Structure of the sets of Nash equilibria of finite games
Applications to the complexity of some decision problems in game theory.

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Finite games

Finite game Γ:

- N players, $N \geq 2$.
- For every player i, a finite action set A^i with cardinal k^i.
- For every player i, a payoff function $g^i : \Pi A^i \to \mathbb{R}$.
- We'll often assume that the game is with integral payoff $g^i : \Pi A^i \to \mathbb{Z}$.
- Players simultaneously choose a mixed action ie a probability x^i dans $\Delta(A^i) = \Delta_{k^i}$ and maximize their payoff expectation.
- Fondamental concept of solution : Nash equilibrium. $(x^1, \cdots, x^n)$ NE iff for all i,

$$x^i \in Argmax g^i(x^1, \cdots, x^{i-1}, \cdot, x^{i+1}, \cdots, x^n).$$

- $NE(\Gamma)$ set of Nash equilibria of Γ ; $NEP(\Gamma) \subset \mathbb{R}^N$ set of vectors of Nash equilibrium payoffs.

Set of equilibria and equilibrium payoffs

$$
\begin{aligned}
NE(\Gamma) &= \{x \in \Pi_i \Delta_{k^i} | \forall i \in N, \forall 1 \leq l \leq k^i, x_l = 0 \text{ or } g^i(a^i_l, x^{-i}) = g^i(x)\} \\
NEP(\Gamma) &= g(NE(\Gamma)) \subset \mathbb{R}^N
\end{aligned}
$$

These sets are always

- Compact (clear)
- Nonempty (Nash 1950)
- Semialgebraic (clear for NE, consequence of Tarski-Seidenberg for NEP)

Recall that a finite dimensional set is semialgebraic if one can write it as union and intersection of finitely manysets of the form $\{P_j(x) \leq 0\}$ or $\{P_j(x) < 0\}$ where the P_j are multivariate polynomials.

We will say that a set is $\mathbb{Z}-$semialgebraic if the coefficients of the P_j are in $\mathbb{Z}$.

Inverse problem

Given a set E, when can we find a game such that E is its set of equilibria (or equilibrium payoffs) ?
We will show that for example:

Proposition (V. 2021)

If $N \geq 3$, $E \subset \mathbb{R}^N$ is $NEP(\Gamma)$ for some finite game Γ with N players (resp. and with integral payoff) iff E is nonempty, compact and semialgebraic (resp. $\mathbb{Z}$-semialgebraic).

That is for equilibrium payoffs the converse of previous slide is true as soon as $N \geq 3$.

Motivations

- Theoretical : “equivalence” between games and semialgebraic sets. Being the set of NEP of a game is a canonical certificate of nonemptiness for semialgebraic sets.
- Analogy with the link between convex functions and convex sets. Minimizing a function on a semialgebraic set can be rewritten as a game theoretic question.
- Implications on the complexity or computability of some decision problems on games.
  Remark : I will only consider decision problems for which the answer is either “Yes” or “No”.

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Structure of equilibria

- Every 2-player game with integral payoff admits an equilibrium where each player plays each action with a rational probability.
- Every 2-player game with integral payoff admits an equilibrium with rational payoff for each player.
- Conversely, for every $e \in \mathbb{Q}^2$ there exists a 2-player game with integral payoff and with a unique equilibrium, of payoff e.
- A set $E \in \mathbb{R}^2$ is $NEP(\Gamma)$ of some 2-player game (resp. with integral payoff) iff it is a finite union of (non necessarily disjoints) sets of the form $[a,b] \times [c,d]$ (resp. with a,b,c,d in $\mathbb{Q}$) (Lehrer Solan Viossat 2011).

Complexity in the 2 player case

Typically, non trivial decision problems about 2 player games are **NP**-complete.
For example (Gilboa Zemel 1989) :

- Is there at least 2 Nash equilibrium in this game ?
- Is there a Nash equilibrium with positive payoff for everyone ?
- Is there a Nash equilibrium in which each player plays his first action with probability 0 ?
- Is there a Nash equilibrium in which each player plays his first action with positive probability ?

Reminder on NP

Decision problem is in $\mathbf{NP}$ if one can solve it in polynomial time (in the data) with a **non deterministic** Turing machine.
Equivalently : there exists an integer k, a set of certificate Y, and a deterministic Turing machine $M(x,y)$, which anwers "yes" or "no" (in polynomial time) to an input x and a certificate $y \in Y$ such that

- If the answer to the initial problem is no for x then $M(x,y)$ answers "no" for any certificate y.
- If the answer to the initial problem is yes for x of size n then $M(x,y)$ answers "yes" for at least one certificate y of size less than n^k.

