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Homogénéisation de modèles cinétiques dans des espaces des phases étendus

THÈSE

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Introduction

Illud in his quoque te rebus cognoscere auemus,
corpora cum deorsum rectum per inane feruntur
ponderibus propriis, incerto tempore ferme
incertisque locis spatio depellere paulum,
tantum quod momen mutatum dicere possis.

Lucrèce, *De rerum natura*, II, 216-220

INTRODUCTION

1. Cadre général : l'équation de Boltzmann linéaire avec une distribution périodique de trous

1.1. **L'équation de Boltzmann linéaire.** Le travail présenté ici est essentiellement consacré au problème de l'homogénéisation de l'équation de Boltzmann linéaire dans un domaine perforé. On commence par rappeler succinctement le modèle mathématique. L'équation de Boltzmann linéaire s'utilise dans divers contextes, par exemple le transport des neutrons dans un matériau fissile, ou bien la diffusion d'un gaz léger dans un gaz lourd — voir le paragraphe 11 dans [18]. Le gaz léger est vu comme une population de particules — des masses ponctuelles — décrite par sa fonction de distribution $f \equiv f(t,x,v)$, densité des particules se trouvant à la position $x \in \Omega$, à l'instant $t \in \mathbb{R}_+$, et se déplaçant à la vitesse $v \in V$. On néglige les collisions entre les molécules du gaz léger, le parcours du point matériel ne dépend donc que du milieu où il se déplace. Le changement de vitesse de la masse ponctuelle est régi par deux fonctions positives $\sigma \equiv \sigma(x,v)$ et $k \equiv k(v,w)$ avec

$$k(v,w) = k(w,v) \text{ et } \int_{v\in V} k(v,w)dv = 1.$$

La quantité $\sigma(x,v)$ représente la fréquence de collision à la position x et à la vitesse v tandis que k est la probabilité pour une molécule de gaz léger d'avoir la vitesse v après collision sachant que sa vitesse avant collision est w.

L'équation de Boltzmann linéaire régissant l'évolution du gaz est donc

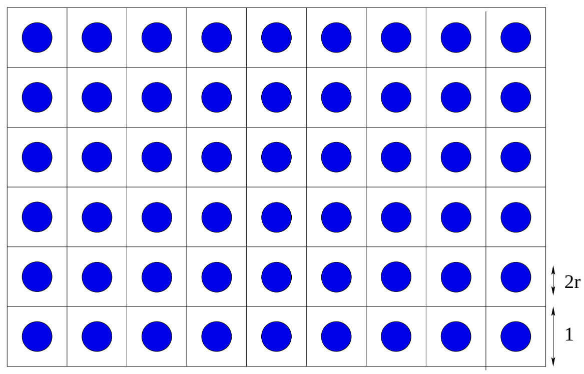
$$(\partial_t + v\cdot\nabla_x)f(t,x,v) + \sigma(x,v)f(t,x,v) = \sigma(x,v)Kf(t,x,v),$$

avec

$$Kf(t,x,v) := \int k(v,w)f(t,x,w)dw.$$

L'ensemble V des vitesses admissibles peut être soit $\mathbb{R}^N$, soit une sphère (par invariance galiléenne et après un choix convenable des unités, on peut toujours se ramener au cas où $V = \mathbb{S}^{N-1}$) pour un gaz de particules monocinétiques, soit une partie finie de $\mathbb{R}^N$ (modèle cinétique à vitesses discrètes).

1.2. **Domaine spatial avec distribution périodique de trous.** Le milieu dans lequel les particules évoluent est l'espace $\mathbb{R}^N$ perforé périodiquement.



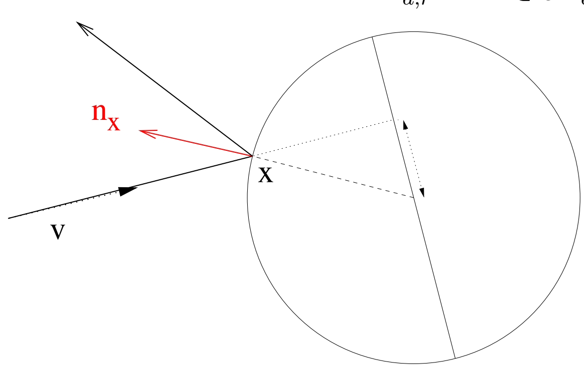
Le terme de « trou » désigne une boule complètement absorbante, au sens où toute particule la rencontrant disparaît à jamais. Le milieu $Z_{d,r}$ est alors décrit par :

$$Z_{d,r} := \left\{x \in \mathbb{R}^N \,\middle|\, \mathrm{dist}\left(x, d\mathbb{Z}^N\right) > r\right\},$$

où r est le rayon des trous et d la distance entre noeuds voisins du réseau périodique formé par les centres des trous. Le fait que les trous « absorbent » les particules est exprimé par la condition au bord pour la fonction de distribution :

$$f(t,x,v) = 0, \quad (t,x,v) \in \mathbb{R}_+^* \times \partial Z_{d,r} \times V, \quad \text{dès que } n_x \cdot v > 0,$$

où n_x est le vecteur normal entrant dans $Z_{d,r}$ en $x \in \partial Z_{d,r}$.



Enfin on supposera qu'à l'instant $t = 0$, on connaît la distribution initiale de la population particulaire f^{in}. Nous avons donc le problème de Cauchy avec condition au bord suivant :

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \sigma(f - Kf) = 0, & (x,v) \in Z_{d,r} \times V, \quad t > 0, \\ f(t,x,v) = 0, \text{ si } v \cdot n_x > 0, & (x,v) \in \partial Z_{d,r} \times V, \quad t > 0, \\ f(0,x,v) = f^{in}(x,v), & (x,v) \in Z_{d,r} \times V. \end{cases}$$

1.3. **Le problème.** Le problème d'homogénéisation consiste à décrire l'évolution de la population de particules de distribution f lorsque le nombre de trous par unité de volume est grand ($d \gg 1$) et que la taille des trous est en même temps petite ($r \ll 1$); et de trouver si possible une équation « équivalente » posée sur l'espace euclidien $\mathbb{R}^N$ sans les trous régissant cette évolution.

Considérons d'abord le cas sans collision, $\sigma = 0$, déjà étudié par E. Caglioti et F. Golse dans [8], c'est-à-dire le cas où les particules ne se déplacent qu'en ligne droite. Posons dorénavant $d = \varepsilon$ et $r_\varepsilon = \varepsilon^\gamma$ le rayon des trous. Alors le milieu extérieur est

$$Z_\varepsilon := \left\{ x \in \mathbb{R}^N \,\middle|\, \mathrm{dist}\left(x, \varepsilon\mathbb{Z}^N\right) > \varepsilon^\gamma \right\},$$

et on considère le problème de Cauchy :

$$(T_\varepsilon) \begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon = 0, & (x,v) \in Z_\varepsilon \times \mathbb{S}^{N-1}, \quad t > 0, \\ f_\varepsilon(t,x,v) = 0, \text{ si } v \cdot n_x > 0, & (x,v) \in \partial Z_\varepsilon \times \mathbb{S}^{N-1}, \quad t > 0, \\ f_\varepsilon(0,x,v) = f^{in}(x,v), & (x,v) \in Z_\varepsilon \times \mathbb{S}^{N-1}. \end{cases}$$

Il existe un exposant critique $\gamma_c = \frac{N}{N-1}$ — voir [6, 12] — au sens où :

– Si $\gamma < \gamma_c$ alors

$$f_\varepsilon \to 0 \text{ dans } L^p_{loc}\left(\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{S}^{N-1}\right) \text{ pour tout } p \in [1, +\infty[.$$

– Si $\gamma > \gamma_c$ alors

$$f_\varepsilon \to f \text{ dans } L^\infty\left(\mathbb{R}_+ \mathbb{R}^N \times \mathbb{S}^{N-1}\right) \ * -\text{faiblement}$$

où f est solution de l'équation du transport libre

$$(T_\varepsilon) \begin{cases} \partial_t f + v \cdot \nabla_x f = 0, & (x,v) \in \mathbb{R}^N \times \mathbb{S}^{N-1}, \quad t > 0, \\ f(0,x,v) = f^{in}(x,v), & (x,v) \in \mathbb{R}^N \times \mathbb{S}^{N-1}. \end{cases}$$

Autrement dit, dans le premier cas, la fonction de distribution tend vers zéro : les particules sont absorbées instantanément, les trous étant trop gros par rapport à leur espacement. Et dans le deuxième cas, la fonction de distribution tend vers la solution de la même équation de transport libre mais dans l'espace tout entier : les trous n'ont aucun effet, leur taille étant trop petite par rapport à leur espacement.

Enfin dans le cas critique $\gamma = \gamma_c$, la solution f_ε converge *-faiblement dans $L^\infty\left(\mathbb{R}_+ \times \mathbb{R}^N \times \mathbb{S}^{N-1}\right)$ vers la solution f de l'équation non autonome

$$\partial_t f + v \cdot \nabla_x f = \frac{\dot{p}(t)}{p(t)} f,$$

où p est une fonction strictement décroissante se comportant comme $2/\pi^2 t$ pour $t \to +\infty$ — voir [8] pour la preuve du comportement asymptotique. On a donc ici l'apparition d'un « terme étrange venu d'ailleurs ».

L'expression que nous venons d'employer est le titre d'un article de Cioranescu et de Murat [7] qui traite un problème similaire pour une équation de la diffusion. Rappelons un des principaux résultats de [7].

Soit Ω un ouvert de $\mathbb{R}^N$ perforé périodiquement :

$$\Omega_\varepsilon := \left\{ x \in \Omega \,\middle|\, \mathrm{dist}(x, \varepsilon\mathbb{Z}^N) > r_\varepsilon \right\},$$

où r_ε est le rayon des trous. On considère alors u_ε solution dans $H_0^1(\Omega_\varepsilon)$ du problème de Dirichlet

$$(H_\varepsilon) \begin{cases} -\Delta u_\varepsilon(x) = f(x), & x \in \Omega_\varepsilon, \\ \\ u_{\varepsilon|\partial\Omega_\varepsilon} = 0. \end{cases}$$

Nous notons de même $\{u_\varepsilon\}$ le prolongement de u_ε par zéro dans les trous. Comme dans le cas du transport, il existe une taille critique r_ε^c avec

$$r_\varepsilon^c := \begin{cases} e^{-\frac{1}{\varepsilon^2}}, & \text{si } N = 2, \\ \\ \varepsilon^{\frac{N}{N-2}}, & \text{si } N \geq 3, \end{cases}$$

telle que

- Si $r_\varepsilon > r_\varepsilon^c$ alors $\{u_\varepsilon\} \rightharpoonup 0$ dans $H_0^1(\Omega)$ faible. Autrement dit, les trous sont trop gros et absorbent tout à la limite ;
- Si $r_\varepsilon < r_\varepsilon^c$ alors $\{u_\varepsilon\} \rightharpoonup u$ dans $H_0^1(\Omega)$ fort où u est solution du problème de Dirichlet

$$\begin{cases} -\Delta u(x) = f(x), & x \in \Omega, \\ \\ u_{|\partial\Omega} = 0. \end{cases}$$

  Autrement dit, les trous sont trop petits, et ont un effet négligeable sur u_ε pour $\varepsilon \ll 1$.

Enfin, dans le cas critique, $r_\varepsilon = r_\varepsilon^c$ nous avons

$$\{u_\varepsilon\} \rightharpoonup u \text{ dans } H_0^1(\Omega) - \text{faible},$$

où u est solution du problème de Dirichlet

$$\begin{cases} -\Delta u(x) + \mu u(x) = f(x), & x \in \Omega, \\ \\ u_{|\partial\Omega} = 0, \end{cases}$$

avec

$$\mu = \begin{cases} \dfrac{\pi}{2}, & \text{si } N = 2, \\ \\ \dfrac{\left|\mathbb{S}^{N-1}\right|(N-2)}{2^N}, & \text{si } N \geq 3, \end{cases}$$

On obtient ainsi une équation avec un terme d'absorption supplémentaire. Le résultat est similaire pour d'autres équations de diffusion comme l'équation de Stokes ou de Navier-Stokes — voir par exemple [1, 2, 3]. Notons à ce propos que dans ce dernier cas, le terme supplémentaire dans la limite d'homogénéisation s'interprète comme un terme de friction de Brinkman – voir notamment [3].

