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Multiple Exchange Property of Gross Substitutes Valuations with a Proof by Discrete Convex Analysis

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Abstract

Discrete convex analysis (DCA) offers a general framework of discrete optimization, combining the ideas from matroid theory and convex analysis. It has found applications in many different areas including operations research, mathematical economics, and game theory. The interaction between DCA and mathematical economics was initiated by Danilov, Koshevoy, and Murota (2001), and accelerated by the crucial observation of Fujishige and Yang (2003) that M-natural-concavity is equivalent to the gross substitutes (GS) property of Kelso and Crawford (1982).

In this talk we show how an old question in economics was settled with the DCA machinery. More concretely, we explain how the equivalence of the gross substitutes condition to the strong no complementarities (SNC) condition of Gul and Stacchetti (1999) can be proved with the use of the Fenchel-type duality theorem and the conjugacy theorem in DCA. The SNC condition means the multiple exchange property of a set function f, saying that, for two subsets X and Y and a subset I of X-Y, there exists a subset J of Y-X such that f((X-I)+J) +f((Y-J)+I) is not smaller than f(X) + f(Y).

Summary of Results

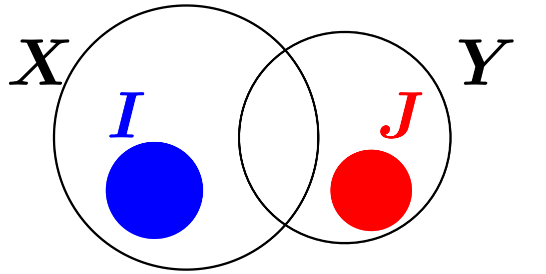
Result: M$^\natural$-concave functions have:

Multiple exchange property (f: set fn)

For any X, Y and $I \subseteq X \setminus Y$,

there exists $J \subseteq Y \setminus X$ s.t.

$$f(X) + f(Y) \leq f(X - I + J) + f(Y + I - J)$$



Implication in Economics: (open since 1999)

Strong No Complementarities (SNC)

$\Longleftrightarrow$ Gross Substitutes (GS)

Wikipedia: "Gross substitutes (indivisible items)"

Corollary: Valuated matroids have multiple EXC

- Generalizing a classical result for matroid bases
- Proof relying on Fenchel-type duality

Significance in Economics

Gross Substitutes Property

Valuation function $f : 2^N \to \mathrm{R}$ (Kelso–Crawford 1982)

Price vector p

Demand correspondence $D(p) = \mathrm{argmax}\, f[-p]$

where $f[-p](X) = f(X) - \sum_{i \in X} p_i$

Gross Substitutes Condition (GS)

$X \in D(p)$, $p \le q \Longrightarrow \exists\, Y \in D(q)$: $\{i \in X \mid p_i = q_i\} \subseteq Y$

Goods are substitute ($\approx$ similar). When the price of some items increases, the demand for other items does not decrease.

Meta Theorem:

In economies with indivisibility, equilibria exist under gross substitutes condition

GS Conditions for Demand

Price vector p

Demand correspondence $D(p) = \mathrm{argmax}\, f[-p]$

- Gross substitutes (GS) (Kelso–Crawford 82)

$X \in D(p),\ \ p \leq q \Longrightarrow \exists\, Y \in D(q) : \{i \in X \mid p_i = q_i\} \subseteq Y$

- No Complementarities (NC) (Gul–Stacchetti 99)

$X, Y \in D(p),\ I \subseteq X \setminus Y \Longrightarrow \exists\, J \subseteq Y \setminus X:\ (X \setminus I) \cup J \in D(p)$

GS Conditions for Valuation Function

$f : 2^N \to \mathrm{R}$ (set function)

- **Single Improvement (SI)** (Gul–Stacchetti 99)

$X \not\in D(p) \Longrightarrow \exists\, Y$:

$$f[-p](X) < f[-p](Y), \quad |X \setminus Y| \leq 1,\ |Y \setminus X| \leq 1$$

- **Strong No Complementarities (SNC)** (Gul–Stacchetti 99)

For any X, Y, any $I \subseteq X \setminus Y$, $\exists\, J \subseteq Y \setminus X$:

$$f(X) + f(Y) \leq f((X \setminus I) \cup J) + f((Y \setminus J) \cup I)$$

- **Submodular & Exchange** (Reijnierse–van Gallekom–Potters 02)

(i) $f(S \cup \{i,j\}) + f(S) \leq f(S \cup \{i\}) + f(S \cup \{j\})$

(ii) $f(S \cup \{i,j\}) + f(S \cup \{k\}) \leq$

$$\max[f(S \cup \{i,k\}) + f(S \cup \{j\}), f(S \cup \{j,k\}) + f(S \cup \{i\})]$$

- **M$^\natural$-concavity (in DCA)** (Fujishige–Yang 03)

Exchange Properties (1)

$f : 2^N \to \mathbf{R}$

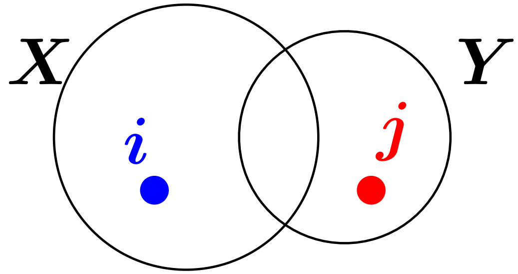
Def: M$^\natural$-concavity (M$^\natural$-EXC)

For any X, Y and $i \in X \setminus Y$,

$$f(X) + f(Y) \leq f(X - i) + f(Y + i)$$

or there exists $j \in Y \setminus X$ s.t.

