Séminaire de Mathématiques Appliquées du CERMICS

École des Ponts ParisTech

State-constrained Linear-Quadratic Optimal Control as a Kernel Regression with Hard Shape Constraints

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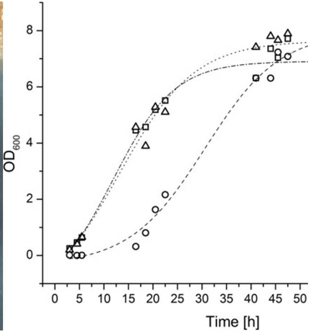
What are shape/state constraints?

Estimation



Side information

$\hookrightarrow$ compensates small number of samples or excessive noise

Control

Physical constraints

$\hookrightarrow$ provides feasible trajectories in path-planning

Ubiquitous and both handled as a constrained optimization problem

Table of Contents

Based on

- *Kernel Regression for Vehicle Trajectory Reconstruction under Speed and Inter-vehicular Distance Constraints*, PCAF, Nicolas Petit and Zoltán Szabó, *IFAC World Congress 2020*
- *Hard Shape-Constrained Kernel Machines*, PCAF and Zoltán Szabó, NeurIPS 2020, `https://arxiv.org/abs/2005.12636`
- *Linearly-constrained Linear Quadratic Regulator from the viewpoint of kernel methods*, PCAF, June 2020, Accepted with minor revision in SIAM Journal on Control and Optimization

Problem statement

Given samples $(x_n, y_n)_{n\in[N]} \in (\mathcal{X}\times\mathbb{R})^N$, a *loss* $L : (\mathcal{X}\times\mathbb{R}\times\mathbb{R})^N \to \mathbb{R}\cup\{\infty\}$, a *regularizer* $\Omega : \mathbb{R}_+ \to \mathbb{R}$. For $x \in \mathcal{X} \subset \mathbb{R}^d$, $f \in \mathcal{C}^s(\mathcal{X},\mathbb{R})$, consider

$$\begin{aligned} \bar{f} \in \underset{f\in\mathcal{F}}{\arg\min}\ & \mathcal{L}(f) = L\left(\left(x_n, y_n, f(x_n)\right)_{n\in[N]}\right) + \Omega\left(\|f\|_{\mathcal{F}}\right) \\ \text{s.t.}\ & b_i \le D_i f(x), \quad \forall\, x \in \mathcal{K}_i, \forall i \in [\mathcal{I}] = [\![1,\mathcal{I}]\!]. \end{aligned}$$

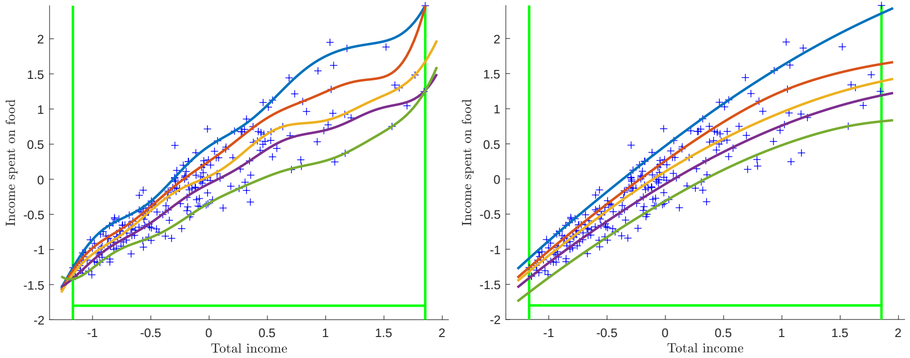
where $\mathcal{F}$ is a Hilbert space of smooth functions from $\mathcal{X}$ to $\mathbb{R}$, D_i is a differential operator ($D_i = \sum_j \gamma_j \partial^{r_j}$), $b_i \in \mathbb{R}$ is a lower bound, $\mathcal{K}_i$ is compact.

For non-finite $\mathcal{K}_i$, we have an infinite number of constraints!

How can we make this optimization problem computationally tractable?

In practice: nonparametric estimation under constraints

In statistics: nonnegative densities, non-crossing quantiles



non-crossing+increasing non-crossing+increasing+concave

Qualitative priors have a great effect on the shape of solutions!

Glimpse of content of the talk

From dealing with a real-valued problem $f : x \in \mathcal{X} \subset \mathbb{R}^d \to y \in \mathbb{R}$

$$\bar{f} \in \underset{f \in \mathcal{F}}{\arg\min}\ \mathcal{L}(f) = L\left(\left(x_n, y_n, f(x_n)\right)_{n\in[N]}\right) + \Omega\left(\|f\|_{\mathcal{F}}\right)$$
$$\text{s.t.} \qquad b_i \leq D_i f(x), \quad \forall\, x \in \mathcal{K}_i, \forall i \in [\mathcal{I}] = [\![1, \mathcal{I}]\!].$$

ex: least-squares with monotonicity constraint

to a path-planning vector-valued problem $f : t \in [0, T] \to y \in \mathbb{R}^Q$

Take $\mathcal{F}$ to be a Hilbert space of trajectories (e.g. Sobolev space)

$$\min_{f(\cdot) \in \mathcal{F}} \ g(f(T)) + \|f\|_{\mathcal{F}}^2$$
$$\text{s.t.} \qquad f(0) = y_0,$$
$$c_i(t)^\top f(t) \leq d_i(t), \quad \forall\, t \in [0, T], \forall i \in [\mathcal{I}].$$

ex: $g(f(T)) = \|y_T - f(T)\|_{\mathbb{R}^Q}^2$

Dealing with an infinite number of constraints: an overview

$$\bar{f} \in \arg\min_{f \in \mathcal{F}} \mathcal{L}(f) \text{ s.t. } "b_i \leq D_i f(x),\ \forall\, x \in \mathcal{K}_i,\ \forall i \in [\mathcal{I}]",\ \mathcal{K}_i \text{ non-finite}$$

Relaxing

- Discretize constraint at "virtual" samples $\{\tilde{x}_{m,i}\}_{m \leq M} \subset \mathcal{K}_i$,
  $\hookrightarrow$ no guarantees out-of-samples [Agrell, 2019, Takeuchi et al., 2006]
- Add constraint-inducing penalty, $\Omega_{\text{cons}}(f) = -\lambda \int_{\mathcal{K}_i} \min(0, D_i f(x) - b_i)\mathrm{d}x$
  $\hookrightarrow$ no guarantees, changes the problem objective [Brault et al., 2019]

Tightening

- Replace $\mathcal{F}$ by algebraic subclass of functions satisfying the constraints
  $\hookrightarrow$ hard to stack constraints, $\Phi(x)^\top A \Phi(x)$, Sum-Of-Squares [Hall, 2018]
- Use only spaces $\mathcal{F}$ s.t. constraints have a "simple" writing, e.g. splines
  $\hookrightarrow$ highly restricted functions classes [Papp and Alizadeh, 2014]
- **Our solution:** discretize $\mathcal{K}_i$ but replace b_i using RKHS geometry

Reproducing kernel Hilbert spaces (RKHS) at a glance (1)

A RKHS $(\mathcal{F}_k, \langle\cdot,\cdot\rangle_{\mathcal{F}_k})$ is a Hilbert space of real-valued[1] functions over a set $\mathcal{X}$ if one of the following equivalent conditions is satisfied [Aronszajn, 1950]

$\exists\, k : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ s.t. $k_x(\cdot) = k(x,\cdot) \in \mathcal{F}_k$ and $f(x) = \langle f(\cdot), k_x(\cdot)\rangle_{\mathcal{F}_k}$

the topology of $(\mathcal{F}_k, \langle\cdot,\cdot\rangle_{\mathcal{F}_k})$ is stronger than pointwise convergence i.e. $\delta_x : f \mapsto f(x)$ is continuous for all x for $f \in \mathcal{F}_k$.

$$|f(x) - f_n(x)| = |\langle f - f_n, k_x\rangle_k| \leq \|f - f_n\|_k \|k_x\|_k = \|f - f_n\|_k \sqrt{k(x,x)}$$

k is s.t. $\exists\, \Phi_k : \mathcal{X} \to \mathcal{F}_k$ s.t. $k(x,y) \;=\; \langle \Phi_k(x), \Phi_k(y)\rangle_{\mathcal{F}_k}$, $\Phi_k(x) = k_x(\cdot)$

k is s.t. $\mathbf{G} = [k(x_i, x_j)]_{i,j=1}^n \succcurlyeq 0$ and $\mathcal{F}_k := \overline{\operatorname{span}(\{k_x(\cdot)\}_{x\in\mathcal{X}})}$, i.e. the completion for the pre-scalar product $\langle k_x(\cdot), k_y(\cdot)\rangle_{k,0} = k(x,y)$

[1] There is a natural extension to vector-valued RKHSs (more on this later).