Cela étant dit, il y a une différence significative entre le cas du transport libre et le cas de la diffusion. Pour les équations de la diffusion, l'effet des trous de taille critique se traduit par un coefficient d'amortissement constant tandis que dans le cas du transport libre avec distribution périodique de trous, le coefficient d'amortissment est une fonction $t \mapsto \left|\frac{\dot{p}}{p}(t)\right|$ qui n'est pas constante puisque p décroît comme $Const./t$ pour $t \to +\infty$.

Si on remplace l'hypothèse de trous répartis périodiquement par des trous de même taille mais dont les centres suivent une distribution de Poisson, la fonction p est une fonction exponentielle, de sorte que le coefficient $\frac{\dot{p}}{p}$ est constant.

Il y a donc une difficulté spécifique propre à la fois au cas périodique **et** au transport libre, due au fait que des particules dont les directions sont très proches de vecteurs à coordonnées rationnelles peuvent mettre très longtemps à rencontrer un obstacle. Nous renvoyons à l'article [6] pour une discussion approfondie de ce phénomène.

C'est la raison pour laquelle on étudie dans cette thèse le problème de l'homogénéisation de l'équation de Boltzmann linéaire (avec $\sigma > 0$) dans une distribution périodique de trous ayant la taille critique.

En effet, cette équation est en quelque sorte intermédiaire entre l'équation de diffusion — considérée par Cioranescu-Murat — et l'équation de transport libre — étudiée par Caglioti-Golse.

Comme on va le voir, la taille critique des trous pour l'équation de Boltzmann est la même que pour l'équation de transport libre ; mais contrairement au cas l'équation de transport libre, le nombre total de particules décroît exponentiellement lorsque le temps $t \to +\infty$, car les collisions avec le milieu ambiant détruisent les longues trajectoires responsables de la décroissante lente.

Nous allons maintenant décrire plus en détail le contenu de cette thèse.

2. Chapitre 1 : majoration de la masse, cas monocinétique

Dans le premier chapitre, nous considérons pour $\varepsilon \in (0, 2^{1-N})$ fixé, la masse totale

$$M_{\sigma,\varepsilon}(t) := \iint_{Z_\varepsilon \cap [0,1]^N \times \mathbb{S}^{N-1}} f_\varepsilon(t,x,v)dxdv,$$

où f_ε est solution du problème de Cauchy avec condition d'absorption au bord

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \sigma\left(f_\varepsilon - K f_\varepsilon\right) = 0, & (x,v) \in Z_\varepsilon \times \mathbb{S}^{N-1}, t > 0, \\ f_\varepsilon(t,x,v) = 0, \text{ si } v \cdot n_x > 0, & (x,v) \in \partial Z_\varepsilon \times \mathbb{S}^{N-1}, \\ f_\varepsilon(0,x,v) = f^{in}(x,v), & (x,v) \in Z_\varepsilon \times \mathbb{S}^{N-1}. \end{cases}$$

Nous supposons pour simplifier que $f^{in} \in L^\infty\left(\mathbb{R}^N \times \mathbb{S}^{N-1}\right)$ et est 1-périodique en sa variable spatiale. Nous obtenons alors une majoration de la masse indépendante du rayon des trous et de la taille du réseau.

Théorème 2.1. *Soit $\sigma > 0$ alors il existe $C_\sigma > 0$ et $\eta_\sigma \in (-\sigma, 0)$ tel que pour tout $\varepsilon > 0$ et pour tout $t \geq 0$*

$$M_{\sigma,\varepsilon}(t) \leq C_\sigma e^{\eta_\sigma t} \left\| f^{in} \right\|_{L^\infty(\mathbb{R}^N \times \mathbb{S}^{N-1})}.$$

De plus,

$$\eta_\sigma \sim -\sigma \text{ lorsque } \sigma \to 0^+.$$

Ce résultat nous dit que, contrairement au cas du transport libre où elle est à décroissance algébrique, la masse totale du système de particules décroît à vitesse au moins exponentielle dès que $\sigma > 0$.

2.1. Esquisse de la démonstration.

L'idée de la démonstration est de s'appuyer sur l'interprétation probabiliste de l'équation de Boltzmann linéaire afin d'obtenir une formule explicite pour la solution en fonction de σ, de k et du temps $\tau_\varepsilon(x,v)$ de sortie du domaine Z_ε d'une particule libre partant de x dans la direction v. On obtient ainsi une majoration de la masse par une fonction indépendante de ε grâce à un théorème d'estimation du temps de sortie dû à J. Bourgain, F. Golse et B. Wennberg [6]. Ensuite, on remarque que cette fonction majorante vérifie une équation intégrale de type renouvellement sur $\mathbb{R}_+$:

$$f(t) = g(t) + \int_0^\infty f(t-s)g(s)ds$$

où f est l'inconnue et g une fonction intégrable. Ce type d'équation a été intensivement étudié en théorie des probabilités — voir par exemple le chapitre correspondant dans [13]. On dispose en particulier de théorèmes sur le comportement asymptotique de la solution qui nous permettent de conclure.

3. Chapitre 2 : homogénéisation, cas monocinétique

3.1. **Le problème.** On étudie dans ce chapitre le problème de l'homogénéisation proprement dit pour l'équation de Boltzmann dans le cas monocinétique et bidimensionnel. Les particules évoluent donc dans l'espace $\mathbb{R}^2$ perforé périodiquement,

$$Z_\varepsilon := \left\{ x \in \mathbb{R}^2 \,\middle|\, \mathrm{dist}\left(x, \varepsilon\mathbb{Z}^2\right) > \varepsilon^2 \right\},$$

et on suppose de plus que la fréquence de collision σ est une constante strictement positive. Enfin, la condition initiale f^{in} vérifie

$$f^{in} \geq 0 \text{ dans } \mathbb{R}^2 \times \mathbb{S}^1,$$

$$\text{et} \quad \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv + \sup_{(x,v)\in\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v) < +\infty.$$

On considère donc le problème de Cauchy avec condition au bord :

$$(\Xi_\varepsilon) \begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \sigma\left(f_\varepsilon - Kf_\varepsilon\right) = 0, & (x,v) \in Z_\varepsilon \times \mathbb{S}^1,\ t > 0, \\ f_\varepsilon(t,x,v) = 0,\ \text{si } v\cdot n_x > 0, & (x,v) \in \partial Z_\varepsilon \times \mathbb{S}^1,\ t > 0, \\ f_\varepsilon(0,x,v) = f^{in}(x,v), & (x,v) \in Z_\varepsilon \times \mathbb{S}^1. \end{cases}$$

3.2. **Les résultats.** On note $F \equiv F(t,s,x,v)$ la solution du problème de Cauchy

$$(\Xi) \begin{cases} \left(\partial_t + v\cdot\nabla_x + \partial_s\right)F + \sigma F = \dfrac{\dot{p}(t\wedge s)}{p(t\wedge s)}F, \\ F(t,0,x,v) = \sigma\int_0^\infty KF(t,\tau,x,v)d\tau, \\ F(0,s,x,v) = \sigma e^{-\sigma s} f^{in}(x,v), \end{cases}$$

pour $(t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$. Et pour chaque fonction $\phi \equiv \phi(x)$ définie presque partout dans Z_ε, on pose

$$\{\phi\}(x) = \begin{cases} \phi(x), & \text{si } x \in Z_\varepsilon, \\ 0, & \text{sinon.} \end{cases}$$

Nous ne démontrons pas que f_ε ne converge pas vers une solution d'une équation fermée, mais vers l'intégrale sur un temps « supplémentaire » de la fonction F qui est solution d'un problème dans un espace des phases plus grand.

Théorème 3.1. *Soit $(f_\varepsilon)_{\varepsilon>0}$ la famille de solutions de Ξ_ε, alors*

$$\{f_\varepsilon\} \rightharpoonup \int_0^\infty F ds,$$

*dans $L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ *-faiblement lorsque $\varepsilon \to 0^+$, et où F est l'unique solution de* (Ξ).

Signalons immédiatement que le théorème est valide en toutes dimensions, la seule différence étant que nous ne disposons pas d'une formule explicite pour p en dehors du cas bidimensionnel — voir [4] pour une telle formule — même si son existence est connue grâce à [19]. Nous étudions ensuite le comportement asymptotique en temps long de la masse totale de la population particulaire dans la limite homogénéisée. Le théorème suivant nous dit que c'est la masse totale du système particulaire lorsque $\varepsilon \to 0^+$ et nous donne un équivalent asymptotique en temps long.

Théorème 3.2. *Avec les mêmes hypothèses et notations que pour le théorème précédent,*

(1) *il existe une fonction $M \in L^1(\mathbb{R}_+)$ telle que*

$$\frac{1}{2\pi} \iint_{Z_\varepsilon \times \mathbb{S}^1} f_\varepsilon(t,x,v) dx dv \to M(t),$$

dans $L^1_{loc}(\mathbb{R}_+)$ lorsque $\varepsilon \to 0^+$, et p.p. en $t \geq 0$ à extraction d'une sous-suite près ;

(2) *la masse totale limite a une représentation explicite*

$$M(t) = \left(\frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v) dx dv \right) \sum_{n \geq 1} \kappa^{*n}(t), \ t > 0,$$

avec

$$\kappa(t) := \sigma e^{-\sigma t} p(t) \mathbf{1}_{t \geq 0}, \quad \kappa^{*n} := \underbrace{\kappa * \cdots * \kappa}_{n \text{ facteurs}},$$

où $*$ désigne le produit de convolution usuel sur $\mathbb{R}$;

(3) *pour tout* $\sigma > 0$, *il existe* $\xi_\sigma \in (-\sigma, 0)$ *tel que*

$$M(t) \sim C_\sigma e^{\xi_\sigma t} \textit{ lorsque } t \to +\infty;$$

(4) *enfin*

$$\xi_\sigma \sim -\sigma \textit{ lorsque } \sigma \to 0^+, \textit{ et } \xi_\sigma \to -2 \textit{ lorsque } \sigma \to +\infty.$$

3.3. **Idée des démonstrations.**

3.3.1. *L'homogénéisation.* Tout d'abord, on note $F_\varepsilon \equiv F_\varepsilon(t, s, x, v)$ la solution pour tout $\varepsilon > 0$ de

$$\begin{cases} (\partial_t + v \cdot \nabla_x + \partial_s) F_\varepsilon + \sigma F_\varepsilon = 0, & (x, v) \in Z_\varepsilon \times \mathbb{S}^1, \quad t, s > 0, \\ F_\varepsilon(t, s, x, v) = 0, \text{ si } v \cdot n_x > 0, & (x, v) \in \partial Z_\varepsilon \times \mathbb{S}^1, \quad t, s > 0, \\ F_\varepsilon(t, 0, x, v) = \sigma \int_0^\infty KF(t, \tau, x, v) d\tau, & (x, v) \in Z_\varepsilon \times \mathbb{S}^1, \quad t > 0, \\ F_\varepsilon(0, s, x, v) = \sigma e^{-\sigma s} f^{in}(x, v), & (x, v) \in Z_\varepsilon \times \mathbb{S}^1, \quad s > 0, \end{cases}$$

et on montre que la solution f_ε de l'équation de Boltzmann linéaire vérifie

$$f_\varepsilon(t, x, v) = \int_0^\infty F_\varepsilon(t, s, x, v) ds.$$

Il faut noter ici que la variable supplémentaire $s \in \mathbb{R}_+$ peut être interprétée de la façon suivante. À toute particule, on associe sa vitesse v. Lorsqu'elle change de vitesse, on peut dire qu'elle disparaît pour donner naissance à une nouvelle particule de vitesse w. De ce point de vue, la variable s est le temps de vie de la particule de vitesse v. En ce sens, l'équation étendue est une description plus microscopique de l'évolution du système décrit par l'équation de Boltzmann linéaire. Par la méthode des caractéristiques, on écrit une formulation explicite de F_ε et grâce à un lemme de moyenne, on montre que par passage à la limite fort-faible

$$F_\varepsilon \rightharpoonup F \text{ dans } L^\infty - \text{faible } *.$$

Enfin, on montre que f_ε converge elle-même vers $\int_0^\infty F ds$ lorsque ε tend vers zéro. On peut résumer cet argument par le diagramme

$$\begin{array}{ccc} F_\varepsilon & \longrightarrow & F \\ \uparrow & & \downarrow \\ f_\varepsilon & \dashrightarrow & \int_0^\infty F ds \end{array}$$

où la flèche horizontale supérieure est l'homogénéisation dans l'espace des phases étendu. L'idée d'utiliser la variable supplémentaire s pour décrire la limite des f_ε lorsque $\varepsilon \to 0$ provient de l'étude de la limite de Boltzmann-Grad pour le gaz de Lorentz périodique : voir [9, 20].