$$f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j)$$

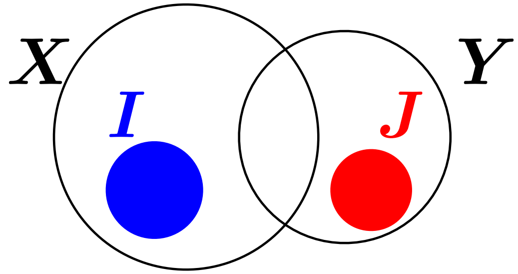


Multiple exchange (M$^\natural$-mEXC) $\equiv$ (SNC)

For any X, Y and $I \subseteq X \setminus Y$,

there exists $J \subseteq Y \setminus X$ s.t.

$$f(X) + f(Y) \leq f(X - I + J) + f(Y + I - J)$$



(M$^\natural$-EXC) $=$ (M$^\natural$-mEXC) with $|I| = 1$, $|J| \leq |I|$

Conditions for Gross Substitution

$$
\begin{array}{ccccccc}
\text{M}^\natural\text{-mEXC} & & & & & & \text{M}^\natural\text{-EXC} \\
\| & & & & & & \Updownarrow \text{(Fujishige–Yang 03)} \\
\text{(SNC)} & \Longrightarrow & \text{(NC)} & \Longleftrightarrow & \text{(GS)} & \Longleftrightarrow & \text{(SI)}
\end{array}
$$

|——— (Gul–Stacchetti 99) ———|

(SNC) Strong No Complementarities (=M$^\natural$-mEXC)
(NC) No Complementarities
(GS) Gross Substitution
(SI) Single Improvement

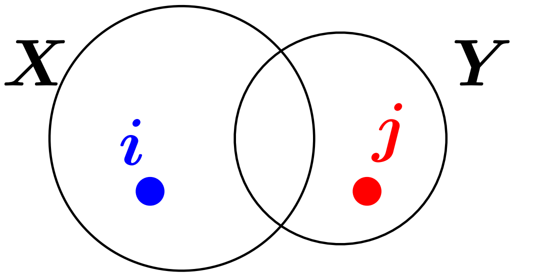
Exchange Properties (2) $f : 2^N \to \mathbf{R}$

($\mathrm{M}^\natural$-EXC) For any X, Y and $i \in X \setminus Y$,

$$f(X) + f(Y) \leq f(X - i) + f(Y + i)$$

or there exists $j \in Y \setminus X$ s.t.

$$f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j)$$

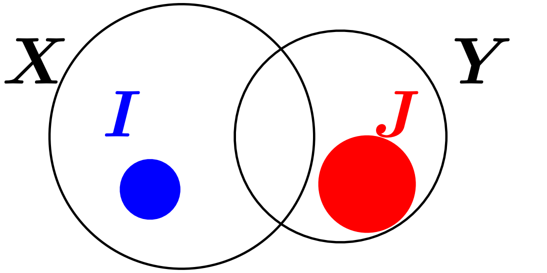


($\mathrm{M}^\natural$-mEXC)

For any X, Y and $I \subseteq X \setminus Y$,

there exists $J \subseteq Y \setminus X$ s.t.

$$f(X) + f(Y) \leq f(X - I + J) + f(Y + I - J)$$

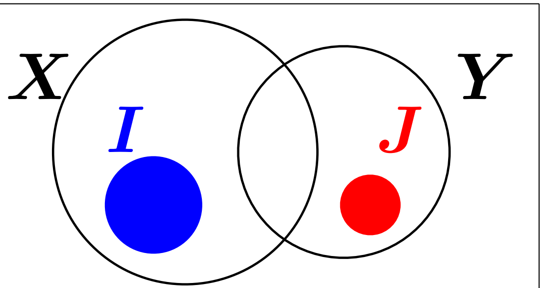


($\mathrm{M}^\natural$-mEXC-strong)

For any X, Y and $I \subseteq X \setminus Y$,

there exists $J \subseteq Y \setminus X$ s.t. $|J| \leq |I|$

$$f(X) + f(Y) \leq f(X - I + J) + f(Y + I - J)$$



Conditions for Gross Substitution

|———(our result)———|

$M^\natural$-mEXC $\Longleftrightarrow$ $M^\natural$-mEXC-str $\Longleftrightarrow$ $M^\natural$-EXC

|| $\Updownarrow$ (Fujishige–Yang 03)

(SNC) $\Longrightarrow$ (NC) $\Longleftrightarrow$ (GS) $\Longleftrightarrow$ (SI)

|——— (Gul–Stacchetti 99) ———|

(SNC) Strong No Complementarities (= $M^\natural$-mEXC)
(NC) No Complementarities
(GS) Gross Substitution
(SI) Single Improvement

Wikipedia "Gross substitutes (indivisible items)"

Introduction to DCA:

- $\text{M}^\natural$-convex function
- $\text{L}^\natural$-convex function
- M-L conjugacy
- Fenchel duality

Two Kinds of Discrete Convexity

Cont. $\mathrm{R}^n \to \mathrm{R}$		Discrete $\mathrm{Z}^n \to \mathrm{R}$
	discr $\longrightarrow$	$\mathrm{L}^\natural$-convex function
convex function		
	discr $\longrightarrow$	$\mathrm{M}^\natural$-convex function

Two Kinds of Discrete Convexity

Cont. $\mathbb{R}^n \to \mathbb{R}$	Discrete $\mathbb{Z}^n \to \mathbb{R}$
midpt convex	
$\Updownarrow$	L$^\natural$-convex function
convex function	
$\Updownarrow$	
equi-dist conv	
	M$^\natural$-convex function