Reproducing kernel Hilbert spaces (RKHS) at a glance (2)

- There is a one-to-one correspondence between kernels k and RKHSs $(\mathcal{F}_k, \langle\cdot,\cdot\rangle_{\mathcal{F}_k})$. Changing $\mathcal{X}$ or $\langle\cdot,\cdot\rangle_{\mathcal{F}_k}$ changes the kernel k.[2]
- for $\mathcal{X} \subset \mathbb{R}^d$, Sobolev spaces $\mathcal{H}^s(\mathcal{X})$ satisfying $s > d/2$ are RKHSs. For $\mathcal{X} = \mathbb{R}^d$ their (Matérn) kernels are well known. Classical kernels include

$$k_{\text{Gauss}}(x,y) = \exp\left(-\|x-y\|^2_{\mathbb{R}^d}/(2\sigma^2)\right) \quad k_{\text{lin}}(x,y) = \langle x, y\rangle_{\mathbb{R}^d}$$

- if $\mathcal{X} \subset \mathbb{R}^d$ is contained in the closure of its interior (e.g. $[0,+\infty[$, for $d=1$), $k \in \mathcal{C}^{s,s}(\mathcal{X}\times\mathcal{X},\mathbb{R})$, $D = \sum_j \gamma_j \partial^{\mathbf{r}_j}$ a differential operator of order at most s, then $\mathcal{F}_k \subset \mathcal{C}^s(\mathcal{X},\mathbb{R})$ and reproducing formula for derivatives:

$$D_x k(x,\cdot) \in \mathcal{F}_k \quad ; \quad Df(x) = \langle f(\cdot), D_x k(x,\cdot)\rangle_{\mathcal{F}_k}$$

[2] It is hard to identify $\mathcal{F}_k$ given k, or k given $\mathcal{F}_k$ (more on this later).

Two essential tools for computations

Representer Theorem (e.g. [Schölkopf et al., 2001])

Let $L : (\mathcal{X} \times \mathbb{R} \times \mathbb{R})^N \to \mathbb{R} \cup \{\infty\}$, strictly increasing $\Omega : \mathbb{R}_+ \to \mathbb{R}$, and

$$\bar{f} = \underset{f \in \mathcal{F}_k}{\arg\min}\, L\left((x_n, y_n, f(x_n))_{n\in[N]}\right) + \Omega\left(\|f\|_k\right)$$

Then $\exists\, (a_n)_{n\in[N]} \in \mathbb{R}^N$ s.t. $\bar{f}(\cdot) = \sum_{n\in[N]} a_n k(x_n, \cdot)$

$\hookrightarrow$ Optimal solutions lie in a finite dimensional subspace of $\mathcal{F}_k$.

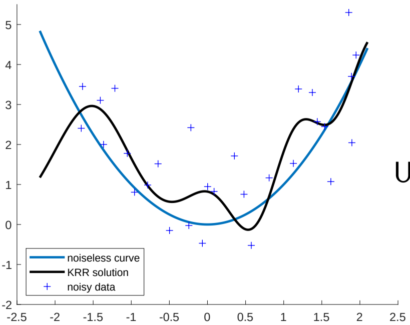
Finite number of evaluations $\implies$ finite number of coefficients

Kernel trick

$$\langle \sum_{n\in[N]} a_n k(x_n, \cdot), \sum_{m\in[M]} a'_m k(x'_m, \cdot)\rangle_k = \sum_{n\in[N]} \sum_{m\in[N']} a_n a'_m k(x_n, x'_m)$$

$\hookrightarrow$ On this finite dimensional subspace, no need to know $(\mathcal{F}_k, \langle\cdot,\cdot\rangle_{\mathcal{F}_k})$.

Example: 1D monotonic kernel ridge regression (KRR)

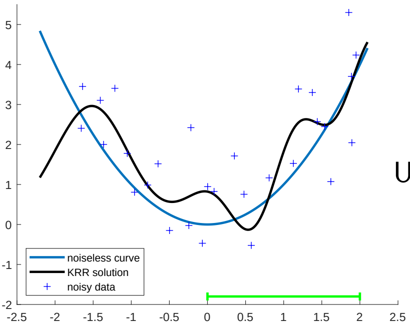


$$\min_{f \in \mathcal{F}_k} \frac{1}{N} \sum_{n=1}^{N} |y_n - f(x_n)|^2 + \lambda \|f\|_{\mathcal{F}_k}^2$$

Applying the representer theorem

Unconstrained KRR $\bar{f} = \sum_{n=1}^{N} \alpha_n k_{x_n}$,
$\boldsymbol{\alpha} = (\mathbf{G} + N\lambda \cdot \mathrm{Id})^{-1}\mathbf{y}$

Example: 1D monotonic kernel ridge regression (KRR)



$$\min_{f \in \mathcal{F}_k} \frac{1}{N} \sum_{n=1}^{N} |y_n - f(x_n)|^2 + \lambda \|f\|_{\mathcal{F}_k}^2$$

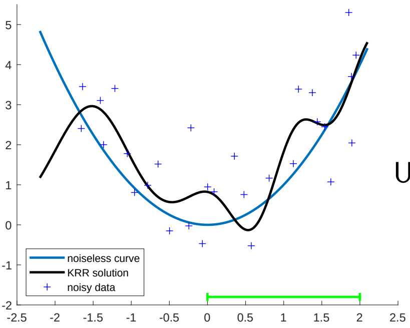
s.t. $\boxed{0 \leq f'(x),\ \forall x \in [0,2]}$

Unconstrained KRR $\bar{f} = \sum_{n=1}^{N} \alpha_n k_{x_n}$,
$\boldsymbol{\alpha} = (\boldsymbol{G} + N\lambda \cdot \mathrm{Id})^{-1} \boldsymbol{y}$

here is not monotonic on $[0,2]$!

Infinite number of evaluations $\Rightarrow$ no representer theorem!
How to modify the problem to ensure constraint satisfaction?

Example: 1D monotonic kernel ridge regression (KRR)



$$\min_{f \in \mathcal{F}_k} \frac{1}{N} \sum_{n=1}^{N} |y_n - f(x_n)|^2 + \lambda \|f\|_{\mathcal{F}_k}^2$$

$$\text{s.t. } \boxed{\eta_m \|f\|_k \le f'(\tilde{x}_m),\ \forall m \in [M]}$$

Unconstrained KRR $\bar{f} = \sum_{n=1}^{N} \alpha_n k_{x_n}$,

$$\boldsymbol{\alpha} = (\boldsymbol{G} + N\lambda \cdot \mathrm{Id})^{-1} \boldsymbol{y}$$

vs

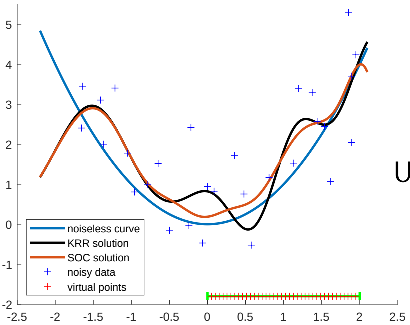
Second-Order Cone (SOC) constrained KRR

Second-Order Cone constraints: $\{f \mid \|Af + b\|_k \le c^\top f + d\}$

SOC comes from adding a buffer, $\eta_m > 0$, to a discretization, $\{\tilde{x}_m\}_{m \in [M]}$

$$\text{LP} \subset \text{QP} \subset \text{SOCP} \subset \text{SDP}$$

Example: 1D monotonic kernel ridge regression (KRR)



$$\min_{f \in \mathcal{F}_k} \frac{1}{N} \sum_{n=1}^{N} |y_n - f(x_n)|^2 + \lambda \|f\|_{\mathcal{F}_k}^2$$

$$\text{s.t. } \boxed{\eta_m \|f\|_k \le f'(\tilde{x}_m),\ \forall m \in [M]}$$

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Second-Order Cone constraints: $\{f \mid \|Af + b\|_k \le c^\top f + d\}$

SOC comes from adding a buffer, $\eta_m > 0$, to a discretization, $\{\tilde{x}_m\}_{m \in [M]}$

$$\text{“}b \le Df(x),\ \forall x \in \mathcal{K}\text{“} \Leftarrow \text{“}b + \eta_m \|f(\cdot)\| \le Df(\tilde{x}_m),\ \forall m \in [M]\text{“}$$

This choice is related to continuity moduli.