3.3.2. *Le comportement asymptotique de la masse totale.* L'étude du comportement asymptotique de la masse totale $\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f_\varepsilon dxdv$ passe donc par l'étude de

$$m(t,s) := \frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{S}^1} F(t,s,x,v)dxdv,$$

dont le comportement est déterminé par la

Proposition 3.3. *Soit*

$$B(t,s) = \sigma - \frac{\dot{p}}{p}(t\wedge s),$$

alors l'EDP de renouvellement

$$\begin{cases} \partial_t\mu(t,s) + \partial_s\mu(t,s) + B(t,s)\mu = 0, & t,s>0,\\ \mu(t,0) = \sigma\int_0^\infty \mu(t,s)ds, & t>0,\\ \mu(0,s) = \sigma e^{-\sigma s}, \end{cases}$$

a une unique solution $\mu \in L^\infty\left([0,T];L^1\left(\mathbb{R}_+\right)\right)$ *pour tout* $T>0$.

Et de plus

$$m(t,s) = \frac{\mu(t,s)}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv \text{ p.p. dans } \mathbb{R}_+\times\mathbb{R}_+.$$

On considère alors

$$M(t) := \int_0^\infty m(t,s)ds,$$

et on montre que

$$\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f_\varepsilon dxdv \to M(t),$$

dans $L^1_{loc}\left(\mathbb{R}_+\right)$. Ensuite, grâce à la méthode des caractéristiques, on a une formulation explicite de μ de laquelle on déduit par intégration en $s\in\mathbb{R}_+$, une équation intégrale satisfaite par

$$\frac{1}{\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv}M,$$

à savoir l'équation intégrale du renouvellement

$$m(t) = \kappa(t) + \int_0^t \kappa(t-s)m(s)ds.$$

C'est une équation bien connue en théorie des probabilités — voir [13] — et pour laquelle on dispose de théorèmes sur le comportement asymptotique de la solution m en temps long, fournissant notamment l'équivalent (3) du théorème 3.2.

4. Chapitre 3 : Approximation par la diffusion de l'équation homogénéisée

Nous étudions ici l'approximation par la diffusion de l'équation homogénéisée (Ξ) obtenue dans le chapitre précédent. C'est-à-dire qu'on s'intéresse à une situation fortement collisionnelle $\sigma \to \frac{\sigma}{\eta^2}$ avec $\eta \ll 1$, et sur une échelle de temps longue, ce qui équivaut à supposer que la vitesse des particules est très grande avec la même échelle de temps. Le scaling de la diffusion correspond au cas où la vitesse des particules est d'ordre $\frac{1}{\eta}$. Autrement dit, on considère $F_\eta \equiv F_\eta(t,s,x,v)$ solution du problème de Cauchy

$$(P_\eta)\begin{cases} \partial_t F_\eta + \frac{v}{\eta}\cdot \nabla_x F_\eta + \partial_s F_\eta + \frac{\sigma}{\eta^2} F = \dfrac{\dot{p}}{p}(t\wedge s)F_\eta, \\ F_\eta(t,0,x,v) = \dfrac{\sigma}{\eta^2}\displaystyle\int_0^\infty \int_{\mathbb{S}^1} F_\eta(t,\tau,x,w)dwd\tau, \\ F_\eta(0,s,x,v) = \frac{\sigma}{\eta^2} e^{-\frac{\sigma}{\eta^2}s}\rho^{in}(x), \end{cases}$$

pour $(t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$, avec $p \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ et strictement décroissante. On suppose de plus que

$$\rho^{in} \in L^1\left(\mathbb{R}^2\right) \cap L^2\left(\mathbb{R}^2\right),$$

et qu'il existe $C > 0$ tel que

$$\sup_{x\in\mathbb{R}^2} \left|\rho^{in}\right| \leq C.$$

Nous avons alors le théorème suivant d'approximation par la diffusion :

Théorème 4.1. *Soit F_η solution de (P_η) alors*

$$\int_0^\infty F_\eta ds \to \rho \text{ dans } L^2_{loc}\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2\times\mathbb{S}^1\right)\right) - \text{faible lorsque } \eta \to 0^+,$$

où ρ est la solution du problème de Cauchy

$$\begin{cases} \partial_t \rho - \frac{1}{2\sigma}\Delta\rho = \dot{p}(0)\rho, \quad t>0, \quad x\in\mathbb{R}^2, \\ \rho(0,x) = \rho^{in}(x). \qquad x \in \mathbb{R}^2. \end{cases}$$

De plus, pour tout $T > 0$ et pour tout compact K de $\mathbb{R}^2$

$$\int_0^\infty \int_{\mathbb{S}^1} F_\eta ds \to \rho \text{ dans } L^{\frac{3}{2}}\left([0,T]; L^{\frac{2}{3}}(K)\right) - \text{fort lorsque } \eta \to 0^+.$$

4.1. **Esquisse de la démonstration.** Il convient de relever que

$$F_\eta\big|_{t=0} \to \rho^{in}\delta_{s=0} \text{ dans } \mathcal{M}\left(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1\right) - \text{faible}$$

lorsque $\eta \to 0^+$. On doit donc s'attendre à une convergence du type

$$F_\eta \to f\delta_{s=0} \text{ dans } \mathcal{M}\left(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1\right) - \text{faible}$$

lorsque $\eta \to 0^+$. Nous sommes ainsi obligés d'écarter les méthodes classiques utilisées en théorie de l'approximation par la diffusion comme le développement de Hilbert (qui utilise la régularité de la solution limite) ou la compacité dans L^2 de la famille F_η. L'idée est d'écrire d'abord la formulation duale du problème, puis d'appliquer la transformée de Fourier. On obtient alors la formulation duale de l'équation désirée en passant à la limite.

Pour la convergence forte, on utilise un lemme de moyenne qui garantit la régularité uniforme de la moyenne en vitesse de $(f_\varepsilon)_{\varepsilon>0}$ dans l'espace $L^p\left([0,T];W_{loc}(x)^{-r,p}\right)$ puis on montre ensuite l'appartenance à un « bon » espace du type $L^p\left([0,T];W_{loc}(x)^{s,p}\right)$ de la famille $\left(\partial_t \int_{\mathbb{S}^1} f_\varepsilon\right)_{\varepsilon>0}$. On conclut enfin avec le lemme d'Aubin.

5. Chapitre 4 : étude du cas non monocinétique

5.1. **Le modèle.** Nous reprenons le problème traité dans le deuxième chapitre dans le cas non monocinétique ; i.e. l'espace des vitesses est la boule unité $\mathbf{B}^2$ au lieu de $\mathbb{S}^1$. Plus précisément, on considère $f_\varepsilon \equiv f_\varepsilon(t,x,v)$ solution du problème de Cauchy

$$(\Xi_\varepsilon)\begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \sigma\left(f_\varepsilon - Kf_\varepsilon\right) = 0, & (x,v)\in Z_\varepsilon\times\mathbf{B}^2, t>0,\\ f_\varepsilon(t,x,v) = 0, \text{ si } v\cdot n_x > 0, & (x,v)\in\partial Z_\varepsilon\times\mathbf{B}^2,\\ f_\varepsilon(0,x,v) = f^{in}(x,v), & (x,v)\in Z_\varepsilon\times\mathbf{B}^2,\end{cases}$$

avec

$$Kf(t,x,v) = \int k(v,w)f(t,x,w)dw\,, \quad k(v,w) = k(w,v) \geq 0\,,\ K1 = 1,$$

et

$$f^{in} \geq 0 \text{ dans } \mathbb{R}^2\times\mathbf{B}^2,$$
$$\text{et } \iint_{\mathbb{R}^2\times\mathbf{B}^2} f^{in}(x,v)dxdv + \sup_{(x,v)\in\mathbb{R}^2\times\mathbf{B}^2} f^{in}(x,v) < +\infty.$$

5.2. **Les résultats.** On note $F \equiv F(t,s,x,v)$ solution du problème de Cauchy suivant

$$(\Xi) \begin{cases} (\partial_t + v\cdot\nabla_x + \partial_s) F + \sigma F = |v| \frac{\dot{p}(|v|(t\wedge s))}{p(|v|(t\wedge s))} F, \\ \\ F(t,0,x,v) = \sigma \int_0^\infty KF(t,\tau,x,v)d\tau, \\ \\ F(0,s,x,v) = \sigma e^{-\sigma s} f^{in}(x,v), \end{cases}$$

pour $(t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbf{B}^2$. Et on rappelle la notation employée ci-dessus :

$$\{\phi\}(x) = \begin{cases} \phi(x), & \text{si } x \in Z_\varepsilon, \\ 0, & \text{sinon.} \end{cases}$$

Théorème 5.1. *Soit $(f_\varepsilon)_{\varepsilon>0}$ la famille de solutions de (Ξ_ε), alors à extraction près,*

$$\{f_\varepsilon\} \rightharpoonup \int_0^\infty F ds,$$

*dans $L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbf{B}^2)$ *-faiblement lorsque $\varepsilon \to 0^+$, où F est l'unique solution de (Ξ).*

Nous étudions ensuite le comportement asymptotique en temps long de la masse totale de la population particulaire dans la limite homogénéisée.

Théorème 5.2. *Avec les même hypothèses et notations que pour le théorème précédent*

(1) *Il existe une fonction $M \in L^1(\mathbb{R}_+)$ telle que*

$$\frac{1}{\pi}\iint_{Z_\varepsilon\times\mathbf{B}^2} f_\varepsilon(t,x,v)dxdv \to M(t)$$

dans $L^1_{loc}(\mathbb{R}_+)$ lorsque $\varepsilon \to 0^+$, et de plus p.p. en $t \geq 0$ à extraction près ;

(2) *Si K est de plus de rang fini, pour tout $\sigma > 0$, il existe un réel $\xi_\sigma \in (-\sigma, 0)$, un entier $n \in \mathbf{N}$ indépendant de σ, et C_σ une constante strictement positive tels que*

$$M(t) \sim C_\sigma t^n e^{\xi_\sigma t} \text{ lorsque } t \to +\infty.$$

5.3. **Esquisse de démonstration.** La démonstration du premier énoncé est essentiellement identique à celle pour le cas monocinétique. Pour le comportement asymptotique de la masse en temps long, l'idée est d'abord de montrer qu'il est déterminé par celui d'une famille de fonction $(\mu_i)_{1\leq i\leq n}$. Ensuite, on montre que cette famille vérifie un système d'équations intégrales de type renouvellement pour lequel nous disposons de théorèmes portant sur le comportement asymptotique. Enfin, nous nous appuyons sur ces théorèmes et sur la structure algébrique du système d'équations pour calculer le comportement asymptotique de $(\mu_i)_{1\leq i\leq n}$, et de là, en déduire celui de M.