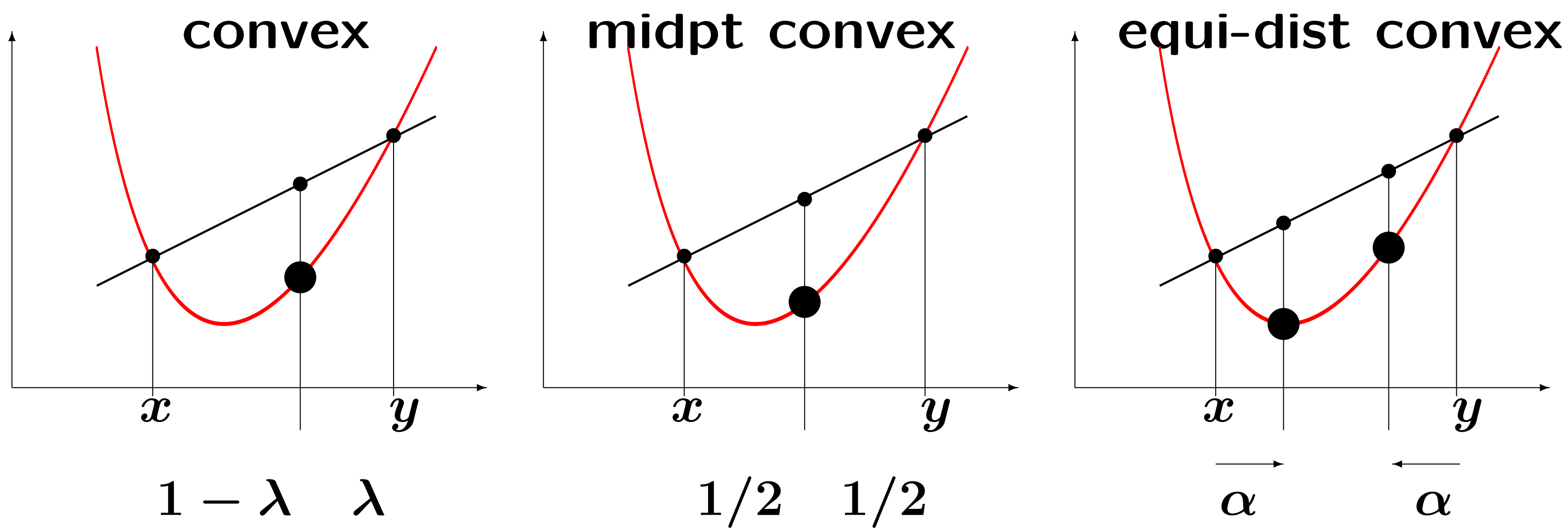
(Ordinary) Convexity

Convex: $\lambda f(x) + (1-\lambda) f(y) \geq f(\lambda x + (1-\lambda) y)$

$\Leftrightarrow$ Midpoint convex: $f(x) + f(y) \geq 2f\left(\frac{x+y}{2}\right)$

$\Leftrightarrow$ Equi-distance convex:

$$f(x) + f(y) \geq f(x - \alpha(x-y)) + f(y + \alpha(x-y))$$

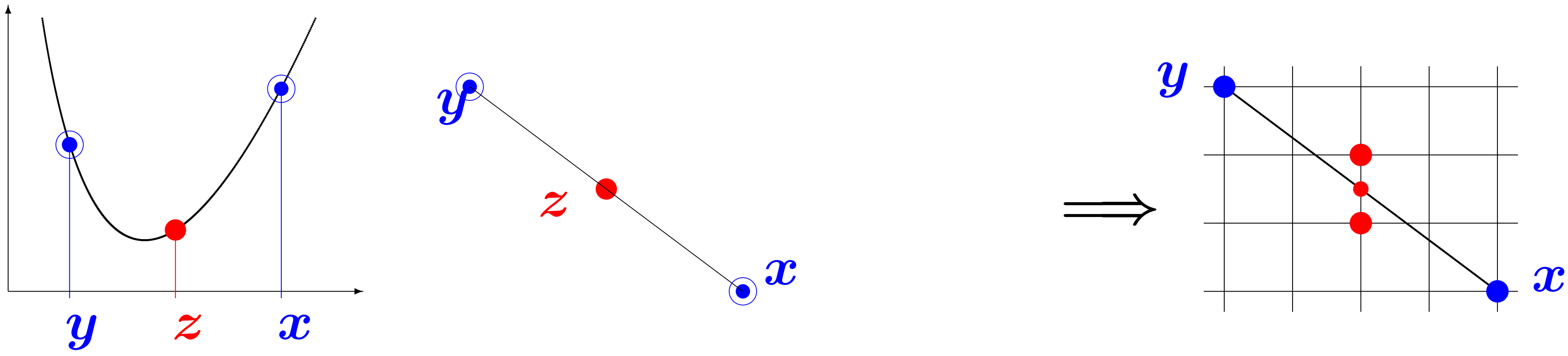


Two Kinds of Discrete Convexity

Cont. $\mathbf{R}^n \to \mathbf{R}$		Discrete $\mathbf{Z}^n \to \mathbf{R}$
midpt convex	discr $\longrightarrow$	
$\Updownarrow$		L$^\natural$-convex function
convex function		
$\Updownarrow$	discr	
equi-dist conv	$\longrightarrow$	
		M$^\natural$-convex function

L♮-convexity from Mid-pt-convexity

(Favati–Tardella 90, Murota 98, Fujishige–Murota 00)



Mid-point convex $(f : \mathrm{R}^n \to \mathrm{R})$:

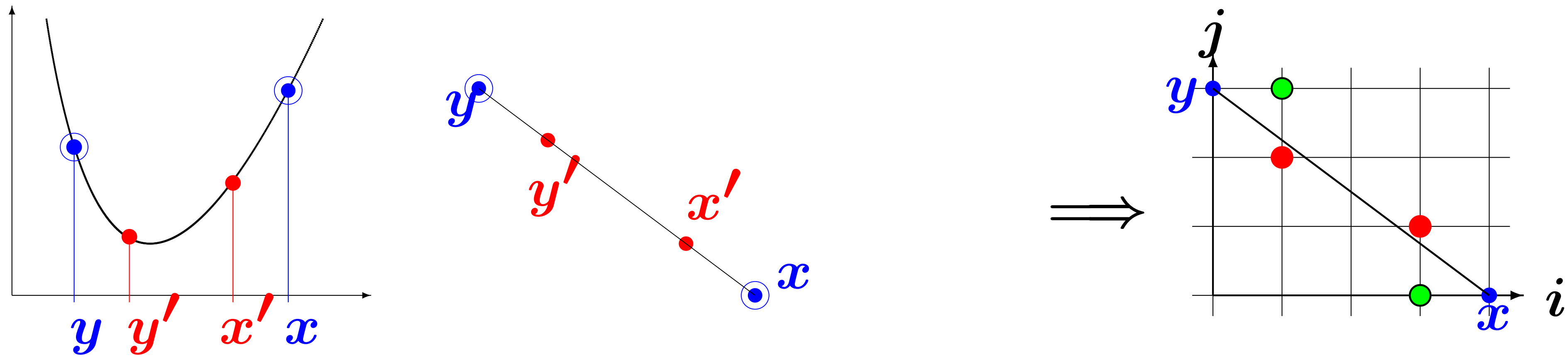
$$f(x) + f(y) \geq 2f\left(\frac{x+y}{2}\right)$$

$\Longrightarrow$ **L♮-convex** $(f : \mathrm{Z}^n \to \mathrm{R})$ **(L = Lattice)**

$$f(x) + f(y) \geq f\left(\left\lceil\frac{x+y}{2}\right\rceil\right) + f\left(\left\lfloor\frac{x+y}{2}\right\rfloor\right)$$

M♮-convexity from Equi-dist-convexity

(Murota 96, Murota–Shioura 99)



Equi-distance convex ($f : \mathrm{R}^n \to \mathrm{R}$):

$$f(x) + f(y) \geq f(x - \alpha(x-y)) + f(y + \alpha(x-y))$$

$\Longrightarrow$ **Exchange** ($f : \mathrm{Z}^n \to \mathrm{R}$) $\quad \forall x, y, \quad \forall i : x_i > y_i$

$$f(x) + f(y) \geq \min \left[f(x - e_i) + f(y + e_i), \right.$$

$$\left. \min_{x_j < y_j} \{ f(x - e_i + e_j) + f(y + e_i - e_j) \} \right]$$

M♮-convex function (M = Matroid)

Two Kinds of Discrete Convexity (Summary)

Cont. $\mathbb{R}^n \to \mathbb{R}$		Discrete $\mathbb{Z}^n \to \mathbb{R}$
midpt convex	$\xrightarrow{\text{discr}}$	discr midpt convex
$\Updownarrow$		L$^\natural$-convex function
convex function		
$\Updownarrow$		
equi-dist conv	$\xrightarrow{\text{discr}}$	simul. exchange
		M$^\natural$-convex function

M-L Conjugacy Theorem

Integer-valued discrete fn $f : \mathbf{Z}^n \to \mathbf{Z} \cup \{+\infty\}$

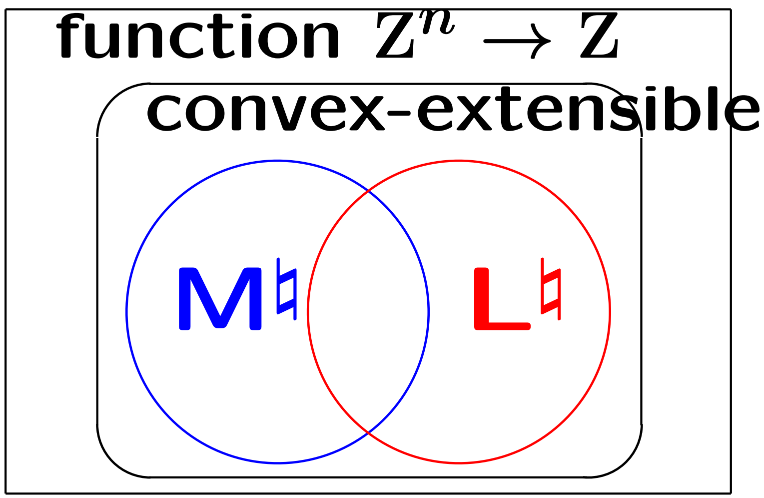
Legendre–Fenchel transform:

$$f^\bullet(p) = \max\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\} \qquad (p \in \mathbf{Z}^n)$$

M♮-convex and L♮-convex are conjugate

$$f \mapsto f^\bullet = g \mapsto g^\bullet = f$$

(Murota 98)



Fenchel-type Duality in DCA

f: M$^\natural$-convex, h: M$^\natural$-concave $(\mathbf{Z}^n \to \mathbf{Z})$

Legendre–Fenchel transform

$$f^\bullet(p) = \max\{\langle p, x\rangle - f(x) \mid x \in \mathbf{Z}^n\} \qquad (p \in \mathbf{Z}^n)$$

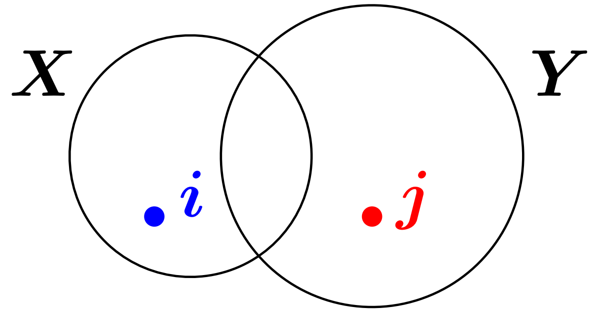
$$h^\circ(p) = \min\{\langle p, x\rangle - h(x) \mid x \in \mathbf{Z}^n\} \qquad (p \in \mathbf{Z}^n)$$