Deriving SOC constraints through continuity moduli

Take $\delta \geq 0$ and x s.t. $\|x - \tilde{x}_m\| \leq \delta$

$$\begin{aligned}|Df(x) - Df(\tilde{x}_m)| &= |\langle f(\cdot), D_x k(x,\cdot) - D_x k(\tilde{x}_m,\cdot)\rangle_k| \\ &\leq \|f(\cdot)\|_k \underbrace{\sup_{\{x \,|\, \|x-\tilde{x}_m\|\leq\delta\}} \|D_x k(x,\cdot) - D_x k(\tilde{x}_m,\cdot)\|_k}_{\eta_m(\delta)}\end{aligned}$$

$$\omega_m(Df,\delta) := \sup_{\{x \,|\, \|x-\tilde{x}_m\|\leq\delta\}} |Df(x) - Df(\tilde{x}_m)| \leq \eta_m(\delta)\|f(\cdot)\|_k$$

For a covering $\mathcal{K} = \bigcup_{m\in[M]} \mathbb{B}_{\mathcal{X}}(\tilde{x}_m, \delta_m)$

$$\text{“}b \leq Df(x),\ \forall x \in \mathcal{K}\text{“} \Leftrightarrow \text{“}b + \omega_m(Df,\delta) \leq Df(\tilde{x}_m),\ \forall m \in [M]\text{“}$$

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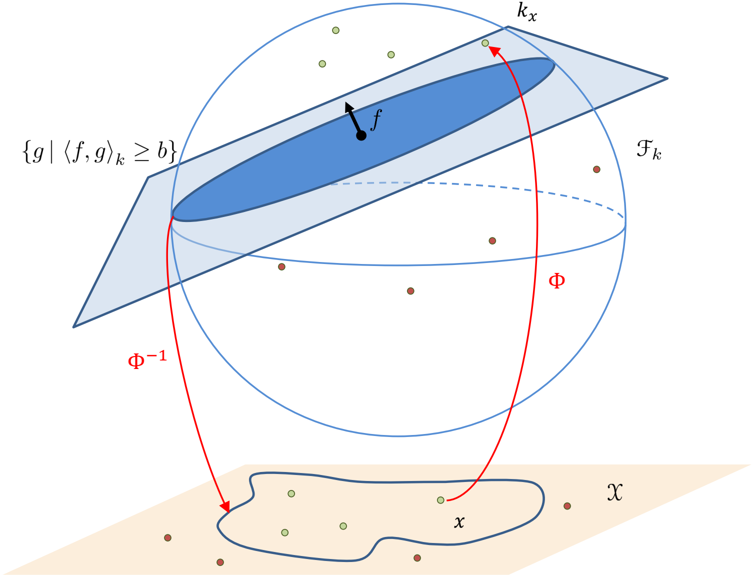
For a covering $\mathcal{K} \subset \bigcup_{m\in[M]} \mathbb{B}_{\mathcal{X}}(\tilde{x}_m, \delta_m)$

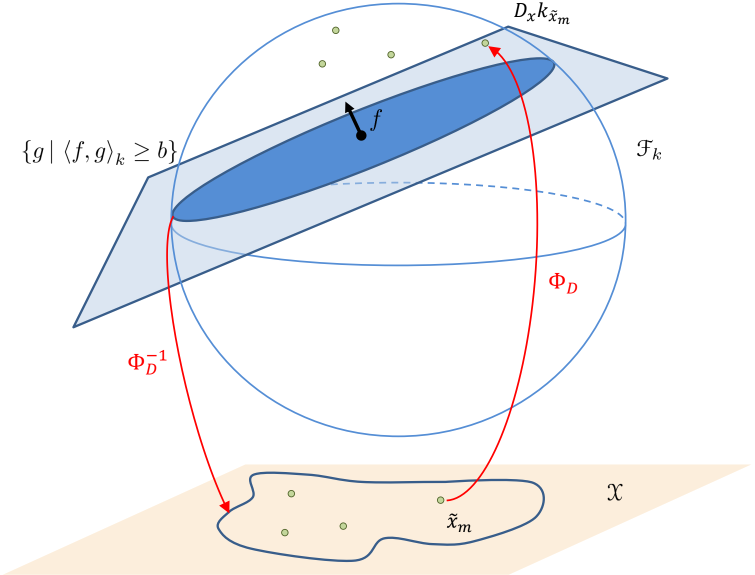
$$\begin{aligned} \text{“}b \leq Df(x),\ \forall x \in \mathcal{K}\text{“} &\Leftarrow \text{“}b + \omega_m(Df,\delta) \leq Df(\tilde{x}_m),\ \forall m \in [M]\text{“} \\ &\Leftarrow \text{“}b + \eta_m\|f(\cdot)\| \leq Df(\tilde{x}_m),\ \forall m \in [M] \end{aligned}$$

Since the kernel is smooth, $\delta \to 0$ gives $\eta_m(\delta) \to 0$.

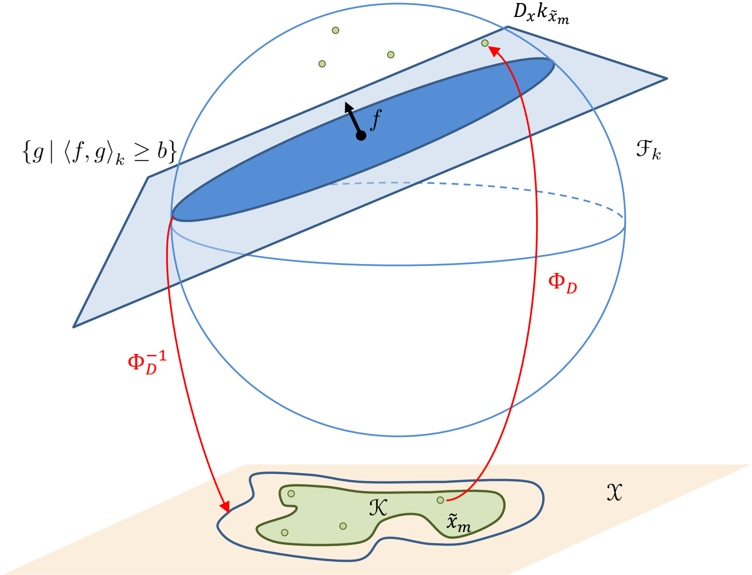
There is also a geometrical interpretation for this choice of η_m.



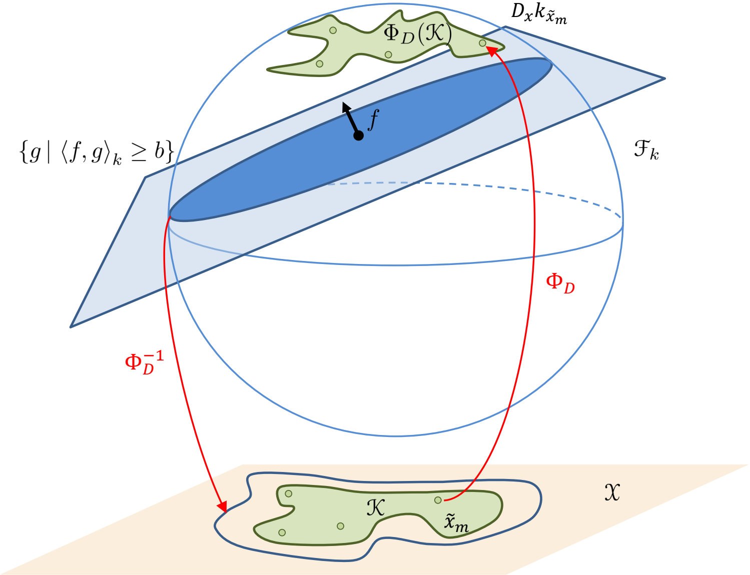
Support Vector Machine (SVM) is about separating red and green points by blue hyperplane.



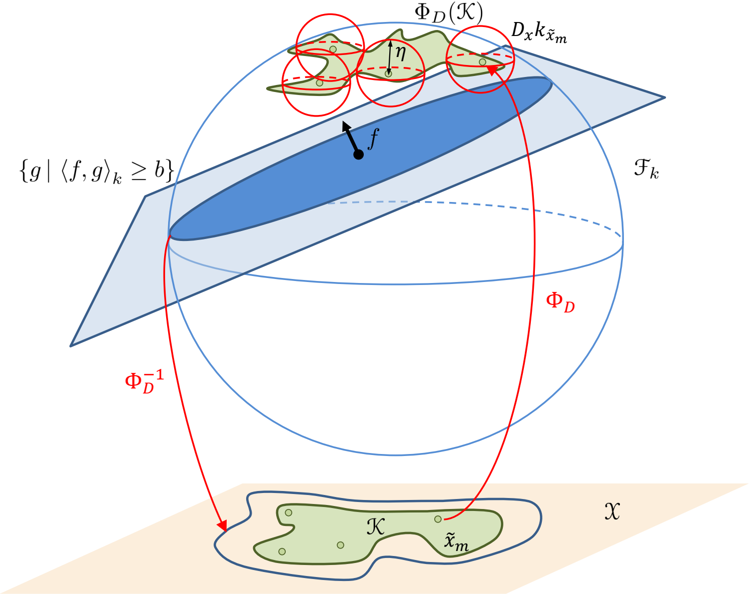
Using the nonlinear embedding $\Phi_D : x \mapsto D_x k(x, \cdot)$, the idea is the same. Consider only the green points, it looks like one-class SVM.



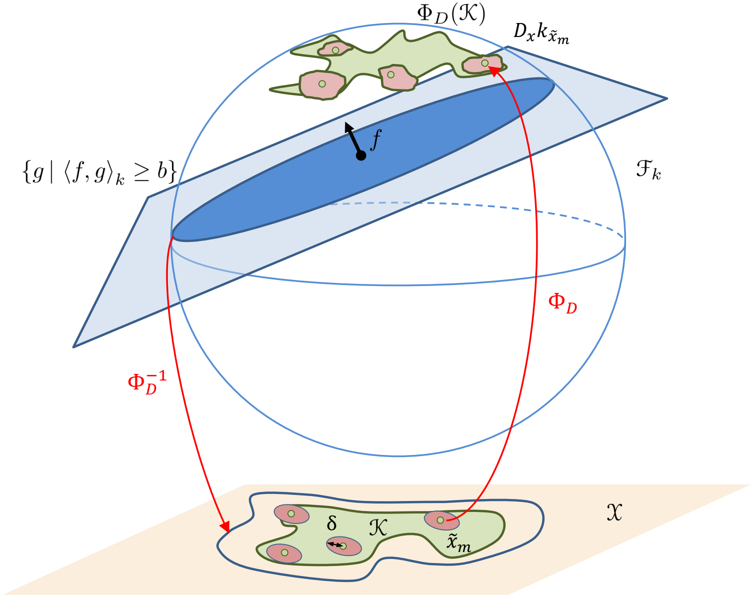
The green points are now samples of a compact set $\mathcal{K}$.