6. Chapitre 5 : homogénéisation et semi-groupes

6.1. **Le problème.** Ce chapitre traite en particulier de « l'homogénéisation des opacités en transfert radiatif » pour reprendre le titre d'un papier de R. Sentis [21], et de l'utilisation de l'idée employée dans le chapitre 2 d'un espace des phases étendu pour étudier ce problème. En effet, la solution d'une équation de transport linéaire dans un milieu dont le coefficient d'absoption a de fortes oscillations converge, lorsque la fréquence d'oscillations tend vers l'infini, vers la solution d'une équation de type intégro-différentiel — voir par exemple [23, 24, 17, 21] — qui traduit un effet de mémoire. Autrement dit la propriété de semi-groupe, qui traduit la markovianité, ou l'absence de mémoire, de la solution de l'équation de départ disparaît à la limite.

On retrouve ce phéomène d'effet de mémoire dans le problème traité dans le second chapitre. Mais pour passer à la limite dans ce cas, nous avions étendu l'espace des phases par l'ajout d'une variable temporelle supplémentaire. Ceci suggère donc de revisiter l'homogénéisation des opacités en transfert radiatif en utilisant cette nouvelle technique afin d'obtenir une équation équivalente dans un espace des phases étendu.

6.2. **Les principaux résultats.** Le premier théorème revisite une remarque importante de L. Tartar [22] qui est le cas canonique de la perte de la propriété de semi-groupe en partant d'une équation différentielle simple. Soit $a_\varepsilon \equiv a_\varepsilon(z)$ une famille bornée de $L^\infty\left(\mathbb{R}^N\right)$ avec $a_\varepsilon \geq \alpha > 0$ p.p. sur $\mathbb{R}^N$. On suppose que lorsque $\varepsilon \to 0^+$, a_ε converge au sens des mesures de Young vers $(\mu_z)_{z\in\mathbb{R}^N}$ famille de mesures de probabilité sur $\mathbb{R}$. On note $u_\varepsilon \equiv u_\varepsilon(t,z)$ la solution pour tout $\varepsilon > 0$ de l'équation différentielle

$$\begin{cases} \dfrac{d}{dt}u_\varepsilon + a_\varepsilon(z)u_\varepsilon = 0, & t>0,\ z\in\mathbb{R}^N, \\ \\ u_\varepsilon(0,z) = u^{in}(z), & z\in\mathbb{R}^N, \end{cases}$$

avec $u^{in} \in L^1\left(\mathbb{R}^N\right) \cap L^\infty\left(\mathbb{R}^N\right)$. On a alors

Théorème 6.1. *Lorsque* $\varepsilon \to 0^+$

$$u_\varepsilon \to \int_0^\infty U ds \text{ dans } L^\infty\left(\mathbb{R}_+\times\mathbb{R}^N\right) \text{ *-faiblement,}$$

où $U \equiv U(t,s,z)$ est la solution de

$$\begin{cases} \partial_t U - \partial_s U = 0, & t,s>0,\ z\in\mathbb{R}^N, \\ U(0,s,z) = -u^{in}(z)\dfrac{d}{ds}\left(\displaystyle\int_0^\infty e^{-s\lambda}d\mu_z(\lambda)\right), & z\in\mathbb{R}^N. \end{cases}$$

Le deuxième résultat applique la technique développée plus haut à une équation du transfert radiatif qui modélise le comportement d'une population de photons dans un milieu gazeux dont l'opacité oscille très fortement. On suppose que la température donnée $T \equiv T(t,x)$ du milieu est bornée, i.e. $T \in [\theta, \Theta]$ avec $0 < \theta \leq \Theta < +\infty$. On rappelle les notations physiques : h la constante de Planck, c la célérité de la lumière dans le vide et k la constante de Boltzmann. On définit $B_\nu \equiv B_\nu(T)$ comme étant l'intensité radiative émise à la fréquence ν par un corps noir à la température T :

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{\frac{h\nu}{kT}}-1}.$$

On pose $\sigma \equiv \sigma(\nu,T)$ la section efficace d'absorption du milieu extérieur à la température T pour un rayonnement incident de fréquence ν et on suppose que

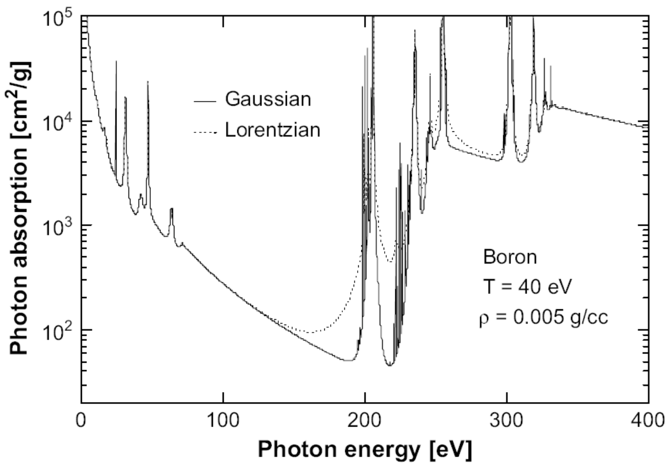
$$0 < m \leq \sigma_\varepsilon(\nu,T) \leq M, \quad \text{pour tout } \nu > 0 \text{ et } T \in [\theta,\Theta].$$

Enfin la fonction $I \equiv I(t,x,\omega,\nu)$ désigne $ch\nu$ fois la densité des photons de fréquence ν à la position x dans la direction ω et au temps t. Elle vérifie l'équation du transfert radiatif :

$$\begin{cases} \frac{1}{c}\partial_t I + \omega\cdot\nabla_x I = \sigma(\nu,T)B_\nu(T) - \sigma(\nu,T)I, \\ I(0,x,\omega,\nu) = I^{in}(x,\omega,\nu), \end{cases}$$

pour $(t,x,\omega,\nu) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}_+^*$ — rappelons que $T \equiv T(t,x)$ est une fonction donnée.

L'une des difficultés pour l'étude de cette équation est que la section efficace d'absorption admet de fortes oscillations comme le montre l'exemple du graphique ci-dessous :



L'idée est de modéliser les oscillations par une suite σ_ε vérifiant

$$\sigma_\varepsilon(\cdot, T) \to \left(\mu_\nu^T\right)_{\nu>0} \tag{1}$$

au sens des mesures de Young, où $\left(\mu_\nu^T\right)_{\nu>0}$ est une famille de probabilités ; et de trouver une équation équivalente décrivant l'intensité radiative à la limite. Autrement dit, on considère l'intensité radiative $I_\varepsilon \equiv I_\varepsilon(t,x,\omega,\nu)$ vérifiant l'équation

$$\begin{cases} \frac{1}{c}\partial_t I_\varepsilon + \omega\cdot \nabla_x I_\varepsilon = \sigma_\varepsilon(\nu,T)B_\nu(T) - \sigma_\varepsilon(\nu,T)I_\varepsilon, \\ I_\varepsilon(0,x,\omega,\nu) = I^{in}(x,\omega,\nu). \end{cases}$$

pour $(t,x,\omega,\nu) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}_+^*$, et où σ_ε vérifie (1). Lorsque μ est une mesure, on note $\tilde{\mu}$ sa transformée de Laplace. Ceci posé, nous avons alors

Théorème 6.2. *Sous les hypothèses et avec les notations ci-dessus*

$$I_\varepsilon \to \int_0^\infty J ds \ \text{dans } L^\infty\left(\mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}_+^*\right) \ \text{*-faiblement,}$$

lorsque $\varepsilon \to 0^+$, *où* $J \equiv J(t,s,x,\omega,\nu)$ *est solution de*

$$\begin{cases} \frac{1}{c}\partial_t J + \omega\cdot \nabla_x J - \partial_s J = \dfrac{d^2\tilde{\mu}_\nu^T}{ds^2} B_\nu(T), \\ J(0,s,x,\omega,\nu) = -I^{in}(x,\omega,\nu)\dfrac{d\tilde{\mu}_\nu^T}{ds}, \end{cases}$$

pour $(t, s, x, \omega, \nu) \in \mathbb{R}_+^* \times \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{S}^2 \times \mathbb{R}_+^*$.

6.3. **Idée des démonstrations.** Les démonstrations sont essentiellement identiques à celle du deuxième chapitre. C'est-à-dire que l'on définit une équation avec une variable supplémentaire dont la solution peut-être écrite explicitement, et on passe à la limite.

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Chapitre 1

The linear Boltzmann equation in a periodic distribution of holes : decay of the mass

CHAPTER I
THE LINEAR BOLTZMANN EQUATION IN A PERIODIC DISTRIBUTION OF HOLES: DECAY OF THE MASS

Introduction

Recent results on the Lorentz gas suggest that the asymptotic behavior of transport equations in the homogenization limit is of a completely different nature according to whether the underlying distribution of obstacles — scatterers or holes — is random or periodic: see [9, 3, 12, 10, 5, 14].

More specifically, for each $\varepsilon \in (0, 2^{1-d})$, define

$$Z_\varepsilon = \{x \in \mathbb{R}^d \,|\, \mathrm{dist}(x, \varepsilon\mathbb{Z}^d) > \varepsilon^{\frac{d}{d-1}}\} = \mathbb{R}^d \setminus \bigcup_{k\in\mathbb{Z}^d} \overline{B(\varepsilon k, \varepsilon^{\frac{d}{d-1}})}\,.$$

For each $f^{in} \equiv f^{in}(x,v) \geq 0$ satisfying (for instance) $f^{in} \in C(\mathbb{R}^d \times \mathbb{S}^{d-1})$ and being $\mathbb{Z}^d-$periodic, define f_ε to be the solution of the initial boundary value problem for the free transport equation:

$$\left\{\begin{array}{ll} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon = 0\,, & x\in Z_\varepsilon\,,\ |v|=1\,,\ t>0\,,\\ f_\varepsilon\left(t, \varepsilon k + \varepsilon^{\frac{d}{d-1}}\omega, v\right) = 0\,, & k\in\mathbb{Z}^d\,,\ |\omega| = |v| = 1\,,\ \omega\cdot v>0\,,\\ f_\varepsilon\big|_{t=0} = f^{in}(x,v)\,, & x\in Z_\varepsilon\,,\ |v|=1\,. \end{array}\right. \tag{0.1}$$

In this problem, f_ε can be thought of as the number density of an ideal gas of point particles that do not see each other. The boundary condition (second equality in the system above) means that no particle can leave the surface of any one of the balls $B(\varepsilon k, \varepsilon^{\frac{d}{d-1}})$. In other words, the particles with distribution function f_ε travel freely at unit speed in the domain Z_ε until they fall into the holes $B(\varepsilon k, \varepsilon^{\frac{d}{d-1}})$, in which case they disappear forever.