Fenchel-type duality thm (Murota 96, 98)

$$\min_{x \in \mathbf{Z}^n}\{f(x) - h(x)\} = \max_{p \in \mathbf{Z}^n}\{h^\circ(p) - f^\bullet(p)\}$$

M♮-concave Set Function

Set function $f : 2^N \to \mathrm{R} \cup \{-\infty\}$



Def: f is M♮-concave $\Longleftrightarrow$ (Murota–Shioura 99)

For any $X, Y \subseteq N$, $i \in X \setminus Y$,

$\exists\ j \in Y \setminus X$:

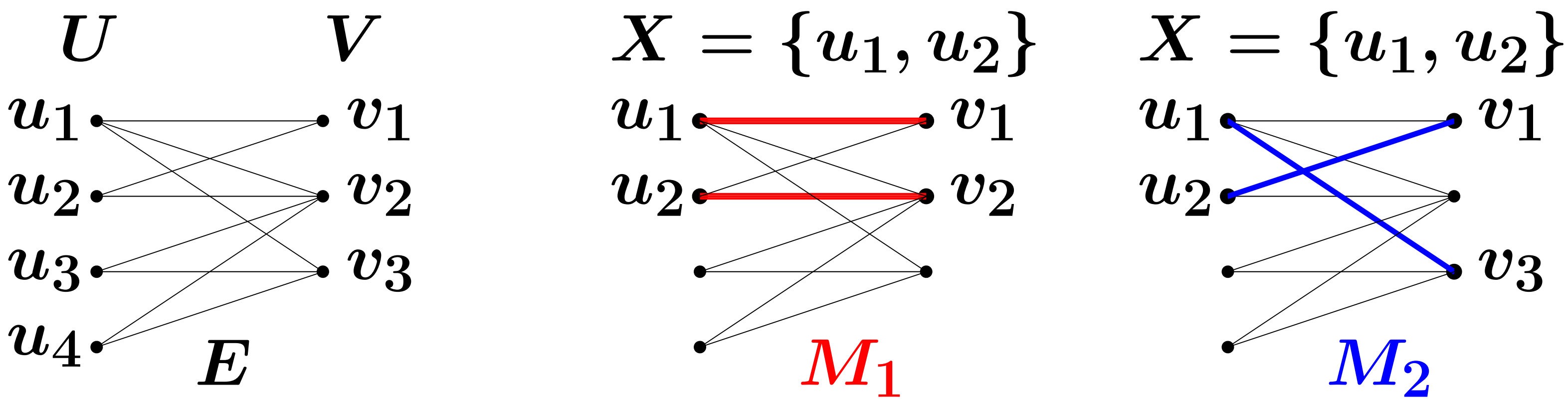
$$f(X) + f(Y) \leq \max \left[f(X - i) + f(Y + i), \max_{j \in Y \setminus X} \{ f(X - i + j) + f(Y + i - j) \} \right]$$

Def: f is valuated matroid $\Longleftrightarrow$ (Dress–Wenzel 90, 92)

For any $X, Y \subseteq N$, $i \in X \setminus Y$, $\exists\ j \in Y \setminus X$:

$$f(X) + f(Y) \leq \max_{j \in Y \setminus X} \{ f(X - i + j) + f(Y + i - j) \}$$

Matching / Assignment



Max weight for $X \subseteq U$ (w: given weight)

$$f(X) = \max\{ \sum_{e \in M} w(e) \mid M\text{: matching, } U \cap \partial M = X\}$$

Max-weight func f is M$^\natural$-concave (cf. Murota 1996)

- Proof by augmenting path

Assignment valuation is GS (cf. Hatfield-Milgrom 2005)

M$^\natural$-concave Set Functions

Quadratic: $f(X) = \sum_{i,j \in X} a_{ij}$ is M$^\natural$-concave

$$\Leftrightarrow a_{ij} \leq 0, \quad a_{ij} \leq \max(a_{ik}, a_{jk}) \ (\forall k \not\in \{i,j\})$$

Max value: $f(X) = \max\{w_i \mid i \in X\}$ [unit preference]

Symmetric concave: $f(X) = \varphi(|X|)$ (φ: concave)

Laminar concave: $f(X) = \sum_A \varphi_A(|X \cap A|)$ (φ_A: concave)

$\{A, B, \ldots.\}$: laminar $\Leftrightarrow$ $A \cap B = \emptyset$ or $A \subseteq B$ or $A \supseteq B$

M$^\natural$-concave Functions from Matroids

Matroid rank: $f(X) = r(X)$ (rank of X) (Fujishige 05)

Matroid rank sum: $f(X) = \sum \alpha_i r_i(X)$

$r_i \leftarrow r_{i+1}$ (strong quotient), $\alpha_i \geq 0$ (Shioura 12)

Weighted matroid: w: weight vector

$f(X) = \max\{w(Y) \mid Y\text{: indep} \subseteq X\}$ (Shioura 12)

Valuated matroid: $\omega : 2^V \to \underline{\mathrm{R}}$

$\Leftrightarrow \omega(X) = f(\chi_X)$ for some M-concave f

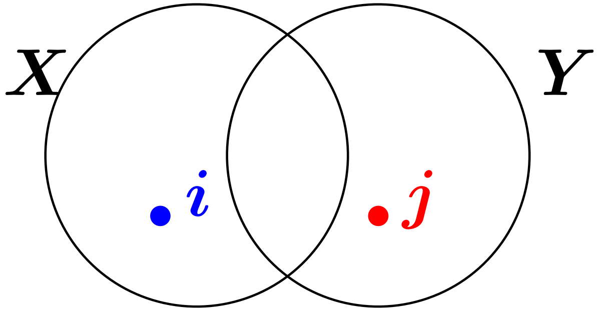
Main Result:

Multiple Exchange Property of M$^\natural$-concave Functions

Matroid Bases

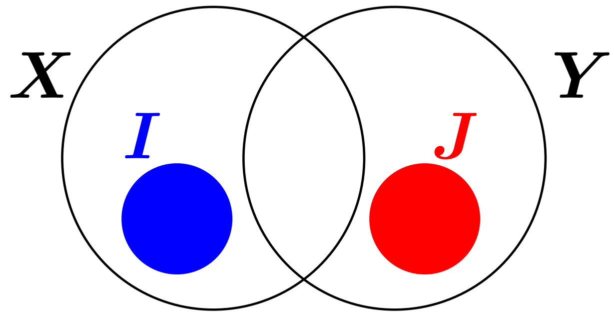
Simultaneous exchange for set family $\mathcal{B}$

For any $X, Y \in \mathcal{B}$, $i \in X \setminus Y$,
$\exists\ j \in Y \setminus X$:
$\underline{X - i + j} \in \mathcal{B}$, $\underline{Y + i - j} \in \mathcal{B}$



dist$= |X \Delta Y|$

Multiple Exchange for Matroids



Multiple exchange for set family $\mathcal{B}$

For any $X, Y \in \mathcal{B}$, $I \subseteq X \setminus Y$,
$\exists\ J \subseteq Y \setminus X$:
$\underline{X - I + J} \in \mathcal{B}$, $\underline{Y + I - J} \in \mathcal{B}$

Theorem: (Brylawski 1973, Green 1973, Woodall 1974)

Matroid bases have multiple exchange property

The essence lies in matroid partition

– constructive proof by successive exchanges
(Brylawski 1973, Green 1973)

– proof by duality (min-max) theorem
(Woodall 1974, McDiarmid 1975)

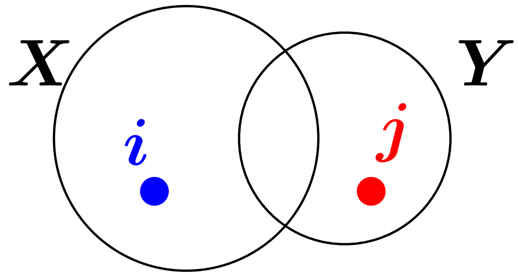
Multiple Exchange Properties $f : 2^N \to \mathbf{R}$

(M$^\natural$-EXC) For any X, Y and $i \in X \setminus Y$,

$$f(X) + f(Y) \leq f(X - i) + f(Y + i)$$

or there exists $j \in Y \setminus X$ s.t.

$$f(X) + f(Y) \leq f(X - i + j) + f(Y + i - j)$$

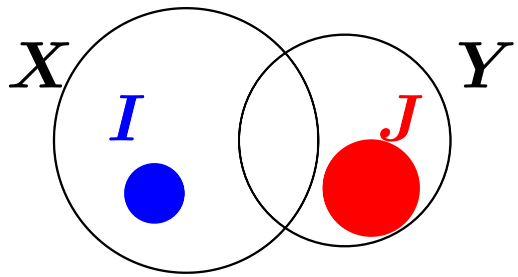


(M$^\natural$-mEXC)

For any X, Y and $I \subseteq X \setminus Y$,

there exists $J \subseteq Y \setminus X$ s.t.

$$f(X) + f(Y) \leq f(X - I + J) + f(Y + I - J)$$

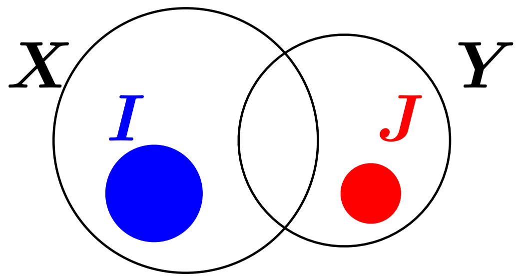


(M$^\natural$-mEXC-strong)

For any X, Y and $I \subseteq X \setminus Y$,

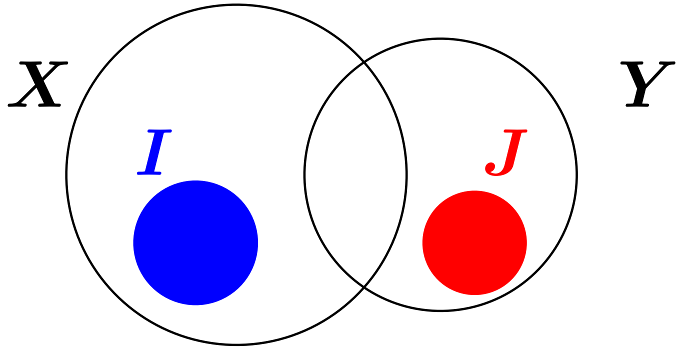
there exists $J \subseteq Y \setminus X$ s.t. $|J| \leq |I|$

$$f(X) + f(Y) \leq f(X - I + J) + f(Y + I - J)$$



Multi-Exchange for M$^\natural$-concave Functions

$f : 2^N \to \mathrm{R} \cup \{-\infty\}$



Multiple Exchange Property

For any $X, Y \subseteq N$, $\quad I \subseteq X \setminus Y$,
$\exists\, J \subseteq Y \setminus X$: $\qquad ((\ |J| \leq |I| \ ))$

$$f(X) + f(Y) \leq f(X - I + J) + f(Y + I - J)$$

Theorem (Murota 18)

(M$^\natural$-EXC) $\Leftrightarrow$ (M$^\natural$-mEXC) $\Leftrightarrow$ (M$^\natural$-mEXC-strong)