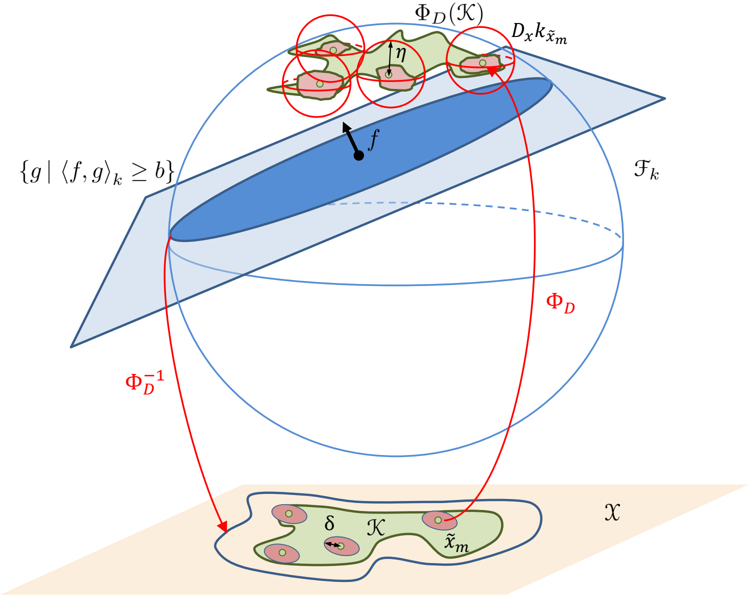
The image $\Phi_D(\mathcal{K})$ looks ugly...



The image $\Phi_D(\mathcal{K})$ looks ugly, can we cover it by balls? How to choose η?



First cover $\mathcal{K} \subset \bigcup\{\tilde{x}_m + \delta\mathbb{B}\}$, and then look at the images $\Phi_D(\{\tilde{x}_m + \delta\mathbb{B}\})$



Cover the $\Phi_D(\{\tilde{x}_m + \delta\mathbb{B}\})$ with tiny balls! This is how SOC was defined.

Main theorem

$$(f_\eta, b_\eta) \in \underset{f \in \mathcal{F}_k, b \in \mathcal{B}}{\arg\min} \quad \mathcal{L}(f) = L\left(b, (x_n, y_n, f(x_n))_{n\in[N]}\right) + \Omega\left(\|f\|_k\right)$$

$$\text{s.t.} \quad b_i + \eta_{i,m}\|f(\cdot)\|_k \leq D_i f(\tilde{x}_{m,i}), \quad \forall\, m \in [M_i], \forall i \in [\mathcal{I}].$$

where $\mathcal{B}$ is a closed convex constraint set over $(b_i)_{i\in[\mathcal{I}]}$. If $\Omega(\cdot)$ is strictly increasing, then

Theoretical guarantees [Aubin-Frankowski and Szabó, 2020]

i) The finite number of SOC constraints is **tighter** than the infinite number of affine constraints.

ii) **Representer theorem** (optimal solutions have a finite expression) $f_\eta = \sum_{i\in[\mathcal{I}],m\in[M_i]} \tilde{a}_{i,m} D_{i,x} k\left(\tilde{x}_{i,m},\cdot\right) + \sum_{n\in[N]} a_n k(x_n,\cdot)$

iii) If $\mathcal{L}$ is μ-strongly convex, we have **bounds**: computable/theoretical[a]

$$\|f_\eta - \bar{f}\|_k \leq \min\left(\sqrt{\frac{2(\mathcal{L}(f_\eta) - \mathcal{L}(f_{\eta=0}))}{\mu}}, \sqrt{\frac{L_{\bar{f}}\|\boldsymbol{\eta}\|_\infty}{\mu}}\right)$$

[a] Assuming $\mathcal{B} = \mathbb{R}^{\mathcal{I}}$ for the *a priori* bound to hold.

Discussion

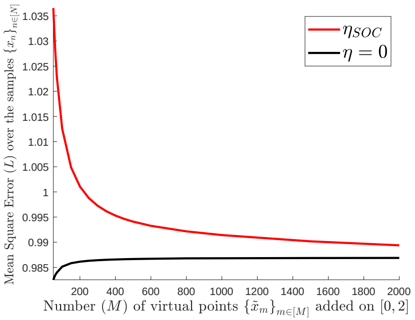
(i) This theorem holds for given samples $(x_n, y_n)_{n \in [N]}$ (optimization rather than statistical properties - no asymptotics)

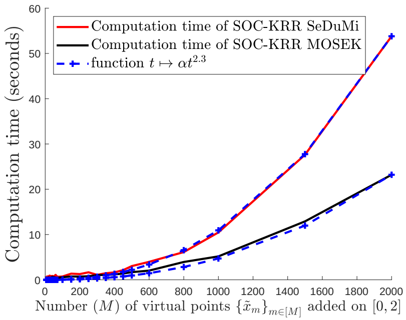
(ii) The representer theorem provides an equivalent finite-dimensional problem of size $N + M$ with SOC constraints $\sim \mathcal{O}((N+M)^3)$

(iii) Better bound $\equiv$ smaller $\eta \equiv$ smaller $\delta \equiv$ larger $M \equiv$ costly in time

(iv) The virtual points can be chosen among the samples (*recycling*)

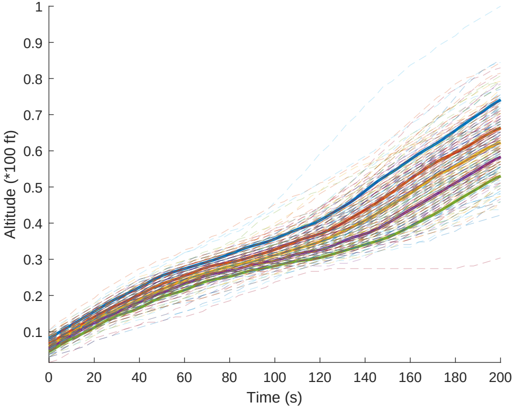
KRR example





Joint quantile regression (JQR): airplane data

Airplane trajectories at takeoff have increasing altitude.



JQR with monotonic constraint over $[x_{\min}, x_{\max}]$:

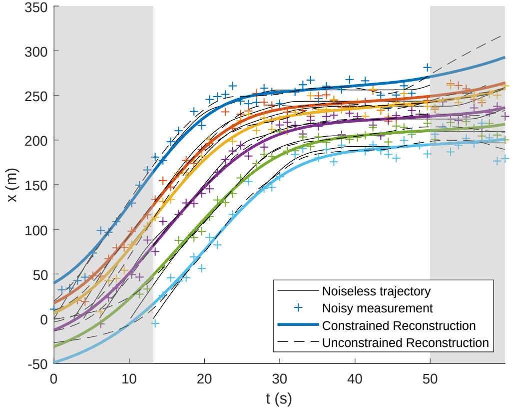
Increasing quantiles should be non-crossing

Data provided by ENAC (flights Paris→Toulouse) [Nicol, 2013]

Two shape constraints jointly handled with $15k$ samples.
Works with higher dimensions too!

Kernel ridge regression (KRR): trajectory reconstruction

Very noisy GPS data: six non-overtaking cars in a traffic jam



KRR with monotonic constraint over $[t_{\min}, t_{\max}]$:

Forward trajectories also maintain security distance

Data from IFSTTAR (MOCoPo Project) [Buisson et al., 2016]

(In *Kernel Regression for Vehicle Trajectory Reconstruction under Speed and Inter-vehicular Distance Constraints*, PCAF and Nicolas Petit and Zoltán Szabó *IFAC World Congress 2020*)

Teaser slide

This approach works as well for

- Other compact coverings than balls, iterative covering
- SDP constraints (e.g. convexity for $d \geq 2$): $0 \preccurlyeq \mathbf{Hess}(f)(x)$
- Vector-valued functions $f : \mathcal{X} \to \mathbb{R}^Q$
- Other applications: finance, control theory,...