A natural question is therefore to estimate the total mass — or equivalently, particle number — of the amount of gas remaining in Z_ε at each time $t > 0$. Integrating the transport equation above along characteristics, we see that the solution f_ε is given for each $t > 0$ by the formula

$$f_\varepsilon\,(t,x,v) = f^{in}(x-tv, v)\mathbb{1}_{t<\tau_\varepsilon(x,v)}\,, \qquad x\in Z_\varepsilon\,,\ |v|=1\,, \tag{0.2}$$

where $\tau_\varepsilon(x,v)$ is the free path length for a particle leaving position $x \in Z_\varepsilon$ in the direction $v \in \mathbb{S}^{d-1}$:

$$\tau_\varepsilon(x,v) = \inf\{t>0 \,|\, x+tv \in Z_\varepsilon\}\,.$$

Obviously, for each $(x,v) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$

$$\tau_\varepsilon(x,v) = \tau_\varepsilon\,(x+\varepsilon k, v) \text{ for each } k\in\mathbb{Z}^d,$$

in other words, $\tau_\varepsilon(\cdot\,, v)$ is $\varepsilon\mathbb{Z}^d-$periodic for each $v \in \mathbb{S}^{d-1}$. Therefore, τ_ε can be viewed equivalently as a function on $\mathcal{Z}_\varepsilon \times \mathbb{S}^{d-1}$, where $\mathcal{Z}_\varepsilon$ is the quotient space

$$\mathcal{Z}_\varepsilon := Z_\varepsilon \diagup \varepsilon\mathbb{Z}^d.$$

For each $r > 0$, we introduce the following punctured torus

$$Y_r := \left(\mathbb{R}^d \diagup \mathbb{Z}^d\right) \setminus B\,(0,r)$$

and the associated free path length for a particle leaving position $x \in Y_r$ in the direction $v \in \mathbb{S}^{d-1}$:

$$\mathcal{T}_r(y,v) := \inf\{t > 0 | y - tv \in \partial Y_r\}.$$

Define μ_r to be the uniform probability measure on $Y_r \times \mathbb{S}^{d-1}$, and p_r the distribution of $\mathcal{T}_r$ under μ_r,i.e.

$$p_r(t) := \mu_r\left(\left\{(x,v) \in Y_r \times \mathbb{S}^{d-1} | \mathcal{T}_r(x,v) > t\right\}\right). \tag{0.3}$$

As mentioned above, the distribution of free path length has been studied in [3, 5, 12, 1]. In particular, we have the following estimate (see [3, 12]).

Proposition 0.1. *(Bourgain-Golse-Wennberg) For $d \geq 2$ there exist $C[d], C'[d] > 0$ such that for each $r > 0$,*

$$\frac{C[d]}{r^{d-1}t} \leq p_r(t) \leq \frac{C'[d]}{r^{d-1}t} \qquad \textit{whenever } t \geq \frac{1}{r^{d-1}}.$$

The lower bound in the case $d = 2$ and the upper bound for all $d \geq 2$ were proved in [3]; the lower bound was later extended to the case of any $d \geq 2$ by Golse and Wennberg see [12]. Notice that

$$\mathcal{Z}_\varepsilon = \varepsilon Y_{\varepsilon^{\frac{1}{d-1}}}, \text{ for each } \varepsilon > 0,$$

which implies that for each $\varepsilon > 0$ and for each $(x,v) \in \mathcal{Z}_\varepsilon \times \mathbb{S}^{d-1}$

$$\tau_\varepsilon(x,v) = \varepsilon \mathcal{T}_{\varepsilon^{\frac{1}{d-1}}}\left(\frac{x}{\varepsilon}, v\right).$$

Consequently, the distribution of τ_ε for (x,v) uniformly distributed in $\mathcal{Z}_\varepsilon \times \mathbb{S}^{d-1}$ is

$$\begin{aligned}
\mathbb{P}_{\text{unif}}(\tau_\varepsilon > t) &= \frac{1}{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}\iint_{\mathcal{Z}_\varepsilon \times \mathbb{S}^{d-1}} \mathbb{1}_{t<\tau_\varepsilon(x,v)} dxdv \\
&= \frac{1}{|\varepsilon Y_{\varepsilon^{\frac{1}{d-1}}}||\mathbb{S}^{d-1}|}\iint_{\varepsilon Y_{\varepsilon^{\frac{1}{d-1}}} \times \mathbb{S}^{d-1}} \mathbb{1}_{t<\varepsilon \mathcal{T}_{\varepsilon^{\frac{1}{d-1}}}\left(\frac{x}{\varepsilon},v\right)} dxdv \\
&= \frac{\varepsilon^d}{|\varepsilon Y_{\varepsilon^{\frac{1}{d-1}}}||\mathbb{S}^{d-1}|}\iint_{Y_{\varepsilon^{\frac{1}{d-1}}} \times \mathbb{S}^{d-1}} \mathbb{1}_{t<\varepsilon \mathcal{T}_{\varepsilon^{\frac{1}{d-1}}}(y,v)} dydv \\
&= \frac{1}{|Y_{\varepsilon^{\frac{1}{d-1}}}||\mathbb{S}^{d-1}|}\iint_{Y_{\varepsilon^{\frac{1}{d-1}}} \times \mathbb{S}^{d-1}} \mathbb{1}_{t<\varepsilon \mathcal{T}_{\varepsilon^{\frac{1}{d-1}}}(y,v)} dydv \\
&= p_{\varepsilon^{\frac{1}{d-1}}}\left(\frac{t}{\varepsilon}\right).
\end{aligned}$$

By Proposition 0.1, for each $\varepsilon > 0$ we have

$$\frac{C[d]}{\left(\varepsilon^{\frac{1}{d-1}}\right)^{d-1}\frac{t}{\varepsilon}} \leq p_{\varepsilon^{\frac{1}{d-1}}}\left(\frac{t}{\varepsilon}\right) \leq \frac{C'[d]}{\left(\varepsilon^{\frac{1}{d-1}}\right)^{d-1} t} \qquad \text{whenever } \frac{t}{\varepsilon} \geq \frac{1}{\left(\varepsilon^{\frac{1}{d-1}}\right)^{d-1}},$$

or equivalently

$$\frac{C[d]}{t} \leq \mathbb{P}_{\text{unif}}(\tau_\varepsilon(x,v) > t) \leq \frac{C'[d]}{t} \quad \text{whenever } t \geq 1. \tag{0.4}$$

Notice that we have just obtained for $\mathbb{P}_{\text{unif}}(\tau_\varepsilon > t)$ two bounds (0.4) which are independent of ε. That being done, as f^{in} is $\mathbb{Z}^d-$periodic we define the total mass of the particle system at time t by

$$M_\varepsilon(t) := \iint_{Z_\varepsilon \cap [0,1]^d \times \mathbb{S}^{d-1}} f_\varepsilon(t,x,v)\, dxdv, \text{ for each } t \geq 0.$$

In view of equality (0.2)

$$M_\varepsilon(t) = \iint_{Z_\varepsilon \cap [0,1]^d \times \mathbb{S}^{d-1}} f^{in}(x - tv, v) \mathbb{1}_{t < \tau_\varepsilon(x,v)} dx dv.$$

Since $f^{in} \in L^\infty\left(\mathbb{R}^d \times \mathbb{S}^{d-1}\right)$, and we assume without loss of generality that ε is of the form $\varepsilon = 1/n$ with $n \in \mathbf{N}^*$. Then

$$\begin{aligned} M_\varepsilon(t) &\le \|f^{in}\|_{L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})} \left(\frac{1}{\varepsilon}\right)^d \iint_{\mathcal{Z}_\varepsilon \times \mathbb{S}^{d-1}} \mathbb{1}_{t<\tau_\varepsilon(x,v)} dx dv \\ &\le \|f^{in}\|_{L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})} \frac{|\mathcal{Z}_\varepsilon|}{\varepsilon^d} |\mathbb{S}^{d-1}| \mathbb{P}_{\text{unif}}\left(\tau_\varepsilon > t\right). \end{aligned}$$

In the same way, if there exists $c > 0$ such that for each $(x, v) \in \mathbb{R}^d \times \mathbb{S}^{d-1}$, $f^{in}(x, v) \ge c$, we have

$$M_\varepsilon(t) \ge c \frac{|\mathcal{Z}_\varepsilon|}{\varepsilon^d} |\mathbb{S}^{d-1}| \mathbb{P}_{\text{unif}}\left(\tau_\varepsilon > t\right).$$

Since $|\mathcal{Z}_\varepsilon| = \varepsilon^d \left(1 - \varepsilon^{\frac{d}{d-1}} |B(0,1)|\right) \sim \varepsilon^d$ as $\varepsilon \to 0^{+,}$ that means that by the estimates on the distribution of τ_r due to Bourgain-Golse-Wennberg [3, 12] as computed above, the two bounds of the total mass M_ε have a algebraic decay rate that is independent of ε.

Now, suppose that, instead of being periodically distributed, the holes are randomly distributed. Specifically, replace Z_ε with

$$\mathcal{Y}_\varepsilon[\{c\}] = \{x \in \mathbb{R}^d \,|\, \operatorname{dist}(x, \{c\}) > \varepsilon\} = \mathbb{R}^d \setminus \bigcup_{c \in \{c\}} \overline{B(c, \varepsilon)}$$

where $\{c\}$ is a countable subset of $\mathbb{R}^d$, distributed under Poisson's law with parameter $\beta_\varepsilon := \frac{\beta}{2\varepsilon}$ with $\beta > 0$. This means that, for each measurable $A \subset \mathbb{R}^d$ with finite measure $|A|$ and each $n \ge 0$ one has

$$\operatorname{Prob}(\#(A \cap \{c\}) = n) = \frac{\beta_\varepsilon^n |A|^n}{n!} e^{-\beta_\varepsilon |A|}.$$

Solving the same Cauchy problem for the free transport equation as above, but with Z_ε replaced with $Y_\varepsilon[\{c\}]$ leads to a particle number density that depends on the countable set $\{c\}$ of hole centers, denoted by $f_\varepsilon \equiv f_\varepsilon(t, x, v, \{c\})$. Defining the total particle number $M_\varepsilon \equiv M_\varepsilon(t, \{c\})$ as above, a straightforward computation (see for instance section 2 of the survey [11]) leads to the estimate

$$\mathbf{E} M_\varepsilon(t) \le C e^{-\beta t}$$

where $\mathbf{E}$ designates the mathematical expectation, i.e. averaging over the hole configuration $\{c\}$.

We thus note a considerable difference in the decay rate for the total particle number according to whether the distribution of holes is periodic or random. The reason for the slower decay in the periodic case is the presence of sufficiently many "channels" (infinite open strips included in Z_ε): see [3, 12, 11].

However, if the free transport equation above is replaced with a linear Boltzmann equation for monokinetic particles, the influence of channels is destroyed by the collisions of the particles with the background medium — more precisely, by the scattering part of these collisions. Specifically, consider the initial boundary value problem

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \sigma(f_\varepsilon - K f_\varepsilon) = 0\,, & x \in Z_\varepsilon\,,\ |v| = 1\,,\ t > 0\,, \\ f_\varepsilon\left(t, \varepsilon k + \varepsilon^{\frac{d}{d-1}}\omega, v\right) = 0\,, & k \in \mathbb{Z}^d\,,\ |\omega| = |v| = 1\,,\ \omega \cdot v > 0\,, \\ f_\varepsilon\big|_{t=0} = f^{in}(x,v)\,, & x \in Z_\varepsilon\,,\ |v| = 1\,, \end{cases}$$

where $\sigma > 0$ and K is an integral operator acting on the only variable v in f_ε, of the form

$$K f_\varepsilon(t,x,v) = \int_{\mathbb{S}^{d-1}} k(v,w) f_\varepsilon(t,x,w) dw\,.$$

The integral kernel $k \equiv k(v,w)$ is the scattering kernel (i.e. the probability of a transition in particle velocity from direction w to direction v): see below the properties satisfied by k.

Because of the term $\sigma K f_\varepsilon$ in the linear Boltzmann equation above, the direction of a typical particle path is piecewise constant. This suggests that, whenever $\sigma > 0$, the total number of particles that remain in Z_ε at time t should decay faster than $O(1/t)$, i.e. as in the case where $\sigma = 0$. Whether this faster decay rate is exponential in spite of the periodic distribution of holes is the subject of the present paper.