We know that $\mathbf{P} \subset \mathbf{NP} \subset \mathbf{EXPTIME}$ with at least one strict inclusion.

NP-hard and **NP**-complete problems

We say that a decision problem A is (many-one polynomialy) reducible to B if there exists f computable in polynomial time such that $A(x)$ is true iff $B(f(x))$ is true. Hence solving A is at least as simple as solving B.
A decision problem B is **NP**-hard if all problems in **NP** are reducible to B, and is **NP**-complete if it is both in **NP** and **NP**-hard.
Exemples of **NP**-complete problems : traveling salesman, 3-SAT, 0-sum subset, cliques of some size in a graph,$\cdots$

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Review of literature

A 3-player game with integral payoff may have only equilibria with non rational payoffs (Nash 1951) .
But existence of at least one equilibrium with an algebraic payoff for each player (by Tarski-Seidenberg).
Partial inverse result

- Bubelis '79 : for every algebraic number e there is a 3-player game with integral payoff, which admits a unique Nash equilibrium, in which the payoff of player 1 is e.

Review of literature

The set of equilibria (resp. equilibrium payoffs) of a 3-player game is nonempty semialgebraic and compact.

- Datta '03 : every algebraic set is isomorphic to the set of completely mixed equilibrium of some 3-player game.
- Balkenborg-Vermeulen '14 : every semialgebraic compact connected set is homeomorphic to one connected component of the set of equilibria of some game.
- Levy '16, Viossat-V '16 : every semialgebraic compact nonempty set in $\mathbb{R}^N$ is the projection of the set of equilibrium payoffs of some game with stricly more than N-players.

Remark : in each case "universality" only modulo something (isomorphism, projection, ...).

Structure of equilibria

New inverse results

- For any e in $\mathbb{A}^N$ (where $\mathbb{A}$ stands for the set of real algebraic numbers) there exists an integral payoff N-player game with a unique equilibrium of payoff e. (V. 2021, generalizes Bubelis 1979).
- $E \subset \mathbb{R}^N$ is $NEP(\Gamma)$ for some N-player game Γ (resp. with integral payoff) iff it is compact nonempty and semialgebraic (resp. $\mathbb{Z}$-semialgebraic). (V. 2021)
- If $E \subset [0,1[^N$ is nonempty, compact and semialgebraic (resp. $\mathbb{Z}$-semialgebraic) then there exists Γ such that $e \in E$ iff there exists an equilibrium of Γ in which each player i plays his first action with probability e_i. (V. 2021)

Constructive !

Constructive proofs, size of the games polynomial in the size of the problem. More precisely, given :

- $E \subset \mathbb{R}^N$ union and intersection of K sets of the form $\{P_k(x) \leq 0\}$ with P_k polynomials of degree at most $\leq d$ in each variable, and with integrak coefficients less than $\leq M$.
- An integral bound C.
- For each player an algebraic number e^i, unique zero of a polynomial of degree $\leq d$ is some given interval of length $\geq 1/M$.

one constructs Γ, such that $NEP(\Gamma) = \{e\} \cup \left(E \cap [-C, C]^N\right)$, in which the number of actions of each player and the bit length of the payoffs are polynomial in $K, d, N, \ln(M), \ln(C)$.

Complexity results

These results on structure implies that, for 3 players (or more) the decision problems on games with integral payoffs are typically $\exists\mathbb{R}$-complete.
For example

- Is there at least 2 Nash equilibrium in this game (Bilo Mavronicolas 2016) ?
- Is there a Nash equilibrium with payoff greater than a given rational number for each player (Bilo Mavronicolas 2016) ?
- Is there a Nash equilibrium with payoff greater than a given algebraic number for each player (V. 2021) ?
- A set F, $\mathbb{Z}$-semi algébrique being fixed (F neither empty nor the whole $\mathbb{R}^N$) : "Is there a Nash equilibrium with payoff in F" ? (V. 2021) ?
- Is there a Nash equilibrium in which each players plays his first action with fixed (algebraic) probability (V. 2021) ?