Control: Take $\mathcal{F}_k$ to be a Hilbert space of trajectories $[0, T] \to \mathbb{R}^Q$

$$\begin{aligned} \min_{x(\cdot) \in \mathcal{F}_k} \quad & g(x(T)) + \|x(\cdot)\|_k^2 \\ \text{s.t.} \quad & x(0) = x_0, \\ & c_i(t)^\top x(t) \leq d_i(t), \quad \forall\, t \in [0, T], \forall i \in [\mathcal{I}]. \end{aligned}$$

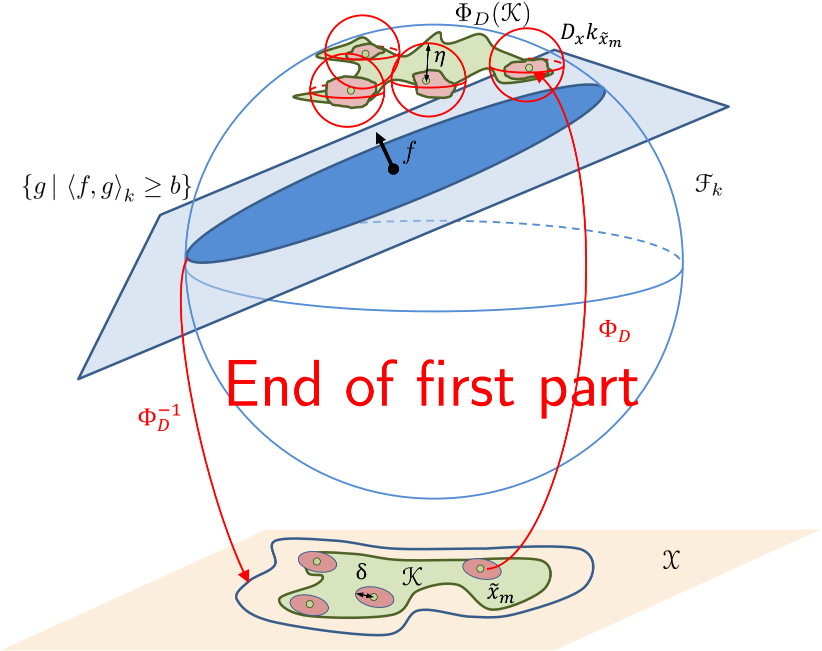
Partial conclusion/Take-home message

We have seen how to tighten an infinite number of affine constraints over a compact set into finitely many SOC constraints in RKHSs
$\hookrightarrow$ we have a representer theorem!

- tightening intractable constraints is the only way to have guarantees
- but tightening is "harder" to perform (here computationally)

Covering schemes suffer from the curse of dimensionality!
$\mathcal{X} \subset \mathbb{R}^d$, $d \gg 1$

But the control problem is only defined over $\mathcal{X} = [0, T]$ ($d = 1$)!


$\Phi_D(\mathcal{K})$
$D_x k_{\tilde{x}_m}$
η
f
$\{g \mid \langle f, g\rangle_k \geq b\}$
$\mathcal{F}_k$
Φ_D
End of first part
Φ_D^{-1}
δ
$\mathcal{K}$
$\tilde{x}_m$
$\mathcal{X}$

Linearly-constrained Linear Quadratic Regulator (LQR)

Problem of time-varying linear quadratic optimal control with finite horizon and affine inequality state constraints

$$\begin{aligned}
\min_{x(\cdot),u(\cdot)} \quad & g(x(T)) + \int_0^T [x(t)^\top Q(t)x(t) + u(t)^\top R(t)u(t)]\mathrm{d}t \\
\text{s.t.} \quad & x(0) = x_0, \\
& x'(t) = A(t)x(t) + B(t)u(t), \text{ a.e. in } [0, T], \\
& c_i(t)^\top x(t) \leq d_i(t), \forall\, t \in [0, T], \forall\, i \in [\![1, P]\!],
\end{aligned}$$

with state $x(t) \in \mathbb{R}^N$, control $u(t) \in \mathbb{R}^M$, $A(\cdot) \in L^1(0, T)$, $B(\cdot) \in L^2(0, T)$, $Q(t) \succcurlyeq 0$ and $R(t) \succcurlyeq r\mathrm{Id}_M$

Linearly-constrained Linear Quadratic Regulator (LQR)

Problem of time-varying linear quadratic optimal control with finite horizon and affine inequality state constraints with $Q \equiv 0$ and $R \equiv \mathrm{Id}_M$

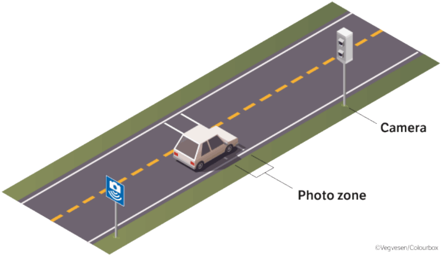
$$\begin{aligned} \min_{x(\cdot),u(\cdot)} \quad & g(x(T)) + \int_0^T \|u(t)\|_{\mathbb{R}^M}^2 \,\mathrm{d}t \\ \text{s.t.} \quad & x(0) = x_0, \\ & x'(t) = A(t)x(t) + B(t)u(t), \text{ a.e. in } [0, T], \\ & c_i(t)^\top x(t) \le d_i(t), \forall\, t \in [0, T], \forall\, i \in \{1, \dots, P\}, \end{aligned}$$

with state $x(t) \in \mathbb{R}^N$, control $u(t) \in \mathbb{R}^M$, $A(\cdot) \in L^1(0, T)$, $B(\cdot) \in L^2(0, T)$, $x(\cdot) : [0, T] \to \mathbb{R}^N$ absolutely continuous and $u(\cdot) \in L^2(0, T)$.

Why are state constraints difficult to study?

- **Theoretical obstacle**: Pontryagine's Maximum Principle involves not only an adjoint vector $p(t)$ but also measures/BV functions $\psi(t)$ supported at times where the constraints are saturated. You cannot just backpropagate the Hamiltonian system from the transversality condition.

- **Numerical obstacle**: Time discretization of constraints may fail e.g.



Speed cameras in traffic control

In between two cameras, drivers always break the speed limit.

Objective: Turn the state-constrained LQR into "KRR"

We have a vector space $\mathcal{S}$ of trajectories $x(\cdot) : [0, T] \to \mathbb{R}^N$

$$\mathcal{S} := \{x(\cdot) \,|\, \exists\, u(\cdot) \in L^2(0, T) \text{ s.t. } x'(t) = A(t)x(t) + B(t)u(t) \text{ a.e. } \}$$

The space of trajectories $\mathcal{S}$ depends on T, $A(\cdot)$, $B(\cdot)$.

LQR (Linear Quadratic Regulator)

$$\min_{\substack{x(\cdot)\in\mathcal{S} \\ u(\cdot)\in L^2(0,T)}} \quad g(x(T)) + \|u(\cdot)\|^2_{L^2(0,T)}$$

$x(0) = x_0$
$c_i(t)^\top x(t) \le d_i(t), t \in [0, T], i \le P$

"KRR" (Kernel Ridge Regression)

$$\min_{x(\cdot)\in\mathcal{S}} \; g(x(T)) + \lambda \|x(\cdot)\|^2_{\mathcal{S}}$$

$x(0) = x_0$
$c_i(t)^\top x(t) \le d_i(t), t \in [0, T], i \le P$

Is $\mathcal{S}$ a RKHS? For which inner product?

Vector-valued reproducing kernel Hilbert space (vRKHS)

Definition (vRKHS)

Let $\mathcal{T}$ be a non-empty set. A Hilbert space $(\mathcal{F}_K, \langle \cdot, \cdot \rangle_K)$ of $\mathbb{R}^N$-vector-valued functions defined on $\mathcal{T}$ is a vRKHS if there exists a matrix-valued kernel $K : \mathcal{T} \times \mathcal{T} \to \mathbb{R}^{N \times N}$ such that the reproducing property holds:

$$K(\cdot, t)p \in \mathcal{F}_K, \quad p^\top f(t) = \langle f, K(\cdot, t)p \rangle_K, \quad \text{for } t \in \mathcal{T},\ p \in \mathbb{R}^N, f \in \mathcal{F}_K$$

Necessarily, K has a Hermitian symmetry: $K(s, t) = K(t, s)^\top$

There is also a one-to-one correspondence between K and $(\mathcal{F}_K, \langle \cdot, \cdot \rangle_K)$ [Micheli and Glaunès, 2014], so changing $\mathcal{T}$ or $\langle \cdot, \cdot \rangle_K$ changes K.

Representer theorem in vRKHSs

Theorem (Representer theorem with SOC constraints)

Let $(\mathcal{F}_K, \langle\cdot,\cdot\rangle_K)$ *be a vRKHS defined on a set* $\mathcal{T}$. *For a "loss"* $L : \mathbb{R}^{N_0} \to \mathbb{R} \cup \{+\infty\}$, *strictly increasing "regularizer"* $\Omega : \mathbb{R}_+ \to \mathbb{R}$, *and constraints* $d_i : \mathbb{R}^{N_i} \to \mathbb{R}$, *consider the optimization problem*

$$\begin{aligned} \bar{f} \in \underset{f \in \mathcal{F}_K}{\arg\min} \quad & L\left(c_{0,1}^\top f(t_{0,1}), \ldots, c_{0,N_0}^\top f(t_{0,N_0})\right) + \Omega\left(\|f\|_K\right) \\ \text{s.t.} \quad & \lambda_i \|f\|_K \le d_i(c_{i,1}^\top f(t_{i,1}), \ldots, c_{i,N_i}^\top f(t_{i,N_i})), \ \forall\, i \in [\![1, P]\!]. \end{aligned}$$

Then there exists $\{p_{i,m}\}_{m \in [\![1,N_i]\!]} \subset \mathbb{R}^N$ *and* $\alpha_{i,m} \in \mathbb{R}$ *such that*

$$\bar{f} = \textstyle\sum_{i=0}^{P} \sum_{m=1}^{N_i} K(\cdot, t_{i,m}) p_{i,m} \text{ with } p_{i,m} = \alpha_{i,m} c_{i,m}.$$

Application to linear control systems with quadratic cost

$$\mathcal{S} := \{x(\cdot) \in W^{1,1} \,|\, \exists\, u(\cdot) \in L^2(0,T) \text{ s.t. } x'(t) = A(t)x(t) + B(t)u(t) \text{ a.e. } \}$$

Given $x(\cdot) \in \mathcal{S}$, for the pseudoinverse $B(t)^{\ominus}$ of $B(t)$, set

$$u(t) := B(t)^{\ominus}[x'(t) - A(t)x(t)] \text{ a.e. in } [0,T].$$

$$\langle x_1(\cdot), x_2(\cdot)\rangle_K := x_1(0)^\top x_2(0) + \int_0^T u_1(t)^\top u_2(t)\mathrm{d}t$$

Lemma

$(\mathcal{S}, \langle\cdot,\cdot\rangle_K)$ is a vRKHS with uniformly continuous $K(\cdot,\cdot)$.