1. The model

Let $d \geq 2$ and consider the monokinetic linear Boltzmann equation which is a classical model for instance in the context of Radiative Transfer:

$$\partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + \sigma(f_\varepsilon - K f_\varepsilon) = 0. \tag{1.1}$$

The unknown function $f(t,x,v)$ is the density at time $t \in \mathbb{R}_+$ of particles with velocity $v \in \mathbb{S}^{d-1}$, located at $x \in \mathbb{R}^d$. It has the following probabilistic interpretation: the probability that the particle be located in an infinitesimal volume dx around the location x with direction in an infinitesimal element of solide angle dv around the direction v at time $t \geq 0$ is $f(t,x,v)dxdv$. For each $\phi \in L^1(\mathbb{R}^d \times \mathbb{S}^{d-1}_v)$, we denote:

$$K\phi(v) := \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} k(v,w)\phi(w)dw,$$

where dw is the uniform measure on the unit sphere $\mathbb{S}^{d-1}$. We henceforth assume that

$$\begin{aligned} &k \in\ L^\infty(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}) \text{ and } \ k(v,w) = k(w,v) \geq 0 \ \text{ a.e. in } v,w \in \mathbb{S}^{d-1}, \\ &\qquad\qquad\qquad \text{with } \tfrac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} k(v,w)dw = 1 \text{ a.e. in } v \in \mathbb{S}^{d-1}. \end{aligned} \tag{1.2}$$

The constant $\sigma > 0$ is the collision frequency, or in other words the average time between two successive collisions in the medium. The linear Boltzmann equation is set on the domain $Z_\varepsilon \times \mathbb{S}^{d-1}$.

We assume an absorption boundary condition on ∂Z_ε

$$f_\varepsilon = 0 \text{ for each } (t,x,v) \in \mathbb{R}_+^* \times \partial Z_\varepsilon \times \mathbb{S}^{d-1},\ \text{whenever } v \cdot n_x > 0,$$

where n_x denotes the inward unit normal vector to ∂Z_ε at $x \in \partial Z_\varepsilon$. As in the case of problem (0.1), this condition means that particles falling into a hole remain there forever.

To sum-up, for each $\varepsilon \in (0, 2^{1-d})$ and each $\sigma > 0$, we consider f_ε, the solution of the initial boundary value problem

$$(\Xi_\varepsilon)\begin{cases} \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon + \sigma(f_\varepsilon - Kf_\varepsilon) = 0, & (x,v) \in Z_\varepsilon \times \mathbb{S}^{d-1}, t > 0, \\ f_\varepsilon = 0, \text{ if } n_x \cdot v > 0, & (x,v) \in \partial Z_\varepsilon \times \mathbb{S}^{d-1}, \\ f_\varepsilon(0,x,v) = f^{in}(x,v), & (x,v) \in Z_\varepsilon \times \mathbb{S}^{d-1}, \end{cases}$$

with $f^{in} \in L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$ and is $\mathbb{Z}^d-$periodic in its first variable.

Contrary to the initial boundary value problem for the free transport equation (0.1), the function f_ε cannot be explicitely computed as in formula (0.2). However, it has the following probabilistic interpretation :

Let $(T_i)_{i\geq 1}$ be independent and identically distributed random variables with distribution

$$\mathbb{P}\left(T_i > t\right) = e^{-\sigma t} \text{ for each } t > 0.$$

Let $(W_i)_{i\geq 1}$ be independent and identically distributed random unit vector fields on $\mathbb{S}^{d-1}$ with the following distribution : for each $I \subseteq \mathbb{S}^{d-1}$:

$$\mathbb{P}\left(W_1(v) \in I\right) = \frac{1}{|\mathbb{S}^{d-1}|}\int_I k(v,w)dw,$$

where dw designates the surface element on $\mathbb{S}^{d-1}$. Moreover the random variables $(T_i)_{i\geq 1}$ and $(W_j)_{j\geq 1}$ are chosen so that $(W_i, T_j)_{i,j\geq 1}$ are mutually independant. Given $x \in Z_\varepsilon$ and $v \in \mathbb{S}^{d-1}$, we define by induction

$$\begin{cases} X_0 = x, \\ V_0 = v, \\ X_n := X_{n-1} - T_n V_{n-1} \text{ for each } n \in \mathbf{N}^*, \\ V_n := W_n(V_{n-1}) \text{ for each } n \in \mathbf{N}^*. \end{cases}$$

Notice that $(V_n)_{n\geq 0}$ is a Markov chain. We denote

$$S_0 := 0,\ S_n := T_1 + \cdots + T_n \text{ for every } n \geq 1.$$

Finally, we set

$$\begin{cases} (X_0, V_0) = (x,v) \in Z_\varepsilon \times \mathbb{S}^{d-1} \\ (X_t, V_t) := (X_n - (t - S_n)V_n, V_n) \text{ if } S_n \leq t < S_n + T_{n+1}. \end{cases}$$

The transport process $(X_t, V_t)_{t\in\mathbb{R}_+}(x,v)$ describes the motion of a particle starting from $x \in Z_\varepsilon$ in the direction $v \in \mathbb{S}^{d-1}$ at time $t = 0$, changing direction according to the law k at exponentially distributed times. Then the solution f_ε of the initial boundary value problem (Ξ_ε) for the linear Boltzmann equation is represented as follows:

(1.3) $\quad f_\varepsilon(t,x,v) = \mathbf{E}_{x,v}\left[f^{in}(X_t,V_t)\mathbb{1}_{t<\theta_\varepsilon(x,v)}\right]$ a.e. in $(t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{S}^{d-1}$

where $\mathbf{E}_{x,v}$ is the expectation for the transport process starting from (x,v) (see [6] pp. 225-226), and $\theta_\varepsilon(x,v)$ is the exit time for the process $(X_t, V_t)_{(x,v)}$. In other words,

$$\theta_\varepsilon(x,v) := \inf\left\{t > 0 | X_t(x,v) \in \partial Z_\varepsilon\right\}.$$

We shall see that this probabilistic interpretation leads to an upper bound for the total mass of the particle system.

2. Main result

We may assume without loss of generality that $f^{in} \in L^{\infty}\left(\mathbb{R}^d \times \mathbb{S}^{d-1}\right)$ and is $\mathbb{Z}^d-$periodic in the space variable x while ε is of the form $\varepsilon = 1/n$ with $n \in \mathbf{N}^*$, so that the solution f_ε of Ξ_ε is also $\mathbb{Z}^d-$periodic in the x variable and belongs to $L^{\infty}\left(\mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^{d-1}\right)$. We define the total mass of the system in any spatial period to be

$$M_{\sigma,\varepsilon}(t) := \iint_{Z_\varepsilon \cap [0,1]^d \times \mathbb{S}^{d-1}} f_\varepsilon(t,x,v)dxdv.$$

Notice that in the non-collisional case ($\sigma = 0$), the mass is controlled by the distribution of free path lengths in Z_ε, as explained in the introduction. Our main result is

Theorem 2.1. *Under the assumptions and with the notations above,*

(1) *for each $\sigma > 0$, there exists $\eta_\sigma \in (-\sigma, 0)$ such that*

$$M_{\sigma,\varepsilon}(t) \le \|f^{in}\|_{L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})} C_\sigma e^{\eta_\sigma t} \text{ for each } \varepsilon \in (0, 2^{1-d}).$$

Moreover, one has

(2) $\eta_\sigma \sim -\sigma$ *as* $\sigma \to 0^+$.

Statement 1 above means that for each $\varepsilon \in (0, 2^{1-d})$ and $\sigma > 0$, $M_{\sigma,\varepsilon}$ decays exponentially fast as $t \to +\infty$ and at exponential speed that is independent of ε. Statement 2 gives the behavior of the characteristic exponent η_σ as $\sigma \to 0^+$ (non collisional regime). In section 3, we give a functional inequality crucial for Theorem 2.1 while the exponential order estimate of the $M_{\sigma,\varepsilon}$ is discussed in section 4.

3. A functional inequality

First, we recall the free path length, or forward exit time, for a particle starting from x in the direction v without changing direction :

$$\tau_\varepsilon(x,v) := \inf\{t > 0 | x - tv \in \partial Z_\varepsilon\}, \tag{3.1}$$

We next introduce the function

$$T_\varepsilon(v) := \sup_{x \in Z_\varepsilon} \tau_\varepsilon(x,v) \tag{3.2}$$

and

$$\mathcal{P}_\varepsilon(t) := \frac{1}{|\mathbb{S}^{d-1}|}\int_{v \in \mathbb{S}^{d-1}} \mathbb{1}_{t < T_\varepsilon(v)} dv \text{ for each } t \ge 0.$$

We recall next Theorem B in [3].

Proposition 3.1. *For each $d > 1$, there exists $C''[d] > 0$ such that, for each $r \in (0, 1/2)$ and each $t > 0$*

$$\frac{|\{v \in \mathbb{S}^{d-1} | \sup_{y \in Y_r} \mathcal{T}_r(y,v) > t\}|}{|\mathbb{S}^{d-1}|} \le \frac{C''[d]}{r^{d-1}t}.$$

Since $\tau_\varepsilon(x,v) = \varepsilon \mathcal{T}_{\varepsilon^{\frac{1}{d-1}}}(\frac{x}{\varepsilon}, v)$, one has $T_\varepsilon = \varepsilon \sup_{y \in Y_{\varepsilon^{\frac{1}{d-1}}}} \mathcal{T}_{\varepsilon^{\frac{1}{d-1}}}(y,v)$ so that, according to the proposition above

$$\begin{aligned}\mathcal{P}_\varepsilon(t) &= \frac{1}{|\mathbb{S}^{d-1}|}\int_{v \in \mathbb{S}^{d-1}} \mathbb{1}_{\varepsilon \sup_{y \in Y_{\varepsilon^{\frac{1}{d-1}}}} \mathcal{T}_{\varepsilon^{\frac{1}{d-1}}}(y,v) > t} dv \\ &\le \frac{C''[d]}{\left(\varepsilon^{\frac{1}{d-1}}\right)^{d-1} \frac{t}{\varepsilon}} = \frac{C''[d]}{t}\end{aligned}$$

for each $t > 0$ and $\varepsilon \in (0, 2^{1-d})$. Besides, since $\mathcal{P}_\varepsilon(t)$ is a probability, one also has $\mathcal{P}_\varepsilon \leq 1$ for each $t \geq 0$ and $\varepsilon \in (0, 2^{1-d})$, so that

$$\mathcal{P}_\varepsilon(t) \leq \inf\left(1, \frac{C''[d]}{t}\right) \text{ for each } t \geq 0 \text{ and } \varepsilon \in (0, 2^{1-d}).$$

We also denote

$$g_{\sigma,\varepsilon}(t) := \sigma e^{-\sigma t}\mathcal{P}_\varepsilon(t)\mathbb{1}_{t>0} \text{ and each } t \in \mathbb{R}$$

and

$$g_\sigma(t) := \sigma e^{-\sigma t} \inf\left(1, \frac{C''[d]}{t}\right)\mathbb{1}_{t>0}, \text{ for each } t \geq 0.$$

Notice that for each $t > 0$ and each $\varepsilon \in (0, 2^{1-d})$

$$g_{\sigma,\varepsilon}(t) \leq g_\sigma(t), \text{ for each } \sigma \geq 0. \tag{3.3}$$

We establish in the present section the following

Proposition 3.2. *Let $\sigma > 0$ and $\varepsilon \in (0, 2^{1-d})$, under the assumptions and with the notations above, we have*

$$\begin{aligned} M_{\sigma,\varepsilon}(t) \leq& \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}\frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d}\mathbb{P}_{unif}(\tau_\varepsilon > t)\, e^{-\sigma t} \\ &+ \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}\frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d}\int_0^t e^{-\sigma(t-s)}\sum_{n\geq 1} g_{\sigma,\varepsilon}^{*n}(s)ds. \end{aligned}$$

for each $t \geq 0$, where $f^{*n} := \underbrace{f * \cdots * f}_{n \text{ factors}}$, and $*$ designates the convolution product on $\mathbb{R}$.

In view of (3.3), this proposition obviously entails

Corollary 3.3. *For each σ and for each $\varepsilon \in (0, 2^{1-d})$, under the assumptions and with the notations above, we have*

$$\begin{aligned} M_{\sigma,\varepsilon}(t) \leq& \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}\frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d}\mathbb{P}_{unif}(\tau_\varepsilon > t)\, e^{-\sigma t} \\ &+ \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}\frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d}\int_0^t e^{-\sigma(t-s)}\sum_{n\geq 1} g_{\sigma}^{*n}(s)ds \end{aligned}$$

for each $t \geq 0$.