The complexity class $\exists\mathbb{R}$

$\exists\mathbb{R}$ is the complexity class of deciding whether a $\mathbb{Z}$-semialgebraic set is nonempty.

- A decision problem is in $\exists\mathbb{R}$ if one can reduce it to deciding whether a $\mathbb{Z}$-semialgebraic is nonempty.
- A decision problem A is $\exists\mathbb{R}$-hard if decing whether a set $\mathbb{Z}$-semialgebraic is nonempty can be reduced to A.
- A decision problem A is $\exists\mathbb{R}$-complete if it is both in $\exists\mathbb{R}$ and $\exists\mathbb{R}$-hard.

By Canny (1988) one has $\mathbf{P} \subset \mathbf{NP} \subset \exists\mathbb{R} \subset \mathbf{PSPACE}$ with at least one strict inclusion.
Some examples : the art gallery problem (guarding a polygon with k gards) ; realisation of a graph with edges of fixed length.

Structure results imply complexity ones

We show for example that decidong whether an game with integral payoff has an equilibrium in which the first player has a payoff of $\sqrt{2}$ is $\exists\mathbb{R}$-complete.

a) In $\exists\mathbb{R}$. For a given N player game, consider the $\mathbb{Z}$-semialgebraic $E \subset \mathbb{R}^N$ of its equilibria. Payoff functions are polynomial, hence we get a $\mathbb{Z}$-semialgebraic set $E' = \{(e,f)\}$ where e is an equilibrium and f the corresponding equilibrium payoff. The answer is “yes” iff the set

$$E' \cap \left(\mathbb{R}^? \times \left(\{\sqrt{2}\} \times \mathbb{R}^{N-1}\right)\right))$$

is nonempty.

Structure results imply complexity ones

b) $\exists\mathbb{R}$-Hard. Assume that one can determine whether a game has an equilibrium in wich the first player has a payoff of $\sqrt{2}$. Let then E be a semialgebraic subset of $\mathbb{R}^N$. One can prove than nonemptiness of E does not change by intersecting it with a ball of some radius R, and replacing any "> 0" par $\geq \varepsilon$, with R and ε of reasonable size with respect to the size of E. Hence we can assume E compact wlog.
Let $E' = (\{\sqrt{2}\} \times E) \cup \{0\}^{N+1}$. E' is compact, semialgebraic and nonempty hence one can construct Γ of a size polynomial in the size of E' (hence of E) such that $NEP(\Gamma) = E'$. And E is nonempty iff Γ has an equilibrium in which the first player gains $\sqrt{2}$.

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We want to show

Proposition (V. 2021)

If $E \subset [0,1[^N$ is nonempty compact and $\mathbb{Z}$-semialgebraic, then there exists Γ with integral payoff such that $e \in E$ iff there is a Nash equilibrium in Γ in which each player i plays its first action with probability e_i.

How to multiply

Let p^i_j be the probability that player i plays its action j. Assume that among all actions of player 1 we constructed two particular ones such that

- One gives a payoff of 1 iff J_2 plays its k-th action, and else a payoff of 0.
- The other gives a payoff of 1 iff J_2 plays its i-th action et J_3 plays its j-th action, and else a payoff of 0.

Assume that for some reason we know that in any equilibrium these two actions are played with positive probability. Then they should give the same payoff thus in any equilibrium,

$$p^2_k = p^2_i \times p^3_j$$

Constructing a monomial

Assume that among all actions of player 1 we constructed two particular ones such that

- One gives a payoff of 1 1 si J_2 plays its i-th action, and else a payoff of 0.
- The other gives a payoff of 1 iff J_3 its j-th action, and else a payoff of 0.

Assume that for some reason we know that in any equilibrium these two actions are played with positive probability. Then they should give the same payoff thus in any equilibrium

$$p_i^2 = p_j^3$$

Combining with the previous slide we get

$$p_k^2 = (p_i^2)^2$$

Constructing polynomial inequalities

Iterating these kinds of arguments one get for example than in any equilibrium $p^2_k = (p^1_l)^3(p^2_i)^2(p^3_j)$ and $p^2_m = (p^1_l)^4(p^2_i)(p^3_j)^2$.
Assume that among all actions of player 1 we constructed two particular ones such that

- One always gives a payoff of 5.
- The other gives a payoff of 1 if J_2 plays its k-th action, of 3 if J_2 plays its m-th action, and else a payoff of 0.