$\|\cdot\|_K$ is a Sobolev-like norm split into two semi-norms

$$\|x(\cdot)\|_K^2 = \underbrace{\|x(0)\|^2}_{\|x(\cdot)\|_{K_0}^2} + \underbrace{\int_0^T \|B(t)^{\ominus}(x'(t) - A(t)x(t))\|^2\mathrm{d}t}_{\|x(\cdot)\|_{K_1}^2}.$$

Splitting $\mathcal{S}$ into subspaces and identifying their kernels

$\mathcal{S}_0 := \{x(\cdot) \,|\, x'(t) = A(t)x(t), \text{ a.e. in } [0, T]\}$ $\qquad \|x(\cdot)\|^2_{K_0} = \|x(0)\|^2$

$\mathcal{S}_u := \{x(\cdot) \,|\, x(\cdot) \in \mathcal{S} \text{ and } x(0) = 0\}$ $\qquad \|x(\cdot)\|^2_{K_1} = \|u(\cdot)\|^2_{L^2(0,T)}.$

As $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_u$, $K = K_0 + K_1$.

Splitting $\mathcal{S}$ into subspaces and identifying their kernels

$\mathcal{S}_0 := \{x(\cdot) \,|\, x'(t) = A(t)x(t), \text{ a.e. in } [0, T]\}$ $\quad \|x(\cdot)\|^2_{K_0} = \|x(0)\|^2$

$\mathcal{S}_u := \{x(\cdot) \,|\, x(\cdot) \in \mathcal{S} \text{ and } x(0) = 0\}$ $\quad \|x(\cdot)\|^2_{K_1} = \|u(\cdot)\|^2_{L^2(0,T)}$.

As $\mathcal{S} = \mathcal{S}_0 \oplus \mathcal{S}_u$, $K = K_0 + K_1$. Since $\dim(\mathcal{S}_0) = N$, for $\Phi_A(t,s) \in \mathbb{R}^{N,N}$ the state-transition matrix $s \to t$ of $x'(\tau) = A(\tau)x(\tau)$

$$K_0(s,t) = \Phi_A(s,0)\Phi_A(t,0)^\top$$

Only using the reproducing property and that for $x(\cdot) \in \mathcal{S}$,

$$x(t) = \Phi_A(t,0)x(0) + \int_0^t \Phi_A(t,\tau)B(\tau)u(\tau)\mathrm{d}\tau. \tag{1}$$

For fixed t, define control matrix $U_t(s) := \begin{cases} B(s)^\top \Phi_A(t,s)^\top & \forall s \le t, \\ 0 & \forall s > t. \end{cases}$

$$\partial_1 K_1(s,t) = A(s)K_1(s,t) + B(s)U_t(s) \text{ a.e. in } [0, T] \text{ with } K_1(0,t) = 0.$$

$$K_1(s,t) = \int_0^{\min(s,t)} \Phi_A(s,\tau)B(\tau)B(\tau)^\top \Phi_A(t,\tau)^\top \mathrm{d}\tau.$$

Examples: controllability Gramian/transversality condition

Steer a point from $(0,0)$ to (T, x_T), with e.g. $g(x(T)) = \|x_T - x(T)\|_N^2$

Exact planning ($x(T) = x_T$)

$$\min_{\substack{x(\cdot)\in\mathcal{S}\\ x(0)=0}} \quad \chi_{x_T}(x(T)) + \frac{1}{2}\|u(\cdot)\|^2_{L^2(0,T)}$$

Relaxed planning ($g \in \mathcal{C}^1$ convex)

$$\min_{\substack{x(\cdot)\in\mathcal{S}\\ x(0)=0}} \quad g(x(T)) + \frac{1}{2}\|u(\cdot)\|^2_{L^2(0,T)}$$

As, $x(0) = 0$, applying the representer theorem: $\exists\, p_T$, $\bar{x}(\cdot) = K_1(\cdot, T)p_T$

Controllability Gramian

$$K_1(T,T) = \int_0^T \Phi_A(T,\tau)B(\tau)B(\tau)^\top \Phi_A(T,\tau)^\top \mathrm{d}\tau$$

$\bar{x}(T) = x_T \Leftrightarrow x_T \in \mathrm{Im}(K_1(T,T))$

Transversality Condition

$$\begin{aligned} 0 &= \nabla\left(p \mapsto g(K_1(T,T)p) + \frac{1}{2}p^\top K_1(T,T)p\right)(p_T) \\ &= K_1(T,T)(\nabla g(K_1(T,T)p_T) + p_T). \end{aligned}$$

Take $p_T = -\nabla g(\bar{x}(T))$

From affine state constraints to SOC constraints

Take (t_m, δ_m) such that $[0, T] \subset \cup_{m \in [\![1,N_P]\!]}[t_m - \delta_m, t_m + \delta_m]$, take

$$\eta_i(\delta_m, t_m) := \sup_{t \in [t_m - \delta_m, t_m + \delta_m] \cap [0,T]} \|K(\cdot, t_m) c_i(t_m) - K(\cdot, t) c_i(t)\|_K,$$

$$d_i(\delta_m, t_m) := \inf_{t \in [t_m - \delta_m, t_m + \delta_m] \cap [0,T]} d_i(t).$$

We have strengthened SOC constraints that enable a representer theorem

$$\eta_i(\delta_m, t_m)\|x(\cdot)\|_K + c_i(t_m)^\top x(t_m) \le d_i(\delta_m, t_m),\ \forall\, m \in [\![1, N_P]\!], \forall\, i \in [\![1, P]\!]$$

$$\Downarrow$$

$$c_i(t)^\top x(t) \le d_i(t),\ \forall\, t \in [0, T], \forall\, i \in [\![1, P]\!]$$

Lemma (Uniform continuity of tightened constraints)

As $K(\cdot,\cdot)$ is UC, if $c_i(\cdot)$ and $d_i(\cdot)$ are $\mathcal{C}^0$-continuous, when $\delta \to 0^+$, $\eta_i(\cdot, t)$ converges to 0 and $d_i(\cdot, t)$ converges to $d_i(t)$, uniformly w.r.t. t.

Main theorem

(H-gen) $A(\cdot) \in L^1(0,T)$ and $B(\cdot) \in L^2(0,T)$, $c_i(\cdot)$ and $d_i(\cdot)$ are $\mathcal{C}^0$.

(H-sol) $c_i(0)x_0 < d_i(0)$ and there exists a trajectory $x^\epsilon(\cdot) \in \mathcal{S}$ satisfying strictly the affine constraints, as well as the initial condition.[3]

(H-obj) $g(\cdot)$ is convex and continuous.

Theorem (Existence and Approximation by SOC constraints)

Both the original problem and its strengthening have unique optimal solutions. For any $\rho > 0$, there exists $\overline{\delta} > 0$ such that for all $(\delta_m)_{m\in[\![1,N_0]\!]}$, with $[0,T] \subset \cup_{m\in[\![1,N_0]\!]}[t_m - \delta_m, t_m + \delta_m]$ satisfying $\overline{\delta} \geq \max_{m\in[\![1,N_0]\!]} \delta_m$,

$$\frac{1}{\gamma_K} \cdot \sup_{t\in[0,T]} \|\bar{x}_\eta(t) - \bar{x}(t)\| \leq \|\bar{x}_\eta(\cdot) - \bar{x}(\cdot)\|_K \leq \rho.$$

with $\gamma_K := \sup_{t\in[0,T],\, p\in\mathbb{B}_N} \sqrt{p^\top K(t,t)p}$.

[3](H-sol) is implied for instance by an inward-pointing condition at the boundary.