M$^\natural$-concave functions have multiple exchange property

K. Murota, Math. Oper. Res. (2018); Japan J. Indust. Appl. Math. (2018)

Proof of

$(\mathrm{M}^\natural\text{-EXC}) \Rightarrow (\mathrm{M}^\natural\text{-mEXC})$

The proof is based on duality and conjugacy

Key Tool: Fenchel-type Duality

Theorem (Murota 98, 03)

M$^\natural$-concave functions $f_1, f_2 : 2^N \to \mathrm{R} \cup \{-\infty\}$

Their (convex) conjugate $g_1, g_2 : \mathrm{R}^N \to \mathrm{R}$

$$g_i(q) = \max_{J \subseteq N} \{ f_i(J) - \sum_{j \in J} q_j \} \qquad (q \in \mathrm{R}^N)$$

Then

$$\max_{J \subseteq N} \{ f_1(J) + f_2(J) \} = \inf_{q \in \mathrm{R}^N} \{ g_1(q) + g_2(-q) \}$$

($= -\infty$ if $\mathrm{dom} f_1 \cap \mathrm{dom} f_2 = \emptyset$)

Matroid intersection formula is a special case:

$f_1(J) = |J|$ if J: M_1-indep, $f_2(J) = 0$ if J: M_2-indep

Proof Step 1: Fenchel duality

(Want to show $\boxed{\leq}$)

$$f(X) + f(Y) \boxed{\leq} \max_{J \subseteq Y \setminus X} \{f(X - I + J) + f(Y + I - J)\}$$

$$= \max_{J \subseteq Y_0} \{ \quad f_1(J) \quad + \quad f_2(J) \quad \}$$

(Fenchel dual) $$= \min_{q \in \mathrm{R}^{Y_0}} \{ \quad g_1(q) \quad + \quad g_2(-q) \quad \}$$

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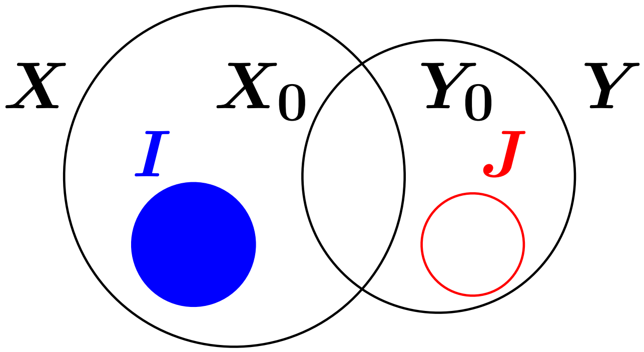
where $Y_0 = Y \setminus X$

$X_0 = X \setminus Y$



$$f_1(J) = f(X - I + J)$$

$$f_2(J) = f(Y + I - J)$$

$$g_1(q) = \max_{J \subseteq Y_0} \{f_1(J) - q(J)\}$$

$$g_2(q) = \max_{J \subseteq Y_0} \{f_2(J) - q(J)\}$$

(conjugate of f_1, f_2)

Proof Step 2: Conjugate Functions

Compute $g_1(q) + g_2(-q)$ to show $\geq \; f(X) + f(Y)$

$$g_1(q) = \max\{f(X - I + J) - q(J) \mid J \subseteq Y_0\}$$

$$g_2(q) = \max\{f(Y + I - J) - q(J) \mid J \subseteq Y_0\}$$

$$g(p) \;= \max\{f(Z) - p(Z) \mid Z \subseteq N\}$$

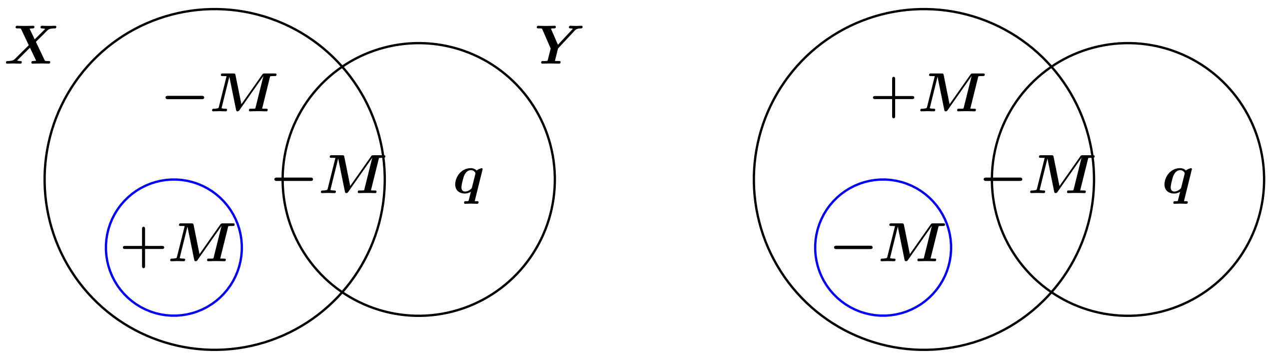
$M > 0$: large

$$g_1(q) \;=\; g(p^{(1)}) - M(|X_0 \setminus I| + |X \cap Y|)$$

$$g_2(-q) = g(p^{(2)}) - M(|I| + |X \cap Y|) + q(Y_0)$$

const

where $p^{(1)}$, $p^{(2)}$ are defined by



$p^{(1)}$ $\qquad$ $p^{(2)}$

Proof Step 3: L$^\natural$-convexity

Conjugate of M$^\natural$ is L$^\natural$-convex, and hence submodular

$$g_1(q) + g_2(-q)$$

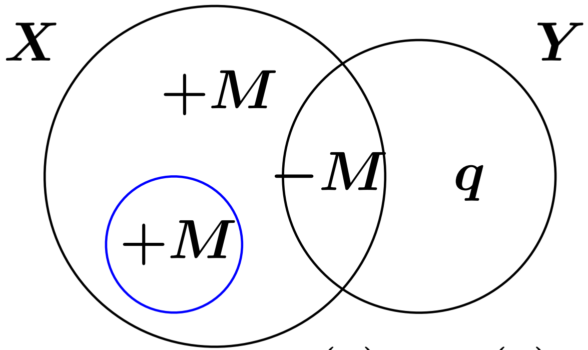
$$= g(p^{(1)}) + g(p^{(2)}) \qquad - M(|X| + |X \cap Y|) + q(Y_0)$$

$$\geq g(p^{(1)} \vee p^{(2)}) + g(p^{(1)} \wedge p^{(2)}) \qquad - M(|X| + |X \cap Y|) + q(Y_0)$$