Assume that for some reason we know that in any equilibrium the second action is played with positive probability. Then it has to give a higher payoff than the first action, hence in any equilibrium

$$(p^1_l)^3(p^2_i)^2(p^3_j) + 3(p^1_l)^4(p^2_i)(p^3_j)^2 - 5 \geq 0$$

Issues

- Doing intersection and unions.
- Being certain that the action we would want to be played with positive probability are indeed played with positive probability.

A word on notations

- Actions denoted by capitals : A_2^1 is an action of J^1 for example.
- A small letter represents the probability with which the corresponding action is played. For example for a fixed profile σ we write a_2^1 pour $\sigma^1(A_2^1)$.
- Games are not given in a matrix (normal) form : the payoff of each player is directly given as a multiaffine function of the actions of the other player.

Example

Instead of writing the following game in the usual normal form

$$\begin{array}{cc} & \begin{array}{cc} A_1^2 & A_2^2 \end{array} \\ \begin{array}{c} A_1^1 \\ A_2^1 \end{array} & \left(\begin{array}{cc} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{array} \right) \\ & A_1^3 \end{array} \qquad \begin{array}{c} \begin{array}{cc} A_1^2 & A_2^2 \end{array} \\ \left(\begin{array}{cc} (0,0,0) & (0,1,1) \\ (0,0,1) & (2,1,1) \end{array} \right) \\ A_2^3 \end{array}$$

we will write

$$\begin{array}{rcl} g^1(A_1^1, \sigma^{-1}) & = & g^2(A_1^2, \sigma^{-2}) = g^1(A_1^3, \sigma^{-3}) = 0 \\ g^1(A_2^1, \sigma^{-1}) & = & 2\sigma^2(A_2^2)\sigma^3(A_2^3) \\ g^2(A_2^2, \sigma^{-2}) & = & \sigma^3(A_2^3) \\ g^2(A_2^3, \sigma^{-3}) & = & 1 - \sigma^1(A_1^1)\sigma^2(A_1^2) \end{array}$$

Example

Instead of writing the following game in the usual normal form

$$
\begin{array}{cc}
 & \begin{array}{cc} A_1^2 & A_2^2 \end{array} \\
\begin{array}{c} A_1^1 \\ A_2^1 \end{array} & \left(\begin{array}{cc} (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) \end{array}\right) \\
 & A_1^3
\end{array}
\qquad
\begin{array}{c}
\begin{array}{cc} A_1^2 & A_2^2 \end{array} \\
\left(\begin{array}{cc} (0,0,0) & (0,1,1) \\ (0,0,1) & (2,1,1) \end{array}\right) \\
A_2^3
\end{array}
$$

ou tout simplement

$$
\begin{aligned}
g^1(A_1^1) &= g^2(A_1^2) = g^1(A_1^3) = 0 \\
g^1(A_2^1) &= 2a_2^2 a_2^3 \\
g^2(A_2^2) &= a_2^3 \\
g^2(A_2^3) &= 1 - a_1^1 a_1^2
\end{aligned}
$$

Example of a set

We prove the proposition
If $E \subset [0,1[^N$ is nonempty compact and $\mathbb{Z}$-semialgebraic, then there exists Γ with integral payoff such that $e \in E$ iff there is a Nash equilibrium in Γ in which each player i plays its first action with probability e_i.
on an example.

- Take $N = 3$ and $E = \{(e_1)^2 + (e_2)^2 + (e_3)^2 + e_1e_2 + e_1e_3 + 2e_2e_3 \leq \frac{1}{400}\} \cap \mathbb{R}^3_+$.
- Semialgebraic, in $[0,1[^3$, closed and nonempty : $(0,0,0) \in E$.

Sets of action

Each player i has two families of actions :

- 11 actions denoted with the letter X : X^i_*, X^i_0, et $X^i_{j,k,l}$ for j, k et l natural integers such that $1 \leq j+k+l \leq 2$. Called "unknowns", all give a payoff identically 0.
- Actions denoted with the letter Y. Called "constraints"? Their payoff, depending only on the probabilities x, is to be constructed.

We say an equilibrium is adapted if all strategies Y of each player are played with probability 0.
We construct Γ such that :

- For any $e \in E$, there exists a unique adapted equilibrium such that $(x^1_*, x^2_*, x^3_*) = e$.
- For any $e \notin E$, there is no adapted equilibrium such that $(x^1_*, x^2_*, x^3_*) = e$.
- There is a unique non adapted equilibrium, in which $(x^1_*, x^2_*, x^3_*) = (0,0,0) \in E$.