Numerical example: constrained pendulum - definition

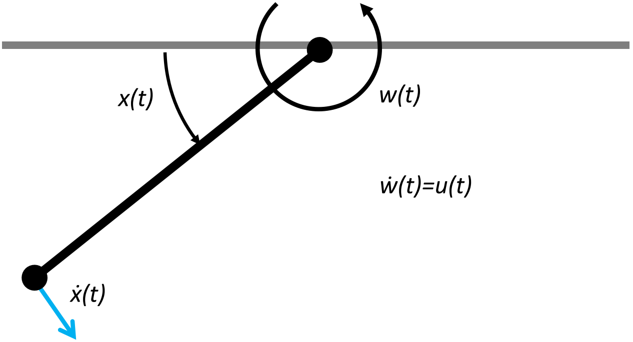
Constrained pendulum when controlling the third derivative of the angle

$$\min_{x(\cdot),w(\cdot),u(\cdot)} \quad -\dot{x}(T) + \lambda \|u(\cdot)\|^2_{L^2(0,T)} \qquad \lambda \ll 1$$

$$x(0) = 0.5, \quad \dot{x}(0) = 0, \quad w(0) = 0, \quad x(T/3) = 0.5, \quad x(T) = 0$$

$$\ddot{x}(t) = -10\,x(t) + w(t), \quad \dot{w}(t) = u(t), \text{ a.e. in } [0, T]$$

$$\dot{x}(t) \in [-3, +\infty[, \quad w(t) \in [-10, 10], \forall\, t \in [0, T]$$



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$$\ddot{x}(t) = -10\,x(t) + w(t), \quad \dot{w}(t) = u(t), \text{ a.e. in } [0, T]$$

$$\dot{x}(t) \in [-3, +\infty[, \quad w(t) \in [-10, 10], \forall\, t \in [0, T]$$

Converting affine state constraints to SOC constraints, applying rep. thm

$$\begin{aligned} \eta_{\dot{x}}\|x(\cdot)\|_K - \dot{x}(t_m) &\le \text{-}3,\\ \eta_w\|x(\cdot)\|_K + w(t_m) &\le 10,\\ \eta_w\|x(\cdot)\|_K - w(t_m) &\le 10 \end{aligned}$$

$$\bar{x}(\cdot) = K(\cdot,0)p_0 + K(\cdot,T/3)p_{T/3} + K(\cdot,T)p_T + \sum_{m=1}^{M} K(\cdot,t_m)p_m$$

Most of computational cost is related to the “controllability Gramians“ $K_1(s,t) = \int_0^{\min(s,t)} e^{(s-\tau)A}BB^\top e^{(t-\tau)A^\top}\mathrm{d}\tau$ which we have to approximate.

Numerical example: constrained pendulum - illustration

Optimal solutions of the constrained pendulum “path-planning“ problem. Red circles: equality constraints. Grayed areas: constraints over $[0, T]$.

Angle $x(\cdot)$ Velocity $\dot{x}(\cdot)$ Couple $w(\cdot)$

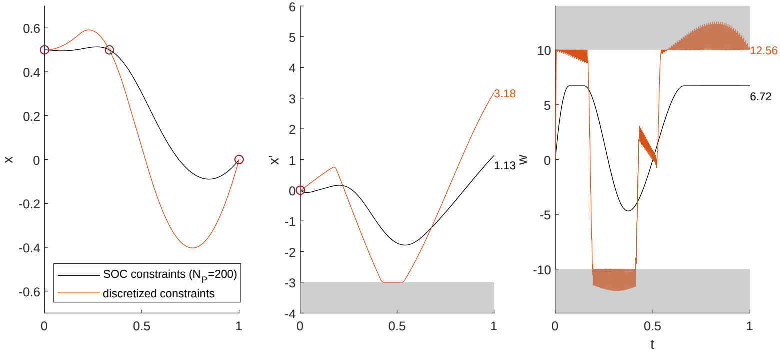


Figure: Comparison of SOC constraints (guaranteed η_w) vs discretized constraints ($\eta_w = 0$) for $N_P = 200$.

Numerical example: constrained pendulum - illustration

Optimal solutions of the constrained pendulum “path-planning“ problem. Red circles: equality constraints. Grayed areas: constraints over $[0, T]$.

Angle $x(\cdot)$ Velocity $\dot{x}(\cdot)$ Couple $w(\cdot)$

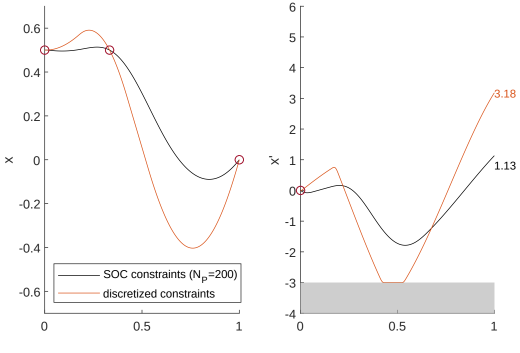


Figure: Comparison of SOC constraints (guaranteed η_w) vs discretized constraints ($\eta_w = 0$) for $N_P = 200$ - Chattering phenomenon!.

Optimal solutions of the constrained pendulum “path-planning“ problem. Red circles: equality constraints. Grayed areas: constraints over $[0, T]$.

Angle $x(\cdot)$ Velocity $\dot{x}(\cdot)$ Couple $w(\cdot)$

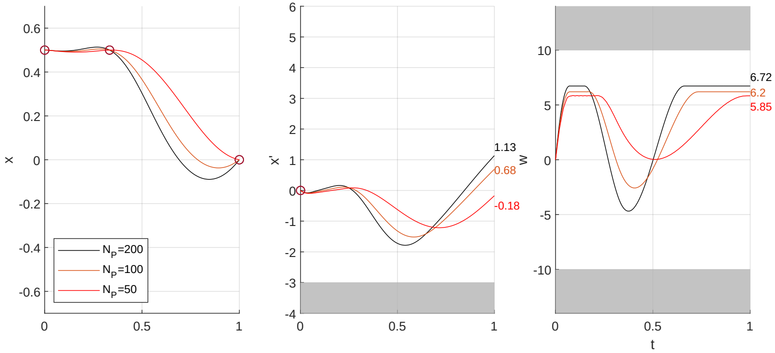


Figure: Comparison of SOC constraints for varying N_P and guaranteed η_w.

Numerical example: constrained pendulum - illustration

Optimal solutions of the constrained pendulum “path-planning“ problem. Red circles: equality constraints. Grayed areas: constraints over $[0, T]$.

Angle $x(\cdot)$ Velocity $\dot{x}(\cdot)$ Couple $w(\cdot)$

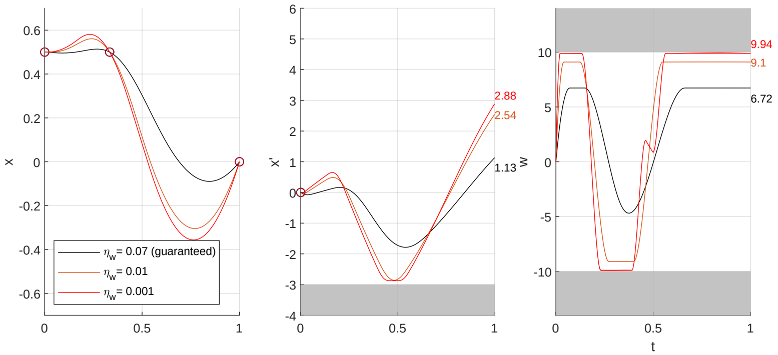


Figure: Comparison of SOC constraints for varying η_w and $N_P = 200$.

Pushing RKHSs beyond/Revisiting classical LQR

For RKHSs

- Control constraints do not correspond to continuous evaluations
  $\hookrightarrow$ limits of RKHS pointwise theory (e.g. $x' = u \in L^2([0, T], [-1, 1])$ a.e.)

- Successive linearizations of nonlinear system lead to changing kernels
  $\hookrightarrow$ a single kernel may not be sufficient (e.g. $x' = f_{[x_n(\cdot)]}x + f_{[u_n(\cdot)]}u$ a.e.)

- Non-quadratic costs for linear systems do not lead to Hilbert spaces
  $\hookrightarrow$ you may need Banach kernels (e.g. $\|u(\cdot)\|^2_{L^2(0,T)} \to \|u(\cdot)\|_{L^1(0,T)}$)

For control theory

- To each evaluation at time t corresponds a covector $p_t \in \mathbb{R}^N$
  $\hookrightarrow$ Representer theorem well adapted for state constraints, but unsuitable for control constraints. Reverts the difficulty w.r.t. PMP approach.

- The Gramian of controllability generates trajectories
  $\hookrightarrow$ This allows for close-form solutions in continuous-time

General conclusion

Shape constraints in RKHSs

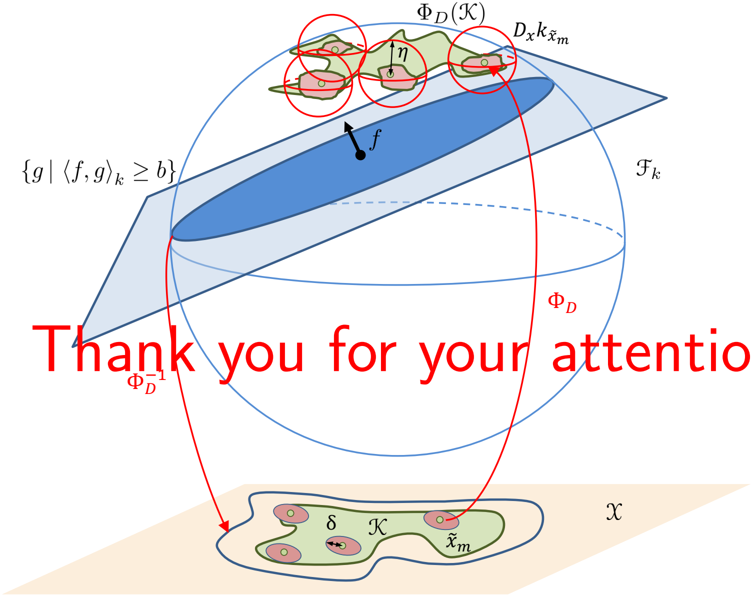
We have seen how to tighten in RKHSs an infinite number of pointwise affine constraints over a compact set into finitely many SOC constraints.