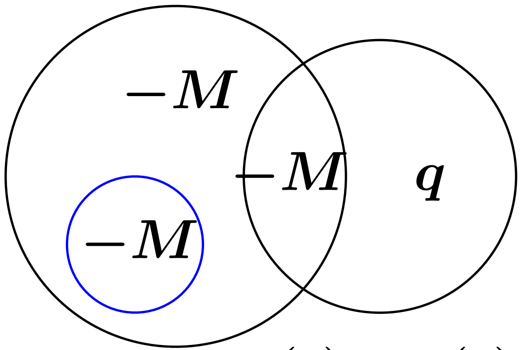
Recall $g(p) = \max\{f(Z) - p(Z)\}$ to observe

$$g(p^{(1)} \vee p^{(2)}) \geq f(Y) - q(Y_0) + M|X \cap Y| \qquad \text{(by } Z = Y\text{)}$$

$$g(p^{(1)} \wedge p^{(2)}) \geq f(X) + M|X| \qquad \text{(by } Z = X\text{)}$$



$p^{(1)} \vee p^{(2)}$



$p^{(1)} \wedge p^{(2)}$

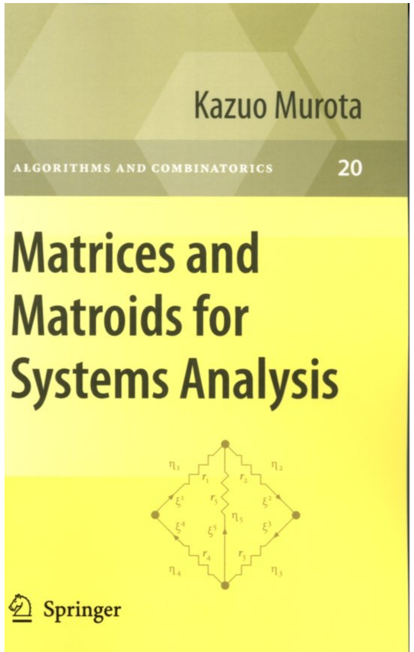
Therefore, $g_1(q) + g_2(-q) \geq f(X) + f(Y)$ (qed)

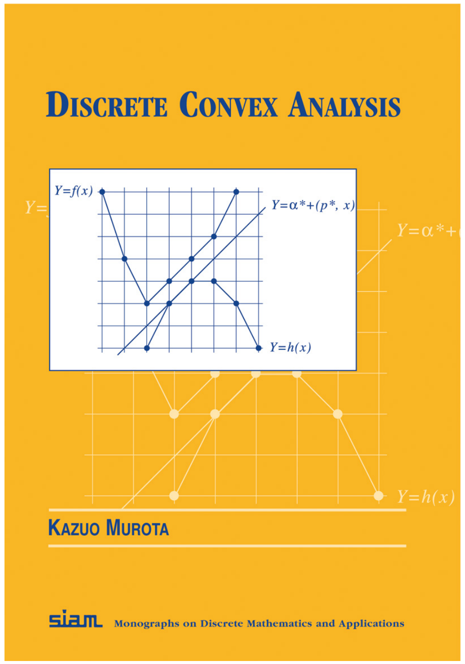
Books

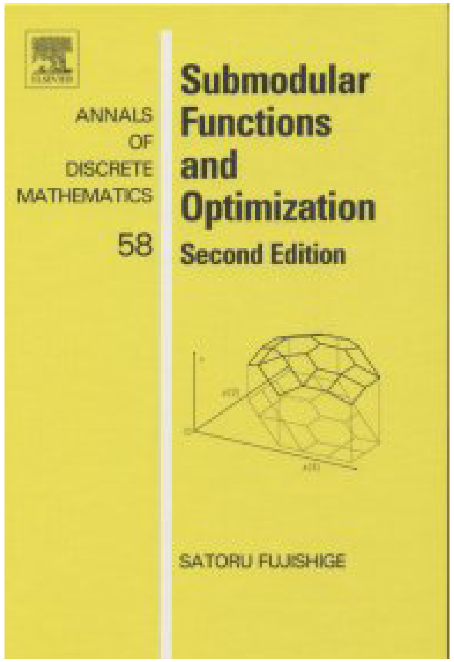
Murota: Matrices and Matroids for Systems Analysis, Springer, 2000/2010 (Chap.5: valuated matroids)

Murota: Discrete Convex Analysis, SIAM, 2003

Fujishige: Submodular Functions and Optimization, 2nd ed., Elsevier, 2005 (Chap. VII)







Survey/Slide/Video/Software

[Survey]

Murota: Recent developments in discrete convex analysis (Research Trends in Combinatorial Optimization, Bonn 2008, Springer, 2009, 219–260)

Murota: Discrete convex analysis: A tool for economics and game theory. Journal of Mechanism and Institution Design 1, 151–273 (2016)

[Slide]

http://www.comp.tmu.ac.jp/kzmurota/publist.html#DCA

[Video]

https://smartech.gatech.edu/xmlui/handle/1853/43257/

https://smartech.gatech.edu/xmlui/handle/1853/43258/

[Software] DCP (Discrete Convex Paradigm)

https://cs.kwansei.ac.jp/~tutimura/DCP/

END