First part : adapted equilibria

We will add contraints such that $e \in E$ iff there is an adapted equilibrium such that $x_*^i = e^i$ pour tout i.
Idea : in an an adapted equilibrium, the payoff of each player is 0 and the strategies Y are not played. Their payoff, depending on the x, is thus nonpositive which gives inequalities.

Initialisation constraints

Role : ensuring that in any adapted equilibrium we have $x^i_{j,k,l} = (x^1_*)^j (x^2_*)^k (x^3_*)^l$ for $j+k+l = 1$, that is $x^i_{1,0,0} = x^1_*$; $x^i_{0,1,0} = x^2_*$; $x^i_{0,0,1} = x^3_*$.

Add 8 strategies for player 1 with payoff

- $\pm(x^2_{0,1,0} - x^2_*)$
- $\pm(x^3_{0,1,0} - x^2_*)$
- $\pm(x^2_{0,0,1} - x^3_*)$
- $\pm(x^3_{0,0,1} - x^3_*)$

Same for player 2 and 3.

Induction constraints

Role : ensuring that in any adapted equilibrium we have $x^i_{j,k,l} = (x^1_*)^j (x^2_*)^k (x^3_*)^l$ for $j+k+l=2$.
Add strategies for player 2 with payoff

- $\pm(x^1_{2,0,0} - x^1_{1,0,0} x^3_{1,0,0})$, which ensures that $x^1_{2,0,0} = (x^1_*)^2$ in any adapted equilibrium
- $\pm(x^1_{1,1,0} - x^1_{1,0,0} x^3_{0,1,0})$, which ensures that $x^1_{1,1,0} = x^1_* x^2_*$ in any adapted equilibrium
- ...

Semialgebraic constraints

Role : ensuring that in any adapted equilibrium we have $(x^1_*, x^2_*, x^3_*) \in E$.

For each player add a strategy with payoff

$x^{i-1}_{2,0,0} + x^{i-1}_{0,2,0} + x^{i-1}_{0,0,2} + x^{i-1}_{1,1,0} + x^{i-1}_{1,0,1} + 2x^{i-1}_{0,1,1} - \frac{1}{400}$

Because of the following slides in any adapted equilibrium the payoff of this strategy, which has to be nonpositive, is

$(x^1_*)^2 + (x^2_*)^2 + (x^3_*)^2 + x^1_* x^2_* + x^1_* x^3_* + 2x^2_* x^3_* - \frac{1}{400})$

adapted equilibria

By construction, in any adapted equilibrium $x_* \in E$.
Conversely, if $x_* \in E$,

- Fix* $x^i_{j,k,l} = (x^1_*)^j (x^2_*)^k (x^3_*)^l$ for $j+k+l \geq 2$
- Let $x^i_0 = 1 - x^i_* - \sum x^i_{j,k,l}$.
- Since $x^i_0 \geq 0$ the strategy is well defined
- All constraints are satisfied

hence we get an adapted equilibrium .

Second part : other equilibria

There could be other equilibria... We did not ever check that E was nonempty !
Add a last contraint Y_*^i with payoff $K(1 - x_0^{i-1} - x_*^{i-1} - \sum x_{j,k,l}^{i-1})$ for K large enough.

- Payoff is 0 in any adapted equilibrium, so does not change anything to the previous construction.
- In any other equilibrium, $y_*^i = 1$ for all i.
- Hence unique additional equilibrium in which $x_* = (0,0,0) \in E$!

General case

More complex in general because

- $N \geq 3$
- Intersections : no major difficulties
- Unions : more difficult
- Second part in general (when $0 \notin E$) : more difficult part of the construction. Not even any reason for E to contain a point with rational coordinates...

Perspectives

- Same kind of results with parametrized games ? For a given semialgebraic correspondance f from $\mathbb{R}^m$ to $\mathbb{R}^N$, is there a N-player game $\Gamma(k)$ whose payoffs depends on the parameter k such that $NEP(\Gamma(k)) = f(k)$?
- Application to dynamic games (stochastic games for example), in which the dynamic programing operator is given par a static game in which some payoffs depend on the estimation of the future.
- Same kind of questions for games with an infinite number of actions (for example sets of actions are interval and payoffs are semialgebraic functions).

Merci pour votre écoute

Merci !

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