- tightening intractable constraints is the only way to have guarantees
- compact coverings in infinite dimensional spaces provide a solution

Linear Quadratic Regulator as a kernel regression

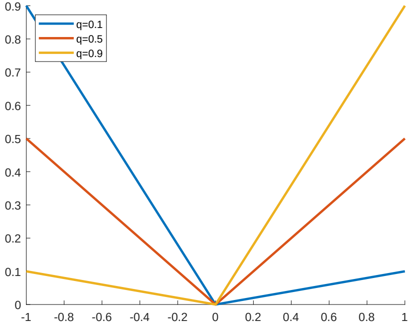
We have seen that state-constrained LQR is a non-trivial 1D example of shape constraints that

- allows to revisit classical notions from the kernel viewpoint
- allows to deal with the difficult problem of state constraints



Thank you for your attention!

Appendix: Joint Quantile Regression (JQR)



$f_\tau(x)$ conditional quantile over (X, Y): $P(Y \leq f_\tau(x)|X = x) = \tau \in]0, 1[$.

Estimation through convex optimization over "pinball loss" $l_\tau(\cdot)$ (i.e. tilted absolute value [Koenker, 2005]).

Known fact: quantile functions can cross when estimated independently.

Joint quantile regression with non-crossing constraints, over $(f_q)_{q\in[Q]}$:

$$\mathcal{L}(f_1, \ldots, f_Q) = \frac{1}{N} \sum_{q\in[Q]} \sum_{n\in[N]} l_{\tau_q}(y_n - f_q(x_n)) + \lambda_f \sum_{q\in[Q]} \|f_q\|_k^2$$

$$\text{s.t. } f_{q+1}(x) \geq f_q(x), \forall q \in [Q-1], \forall x \in [\min x_n, \max x_n]^d.$$

Appendix: JQR performance over UCI datasets

- PDCD = Primal-Dual Coordinate Descent [Sangnier et al., 2016], JQR with parallel/heteroscedatic quantile penalization (see also ITL [Brault et al., 2019] for noncrossing inducer)
- mean $\pm$ std of 100$\times$value of the pinball loss (smaller is better)

Dataset	d	N	PDCD	SOC
engel	1	235	48 ± 8	53 ± 9
GAGurine	1	314	61 ± 7	65 ± 6
geyser	1	299	105 ± 7	108 ± 3
mcycle	1	133	66 ± 9	62 ± 5
ftcollinssnow	1	93	154 ± 16	148 ± 13
CobarOre	2	38	159 ± 24	151 ± 17
topo	2	52	69 ± 18	62 ± 14
caution	2	100	88 ± 17	98 ± 22
ufc	3	372	81 ± 4	87 ± 6

Annex: Green kernels and RKHSs

Let D be a differential operator, D^* its formal adjoint. Define the Green function $G_{D^*D,x}(y) : \Omega \to \mathbb{R}$ s.t. $D^*D\, G_{D^*D,x}(y) = \delta_x(y)$ then, if the integrals over the boundaries in Green's formula are null, for any $f \in \mathcal{F}_k$

$$f(x) = \int_\Omega f(y) D^* D G_{D^*D,x}(y) dy = \int_\Omega Df(y) D G_{D^*D,x}(y) =: \langle f, G_{D^*D,x} \rangle_{\mathcal{F}_k},$$

so $k(x, y) = G_{D^*D,x}(y)$ [Saitoh and Sawano, 2016, p61]. For vector-valued contexts, e.g. $\mathcal{F}_K = W^{s,2}(\mathbb{R}^d, \mathbb{R}^d)$ and $D^*D = (1 - \sigma^2\Delta)^s$ component-wise, see [Micheli and Glaunès, 2014, p9].

Alternatively, in 1D, $D\, G_{D,x}(y) = \delta_x(y)$, the kernel associated to the inner product $\int_\Omega Df(y) Dg(y) dy$ for the space of f "null at the border" writes as

$$k(x, y) = \int_\Omega G_{D,x}(z) G_{D,y}(z) dz$$

see [Berlinet and Thomas-Agnan, 2004, p286] and [Heckman, 2012].

Annex: IPC gives strictly feasible trajectories

(H-sol) $C(0)x_0 < d(0)$ and there exists a trajectory $x^\epsilon(\cdot) \in \mathcal{S}$ satisfying strictly the affine constraints, as well as the initial condition.

(H1) $A(\cdot)$ and $B(\cdot)$ are $\mathcal{C}^0$. $C(\cdot)$ and $d(\cdot)$ are $\mathcal{C}^1$ and $C(0)x_0 < d(0)$.

(H2) There exists $M_u > 0$ s.t. , for all $t \in [0, T]$ and $x \in \mathbb{R}^N$ satisfying $C(t)x \leq d(t)$, and $\|x\| \leq (1 + \|x_0\|)e^{T\|A(\cdot)\|_{L^\infty(0,T)} + TM_u\|B(\cdot)\|_{L^\infty(0,T)}}$, there exists $u_{t,x} \in M_u\mathbb{B}_M$ such that

$$\forall\, i \in \{j \,|\, c_j(t)^\top x = d_j(t)\},\ c_i'(t)^\top x - d_i'(t) + c_i(t)^\top(A(t)x + B(t)u_{t,x}) < 0.$$

This is an inward-pointing condition (IPC) at the boundary.

Lemma (Existence of interior trajectories)

If (H1) and (H2) hold, then (H-sol) holds.

Annex: control proof main idea, nested property

$$\eta_i(\delta, t) := \sup \|K(\cdot, t)c_i(t) - K(\cdot, s)c_i(s)\|_K, \quad \omega_i(\delta, t) := \sup |d_i(t) - d_i(s)|,$$
$$d_i(\delta_m, t_m) := \inf d_i(s), \qquad \text{over } s \in [t_m - \delta_m, t_m + \delta_m] \cap [0, T]$$

For $\overrightarrow{\epsilon} \in \mathbb{R}_+^P$, the constraints we shall consider are defined as follows

$$\mathcal{V}_0 := \{x(\cdot) \in \mathcal{S} \mid C(t)x(t) \leq d(t), \forall\, t \in [0, T]\},$$
$$\mathcal{V}_{\delta,\text{fin}} := \{x(\cdot) \in \mathcal{S} \mid \overrightarrow{\eta}(\delta_m, t_m)\|x(\cdot)\|_K + C(t_m)x(t_m) \leq d(\delta_m, t_m), \forall\, m \in [\![1, N_0]\!]\},$$
$$\mathcal{V}_{\delta,\text{inf}} := \{x(\cdot) \in \mathcal{S} \mid \overrightarrow{\eta}(\delta, t)\|x(\cdot)\|_K + \overrightarrow{\omega}(\delta, t) + C(t)x(t) \leq d(t), \forall\, t \in [0, T]\},$$
$$\mathcal{V}_{\overrightarrow{\epsilon}} := \{x(\cdot) \in \mathcal{S} \mid \overrightarrow{\epsilon} + C(t)x(t) \leq d(t), \forall\, t \in [0, T]\}.$$

Proposition (Nested sequence)

Let $\delta_{max} := \max_{m\in[\![1,N_0]\!]} \delta_m$. *For any* $\delta \geq \delta_{max}$, *if, for a given* $y_0 \geq 0$, $\epsilon_i \geq \sup_{t\in[0,T]}[\eta_i(\delta, t)y_0 + \omega_i(\delta, t)]$, *then we have a nested sequence*

$$(\mathcal{V}_{\overrightarrow{\epsilon}} \cap y_0\mathbb{B}_K) \subset \mathcal{V}_{\delta,inf} \subset \mathcal{V}_{\delta,fin} \subset \mathcal{V}_0.$$

Only the simpler $\mathcal{V}_{\overrightarrow{\epsilon}}$ constraints matter!

Annex: List of shape constraints

- **Monotonicity w.r.t. partial ordering**:

$$\partial^{e_1} f(x) \geq \ldots \geq \partial^{e_d} f(x) \geq 0 \quad (\forall x).$$

$$\partial^{e_j} f(x) \geq 0, \quad (\forall j \in [d], \quad \forall x).$$

- **Supermodularity**: $f(u \vee v) + f(u \wedge v) \geq f(u) + f(v)$, $u, v \in \mathbb{R}^d$, where $u \vee v := (\max(u_j, v_j))_{j \in [d]}$ and $u \wedge v := (\min(u_j, v_j))_{j \in [d]}$. For $f \in C^2$

$$\frac{\partial^2 f(x)}{\partial x_i \partial x_j} \geq 0 \quad (\forall i \neq j \in [d], \forall x).$$

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