The first inequality is based on the representation of the solution in terms of the transport process. Notice that the series in the right-side of the second inequality is independent of ε. It will imply a bound on $M_{\sigma,\varepsilon}$ that depends only on $\sigma > 0$.

3.1. **A first inequality.** First, recall that $(T_i)_{i\geq 1}$ denote independent and identically distributed random variables with distribution

$$\mathbb{P}(T_i > t) = e^{-\sigma t} \text{ for each } t \geq 0,$$

and $(S_n)_{n\geq 0}$ designate the sum of the random variables T_i:

$$S_n := \begin{cases} 0 \text{ if } n = 0, \\ T_1 + \cdots + T_n \text{ otherwise.} \end{cases}$$

The stochastic process $(X_n, V_n)_{n\geq 0}$ is the one defined in Section 1. We denote for the sake of simplicity

$$I_0(t) := \iint_{Z_\varepsilon\cap[0,1]^d\times\mathbb{S}^{d-1}} \mathbf{E}_{x,v}[\mathbb{1}_{t<T_1}\mathbb{1}_{t<\tau_\varepsilon(x,v)}]dxdv,$$

and more generally for every $n \geq 1$,

$$I_n(t) := \iint_{Z_\varepsilon \cap [0,1]^d \times \mathbb{S}^{d-1}} \mathbf{E}_{x,v}[\mathbb{1}_{S_n < t < S_n + T_{n+1}} \prod_{1 \leq i \leq n} \mathbb{1}_{T_i < \tau_\varepsilon(X_{i-1}, V_{i-1})}] dxdv. \tag{3.4}$$

Lemma 3.4. *Under the assumptions and with the notations above,*

$$M_{\sigma,\varepsilon}(t) \leq \|f^{in}\|_{L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})} \sum_{n \geq 0} I_n(t) \text{ for each } t \geq 0. \tag{3.5}$$

Proof. We recall equality (1.3)

$$f_\varepsilon(t,x,v) = \mathbf{E}_{x,v}\left[f^{in}(X_t, V_t) \mathbb{1}_{t < \theta_\varepsilon(x,v)}\right] \text{ a.e. in } (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^d \times \mathbb{S}^{d-1},$$

where $(X_t, V_t)(x,v)$ is the transport process defined starting from $x \in Z_\varepsilon \cap [0,1]^d$ in the direction $v \in \mathbb{S}^{d-1}$. We have

$$\begin{aligned} f_\varepsilon(t,x,v) &= \mathbf{E}_{x,v}\left[f^{in}(X_t, V_t) \mathbb{1}_{t < \theta_\varepsilon(x,v)}\right] \\ &= \mathbf{E}_{x,v}\left[f^{in}(X_t, V_t) \mathbb{1}_{t < \theta_\varepsilon(x,v)} \sum_{n \geq 0} \mathbb{1}_{S_n \leq t < S_{n+1}}\right] \\ &= \sum_{n \geq 0} \mathbf{E}_{x,v}\left[f^{in}(X_t, V_t) \mathbb{1}_{t < \theta_\varepsilon(x,v)} \mathbb{1}_{S_n \leq t < S_{n+1}}\right] \\ &= \mathbf{E}_{x,v}\left[f^{in}(X_t, V_t) \mathbb{1}_{t < \theta_\varepsilon(x,v)} \mathbb{1}_{t < T_1}\right] \\ &\quad + \sum_{n \geq 1} \mathbf{E}_{x,v}\left[f^{in}(X_t, V_t) \mathbb{1}_{t < \theta_\varepsilon(x,v)} \mathbb{1}_{S_n \leq t < S_{n+1}}\right]. \end{aligned}$$

Therefore we have for each $\varepsilon \in (0, 2^{1-d})$ and for each $t \geq 0$, $x \in Z_\varepsilon \cap [0,1]^d$ $v \in \mathbb{S}^{d-1}$

$$\begin{aligned} f_\varepsilon(t,x,v) \leq{}& \|f^{in}\|_{L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})} \mathbf{E}_{x,v}\left[\mathbb{1}_{t < \theta_\varepsilon(x,v)} \mathbb{1}_{t < T_1}\right] \\ &+ \|f^{in}\|_{L^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})} \sum_{n \geq 1} \mathbf{E}_{x,v}\left[\mathbb{1}_{t < \theta_\varepsilon(x,v)} \mathbb{1}_{S_n \leq t < S_{n+1}}\right]. \end{aligned} \tag{3.6}$$

As $\mathbb{1}_{t < T_1}$ implies that the particle has moved from $x \in Z_\varepsilon \cap [0,1]^d$ in the direction $v \in \mathbb{S}^{d-1}$ for $t \in \mathbb{R}_+$ without changing direction meanwhile, we have

$$\mathbf{E}_{x,v}\left[\mathbb{1}_{t < \theta_\varepsilon(x,v)} \mathbb{1}_{t < T_1}\right] = \mathbf{E}_{x,v}\left[\mathbb{1}_{t < \tau_\varepsilon(x,v)} \mathbb{1}_{t < T_1}\right]. \tag{3.7}$$

Besides

$$\mathbb{1}_{t < \theta_\varepsilon(x,v)} \mathbb{1}_{S_n \leq t < S_{n+1}} \leq \mathbb{1}_{S_n < \theta_\varepsilon(x,v)} \mathbb{1}_{S_n \leq t < S_{n+1}} \tag{3.8}$$

and $\mathbb{1}_{S_n < \theta_\varepsilon(x,v)}$ means that the particle starting from x in the direction v has changed direction n times, for each $i \in [|0, n-1|]$ from X_i in the direction V_i without changing direction during T_{i+1} and without falling into any hole. Therefore for each $n \geq 1$

$$\begin{aligned} \mathbf{E}_{x,v}\left[\mathbb{1}_{t < \theta_\varepsilon(x,v)} \mathbb{1}_{S_n \leq t < S_{n+1}}\right] &\leq \mathbf{E}_{x,v}\left[\mathbb{1}_{S_n < \theta_\varepsilon(x,v)} \mathbb{1}_{S_n \leq t < S_{n+1}}\right] \\ &\leq \mathbf{E}_{x,v}\left[\mathbb{1}_{S_n \leq t < S_{n+1}} \prod_{i=1}^n \mathbb{1}_{T_i < \tau_\varepsilon(X_{i-1}, V_{i-1})}\right]. \end{aligned} \tag{3.9}$$

In view of inequalities (3.7), (3.8) and (3.9), inequality (3.6) entails

$$\begin{aligned}
f_\varepsilon(t,x,v) \le \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}&\mathbf{E}_{x,v}\left[\mathbb{1}_{t<\theta_\varepsilon(x,v)}\mathbb{1}_{t<T_1}\right]\\
&+\|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}\sum_{n\ge 1}\mathbf{E}_{x,v}\left[\mathbb{1}_{t<\theta_\varepsilon(x,v)}\mathbb{1}_{S_n\le t<S_{n+1}}\right],\\
&\le \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}\mathbf{E}_{x,v}\left[\mathbb{1}_{t<\tau_\varepsilon(x,v)}\mathbb{1}_{t<T_1}\right]\\
&\quad+\|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}\sum_{n\ge 1}\mathbf{E}_{x,v}\left[\mathbb{1}_{S_n\le t<S_{n+1}}\prod_{i=1}^n\mathbb{1}_{T_i<\tau_\varepsilon(X_{i-1},V_{i-1})}\right].
\end{aligned}$$

We integrate both sides of the inequality above in $x\in Z_\varepsilon\cap[0,1]^d$ and in $v\in\mathbb{S}^{d-1}$. □

3.2. An estimate for I_n.

We first give an estimate for I_n.

Lemma 3.5. *Under the assumptions and with the notations above,*

$$I_0(t) := \frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d}e^{-\sigma t}\mathbb{P}_{unif}\left(\tau_\varepsilon > t\right)\ \text{ and each } t\ge 0,$$

and for every $n\ge 1$,

$$I_n(t) \le \frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d}\int_0^t e^{-\sigma(t-s)}\left(g_{\sigma,\varepsilon}\right)^{*n}(s)ds\ \text{ and each } t\ge 0.$$

Proof. We begin with case $n=0$. Since T_1 has distribution $\mu_{T_1}(ds) := \sigma e^{-\sigma s}ds$, one has

$$\begin{aligned}
I_0(t) &= \iint_{Z_\varepsilon\cap[0,1]^d\times\mathbb{S}^{d-1}}\mathbf{E}_{x,v}[\mathbb{1}_{t<T_1}\mathbb{1}_{t<\tau_\varepsilon(x,v)}]dxdv\\
&= \iint_{Z_\varepsilon\cap[0,1]^d\times\mathbb{S}^{d-1}}\int\sigma e^{-\sigma s}\mathbb{1}_{t<s}\mathbb{1}_{t<\tau_\varepsilon(x,v)}dxdvds\\
&= \left(\int_t^\infty\sigma e^{-\sigma s}ds\right)\iint_{Z_\varepsilon\cap[0,1]^d\times\mathbb{S}^{d-1}}\mathbb{1}_{t<\tau_\varepsilon(x,v)}dxdv\\
&= e^{-\sigma t}\frac{1}{\varepsilon^d}\iint_{\mathcal{Z}_\varepsilon^d\times\mathbb{S}^{d-1}}\mathbb{1}_{t<\tau_\varepsilon(x,v)}dxdv\\
&= \frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d}e^{-\sigma t}\mathbb{P}_{unif}\left(\tau_\varepsilon>t\right).
\end{aligned}$$

Consider next the case $n>0$. Since for each $(x,v)\in Z_\varepsilon\cap[0,1]^d\times\mathbb{S}^{d-1}$ and for each $t\ge 0$,

$$\mathbb{1}_{t<\tau_\varepsilon(x,v)}\le\mathbb{1}_{t<T_\varepsilon(v)},\tag{3.10}$$

we have in view of (3.4)

$$\begin{aligned}
I_n(t) &= \iint_{Z_\varepsilon\cap[0,1]^d\times\mathbb{S}^{d-1}}\mathbf{E}_{x,v}\left[\mathbb{1}_{S_n<t<S_n+T_{n+1}}\prod_{1\le i\le n}\mathbb{1}_{T_i<\tau_\varepsilon(X_{i-1},V_{i-1})}\right]dxdv\\
&\le \iint_{Z_\varepsilon\cap[0,1]^d\times\mathbb{S}^{d-1}}\mathbf{E}_{x,v}\left[\mathbb{1}_{S_n<t<S_n+T_{n+1}}\prod_{1\le i\le n}\mathbb{1}_{T_i<T_\varepsilon(V_{i-1})}\right]dxdv.
\end{aligned}$$

And thus

$$\begin{aligned}
I_n(t)\le\iint_{Z_\varepsilon\cap[0,1]^d\times\mathbb{S}^{d-1}}&\int\mathbb{1}_{t_1+\cdots+t_n<t<t_1+\cdots+t_{n+1}}\prod_{1\le i\le n}\mathbb{1}_{t_i<T_\varepsilon(v_{i-1})}\\
&dxdv\mu_{T_1}(dt_1)\cdots\mu_{T_{n+1}}(dt_{n+1})\mu_{V_1}(v,dv_1)\cdots\mu_{V_{n-1}}(v_{n-2},dv_{n-1}).
\end{aligned}$$

