,\alpha}(\mathbb{R}^d)\right)^{d\times d}$, $\alpha \in]0, 1[$.

Here, $\tilde{a}_\infty$ is the limit L^2-function associated with $\tilde{a}$.

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Existence of the corrector

Theorem 1 : Existence result for the corrector equation

For every $p \in \mathbb{R}^d$, there exists a unique (up to an additive constant) function $w_p \in H^1_{loc}(\mathbb{R}^d)$ such that $\nabla w_p \in \left(L^2_{per}(\mathbb{R}^d) + \mathcal{B}^2(\mathbb{R}^d)\right)^d$, solution to :

$$\begin{cases} -\operatorname{div}((a_{per} + \tilde{a})(p + \nabla w_p)) = 0 \quad \text{in } \mathbb{R}^d, \\ \displaystyle\lim_{|x|\to\infty} \frac{|w_p(x)|}{1+|x|} = 0. \end{cases} \tag{4}$$

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Main Idea : Assume $w_p = w_{p,per} + \tilde{w}_p$ with $\nabla \tilde{w}_p \in \mathcal{B}^2(\mathbb{R}^d)$, then (4) is equivalent to an equation of the form :

$$-\operatorname{div}((a_{per} + \tilde{a})\nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d,$$

where $u = \tilde{w}_p$ and $f = \tilde{a}\,(p + \nabla w_{p,per}) \in \mathcal{B}^2(\mathbb{R}^d)$.

Lemma 1 : Existence result in the case $\tilde{a} = 0$

Let $f \in \mathcal{B}^2(\mathbb{R}^d)$, $\exists u \in H^1_{loc}(\mathbb{R}^d)$ such that $\nabla u \in \mathcal{B}^2(\mathbb{R}^d)$, solution to :

$$-\operatorname{div}(a_{per} \nabla u) = \operatorname{div}(f) \quad \text{in } \mathcal{D}'(\mathbb{R}^d).$$

[3] [Avellaneda, Lin 1991]

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Main idea (for $d \geq 3$) : Use the Green function G_{per} (i.e. the fundamental solution) associated with $-\operatorname{div}(a_{per}.)$ on $\mathbb{R}^d$ to define a solution u :

$$u(x) = \int_{\mathbb{R}^d} \nabla_y G_{per}(x,y) f(y) dy.$$

In order to show that u is well-defined in H^1_{loc} and $\nabla u \in \mathcal{B}^2$, we use pointwise estimates satisfied by G_{per} [3] :

$$|\nabla_y G_{per}(x,y)| \leq C \frac{1}{|x-y|^{d-1}}, \quad |\nabla_x \nabla_y G_{per}(x,y)| \leq C \frac{1}{|x-y|^{d}}.$$

[3] [Avellaneda, Lin 1991]

Lemma 2 : A priori estimate

There exists a constant $C > 0$ such that for every $f \in \mathcal{B}^2(\mathbb{R}^d)$ and u solution in $\mathcal{D}'(\mathbb{R}^d)$ to

$$-\operatorname{div}((a_{per} + \tilde{a})\,\nabla u) = \operatorname{div}(f) \quad \text{in } \mathbb{R}^d,$$

with $\nabla u \in \mathcal{B}^2(\mathbb{R}^d)$, we have the following estimate :

$$\|\nabla u\|_{\mathcal{B}^2(\mathbb{R}^d)} \leq C\|f\|_{\mathcal{B}^2(\mathbb{R}^d)}.$$

- Lemma 2 ensures the continuity of the reciprocal linear operator $f \mapsto \nabla\left(-\operatorname{div} a\nabla\right)^{-1}\operatorname{div}(f)$ from $\mathcal{B}^2(\mathbb{R}^d)$ to $\mathcal{B}^2(\mathbb{R}^d)$.

Main ideas for the proof of Theorem 1 [4] :

- For every $s \in [0,1]$, we consider the assertion $\mathrm{P}(s)$:

  " There exists $u \in H^1_{loc}(\mathbb{R}^d)$ solution in $\mathcal{D}'(\mathbb{R}^d)$ to

  $$-\operatorname{div}\left((a_{per} + s\tilde{a})\nabla u\right) = \operatorname{div}(f),$$

  such that $\nabla u \in \mathcal{B}^2(\mathbb{R}^d)$".

- We define $\mathcal{I} = \{s \in [0,1],\ \mathcal{P}(s) \text{ is true}\}$.
- Lemma 1 $\Rightarrow \mathcal{I}$ is not empty.
- Lemma 2 $\Rightarrow \mathcal{I}$ is both open and closed for the topology of $[0,1]$.
- Using an argument of connexity, $\mathcal{I} = [0,1] \Rightarrow s = 1 \in \mathcal{I}$.

[4] Proof adapted from [Blanc, Le Bris, Lions 2018]

Existence of the corrector

Remarks

- The corrector satisfies $\nabla w_p \in L^2 per + \mathcal{B}^2(\mathbb{R}^d)$, its gradient shares the same structure "periodic + $\mathcal{B}^2$" as the coefficient a.
- The proof of existence is heavily based upon the geometric distribution of the x_k. In particular, Assumption (H2), which ensures the distance between the points x_k to scale exponentially far from the origin, is essential in our approach.

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Application to homogenization

Proposition : Homogenization result

Let u^ε the sequence of solutions in $H_0^1(\Omega)$ to

$$\begin{cases} -\operatorname{div}((a_{per} + \tilde{a})(./\varepsilon)\nabla u^\varepsilon) = f & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{in } \partial\Omega. \end{cases}$$

Then the homogenized (weak-$H^1(\Omega)$ and strong-$L^2(\Omega)$) limit u^* obtained when $\varepsilon \to 0$ is the solution to

$$\begin{cases} -\operatorname{div}(a^*_{per}\nabla u^*) = f & \text{on } \Omega \\ u^*(x) = 0. & \text{in } \partial\Omega. \end{cases}$$

- As expected, the homogenized coefficient is identical to the periodic homogenized coefficient.

Application to homogenization

Theorem 2 : Convergence results

Assume $d \geq 3$ and Ω is a $C^{2,1}$-bounded domain. Let $\Omega_1 \subset\subset \Omega$. We define $u^{\varepsilon,1} = u^* + \varepsilon \sum_{i=1}^{d} \partial_i u^* w_{e_i}(./\varepsilon)$ where w_{e_i} is solution to corrector equation for $p = e_i$ and $u^* \in H^1(\Omega)$ is the homogenized limit. Then $R^\varepsilon = u^\varepsilon - u^{\varepsilon,1}$ satisfies the following estimates :

$$\|R^\varepsilon\|_{L^2(\Omega)} \leq C_1 \varepsilon \|f\|_{L^2(\Omega)},$$

$$\|\nabla R^\varepsilon\|_{L^2(\Omega_1)} \leq C_2 \varepsilon \|f\|_{L^2(\Omega)},$$

where C_1 and C_2 are two positive constants independent of f and ε.

Thank you for your attention !