In view of our assumptions on the random variables $(T_i)_{1\le i\le n+1}$ and $(V_i)_{1\le i\le n}$, the distribution of T_i is $\mu_{T_i}(dt_i) = \sigma e^{-\sigma t_i} dt_i$ while the one of V_i is $\mu_{V_i}(v_{i-1}, dv_i) = \frac{k(v_{i-1},v_i)}{|\mathbb{S}^{d-1}|} dv_i$ for each $1 \le i \le n$. Since $k \le 1$, replace k with 1 in the integrand in the right-hand side of the inequality above and integrate in $v, (v_i)_{1\le i\le n}$ on $\mathbb{S}^{d-1}$ to obtain with inequality (3.10)

$$\begin{aligned} I_n(t) \le & |\mathbb{S}^{d-1}| \int_{Z_\varepsilon \cap [0,1]^d} dx \int \mathbb{1}_{t_1+\cdots+t_n<t<t_1+\cdots+t_{n+1}} \prod_{i=1}^n \mathcal{P}_\varepsilon(t_i) \\ & \mu_{T_1}(dt_1)\cdots\mu_{T_{n+1}}(dt_{n+1}). \end{aligned}$$

Since

$$\int_{Z_\varepsilon \cap [0,1]^d} dx = \frac{1}{\varepsilon^d} \int_{\mathcal{Z}_\varepsilon} dx = \frac{|\mathcal{Z}_\varepsilon|}{\varepsilon^d},$$

one has

$$\begin{aligned} I_n(t) \le & \frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d} \int \mathbb{1}_{t_1+\cdots+t_n<t<t_1+\cdots+t_{n+1}} \prod_{i=1}^n \mathcal{P}_\varepsilon(t_i) \\ & \mu_{T_1}(dt_1)\cdots\mu_{T_{n+1}}(dt_{n+1}). \end{aligned}$$

Now integrate in t_{n+1} on $[t-(t_1+\cdots+t_n), +\infty]$ to obtain

$$\begin{aligned} I_n(t) \le \frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d} \int \mathbb{1}_{t_1+\cdots+t_n<t} e^{-\sigma(t-(t_1+\cdots+t_n))} \prod_{i=1}^n \mathcal{P}_\varepsilon(t_i) \\ \mu_{T_1}(dt_1)\cdots\mu_{T_n}(dt_n) \end{aligned}$$

or equivalently

$$\begin{aligned} I_n &\le \frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d} \int \mathbb{1}_{t_1+\cdots+t_n<t} e^{-\sigma(t-(t_1+\cdots+t_n))} \prod_{i=1}^n \sigma e^{-\sigma t_i} \mathcal{P}_\varepsilon(t_i)\, dt_1\cdots dt_n, \\ &\le \frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d} \int \mathbb{1}_{t_1+\cdots+t_n<t} e^{-\sigma(t-(t_1+\cdots+t_n))} \prod_{i=1}^n g_{\sigma,\varepsilon}(t_i)\, dt_1\cdots dt_n, \\ &\le \frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d} \int_0^t e^{-\sigma(t-s)} \left(g_{\sigma,\varepsilon}\right)^{*n}(s) ds. \end{aligned}$$

□

3.3. **Proof of Proposition** 3.2.

Proof. Lemma 3.4 states that

$$M_{\sigma,\varepsilon}(t) \le \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})} \sum_{n\ge 0} I_n(t).$$

That implies, by Lemma 3.5

$$\begin{aligned} M_{\sigma,\varepsilon}(t) \le & \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})} \frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d} \mathbb{P}_{unif}(\tau_\varepsilon > t) e^{-\sigma t} \\ & + \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})} \frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d} \int_0^t e^{-\sigma(t-s)} \sum_{n\ge 1} g_{\sigma,\varepsilon}^{*n}(s) ds, \end{aligned}$$

which is precisely the conclusion of Proposition 3.2. □

4. The long-time behavior of the total mass

4.1. **Introduction.** By Corollary 3.3, for each $t \geq 0$

$$M_{\sigma,\varepsilon}(t) \leq \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}\frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d}\left(e^{-\sigma t}\mathbb{P}_{unif}\left(\tau_\varepsilon > t\right) + \int_0^t e^{-\sigma(t-s)}\sum_{n\geq 1} g_\sigma^{*n}(s)ds\right),$$

or

$$M_{\sigma,\varepsilon}(t) \leq \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}\frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d}\left(e^{-\sigma t}\mathbb{P}_{unif}\left(\tau_\varepsilon > t\right) + \int_0^t e^{-\sigma(t-s)}F_\sigma(s)ds\right),$$

with

$$F_\sigma(t) := \sum_{n\geq 1}\left(g_\sigma\right)^{*n}(t) \text{ for each } t > 0.$$

We establish in the present section the following

Proposition 4.1. *Let $\sigma > 0$, under the assumptions, and with the notations above, there exists $\eta_\sigma \in (-s, 0)$ and $C > 0$ such that for each $t \geq 0$*

$$F_\sigma(t) \leq Ce^{\eta_\sigma t},$$

with $C_\eta \in L^\infty(\mathbb{R}_+)$ and $C_\eta(t) \to 0$ as $t \to +\infty$. Moreover $\eta_\sigma \sim -\sigma$ as $\sigma \to 0^+$.

We first show how this proposition entails Theorem 2.1.

Proof. Recall that for each $\sigma > 0$ and for each $\varepsilon \in (0, 2^{1-d})$,

$$M_{\sigma,\varepsilon}(t) \leq \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}\frac{|\mathcal{Z}_\varepsilon||\mathbb{S}^{d-1}|}{\varepsilon^d}\left(e^{-\sigma t}\mathbb{P}_{unif}\left(\tau_\varepsilon > t\right) + \int_0^t e^{-\sigma(t-s)}F_\sigma(s)ds\right),$$

or since

$$|\mathcal{Z}_\varepsilon| = \varepsilon^d\left(1 - \varepsilon^{\frac{d}{d-1}}|B(0,1)|\right) \leq \varepsilon^d,$$

and for each $t \geq 0$ $\mathbb{P}_{unif}\left(\tau_\varepsilon > t\right) \leq 1$,

$$M_{\sigma,\varepsilon}(t) \leq \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}|\mathbb{S}^{d-1}|\left(e^{-\sigma t} + \int_0^t e^{-\sigma(t-s)}F_\sigma(s)ds\right).$$

By Proposition 4.1,

$$\begin{aligned}M_{\sigma,\varepsilon}(t) &\leq \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}|\mathbb{S}^{d-1}|\left(e^{-\sigma t} + \int_0^t e^{-\sigma s}e^{\eta_\sigma(t-s)}Cds\right),\\ &\leq \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}|\mathbb{S}^{d-1}|G(t)e^{\eta_\sigma t},\end{aligned}$$

with

$$\begin{aligned}G(t) &:= e^{-(\sigma+\eta_\sigma)t} + C\int_0^t e^{-(\sigma+\eta_\sigma)s}ds,\\ &= e^{-(\sigma+\eta_\sigma)t} + \frac{C}{\sigma+\eta_\sigma}\left(1 - e^{-(\sigma+\eta_\sigma)t}\right).\end{aligned}$$

Since $\eta_\sigma \in (-\sigma, 0)$, we have $\sigma + \eta_\sigma > 0$ so that we have

$$G(t) \leq 1 + \frac{C}{\sigma+\eta_\sigma} \text{ for each } t \geq 0.$$

Hence for each $\sigma > 0$

$$M_{\sigma,\varepsilon}(t) \leq \|f^{in}\|_{L^\infty(\mathbb{R}^d\times\mathbb{S}^{d-1})}Ce^{\eta_\sigma t} \text{ for each } \varepsilon \in (0, 2^{1-d})$$

with

$$C = |\mathbb{S}^{d-1}|\left(1 + \frac{C}{\sigma+\eta_\sigma}\right)$$

which is precisely the conclusion of Theorem 2.1. □

4.2. **A geometric series.**

4.2.1. *The Lapace Transform.* First, for each $\xi \in \mathbb{R}$ and each locally bounded measurable function $f : \mathbb{R}_+ \mapsto \mathbb{R}$, define its Laplace transform

$$\mathcal{L}[f](\xi) := \int_0^\infty e^{-\xi t} f(t) dt.$$

Recall that for each $f, g : \mathbb{R}_+ \mapsto \mathbb{R}$, we have

$$\mathcal{L}[f * g](\xi) = \mathcal{L}[f](\xi)\mathcal{L}[g](\xi)$$

where $*$ denote the convolution product on the half-line, defined by

$$f * g(t) = \int_0^t f(t-s)g(s)ds.$$

4.2.2. *The function F_σ.* Recall that

$$g_\sigma(t) = \sigma e^{-\sigma t} \inf\left(1, \frac{C'[d]}{t^d}\right) \text{ for each } t \geq 0$$

We denote henceforth $g : t \mapsto \inf\left(1, \frac{C'[d]}{t^d}\right)$ so that for each $t \geq 0$

$$g_\sigma(t) = \sigma e^{-\sigma t} g(t).$$

Obviously, g is nonincreasing, positive and nonintegrable.Since g_σ is a locally bounded measurable function vanishing identically on $\mathbb{R}_-^*$, we have for each $\xi > -\sigma$

$$e^{-\xi t} F_\sigma(t) = \sum_{n \geq 1} \left(e^{-\xi t} g_\sigma\right)^{*n}(t) = \sum_{n \geq 1} \sigma^n \left(e^{-(\sigma+\xi)t} g\right)^{*n}(t).$$

As F_σ is a series of nonnegative functions, by monotone convergence we can integrate in t on $\mathbb{R}$ term by term so that

$$\mathcal{L}[F_\sigma](\xi) = \sum_{n \geq 1} \sigma^n \left(\mathcal{L}[g](\xi + \sigma)\right)^n. \tag{4.1}$$

The right-hand side of the equation above is a geometric series of ratio

$$\sigma \mathcal{L}[g](\xi + \sigma),$$

so that $\mathcal{L}[F_\sigma](\xi) < +\infty$ if and only if $\sigma \mathcal{L}[g](\xi + \sigma) < 1$. Before going further in the study of the Laplace Transform of F_σ, we establish some properties of this function.

Proposition 4.2. *Under the assumptions and with the notations above, the function F_σ is continue, integrable and its derivative is piecewise continue.*

Proof. First, we show that the sequence

$$F_{\sigma,N}(t) := \sum_{n=1}^N g_\sigma^{*n}(t)$$

converges pointwise to F_σ. On the first hand, as g_σ is nonnegative, for each $t \geq 0$, the sequence $\left(F_{\sigma,n}(t)\right)_{n \geq 1}$ is nondecreasing. On the second hand, we have for each $t \geq 0$

$$g_\sigma(t) \leq \sigma e^{-\sigma t}$$

thus one obtains by a straightforward computation for each $n \geq 1$ each $t \geq 0$

$$g_\sigma^{*(n+1)}(t) \leq \sigma \frac{\sigma^n}{n!} t^n e^{-\sigma t}$$

so that for each $t \geq 0$

$$\begin{aligned} F_\sigma(t) &= g_\sigma(t) + \sum_{n\geq 2} g_\sigma^{*n}(t) \\ &\leq g_\sigma(t) + \sigma e^{-\sigma t} \sum_{n\geq 1} \frac{\sigma^n}{n!} t^n \\ &\leq g_\sigma(t) + \sigma e^{-\sigma t}\left(e^{\sigma t} - 1\right) \\ &\leq g_\sigma(t) + \sigma\left(1 - e^{-\sigma t}\right). \end{aligned}$$

In other words, for each $t \geq 0$ the sequence $\left(F_{\sigma,n}(t)\right)_{n\geq 1}$ is bounded. Therefore, for each $t \geq 0$, the series converges pointwise to F_σ.

Now we notice that it verifies for each $t \geq 0$

$$F_\sigma = g_\sigma + g_\sigma * F_\sigma$$

which implies, since g_σ is continue, that F_σ is continue.

We show that $F_\sigma \in L^1(\mathbb{R}_+)$. We notice that g is differentiable a.e. with, since, g being nonincreasing

$$\dot{g}(t) \leq 0 \ \forall t \in \mathbb{R}_+.$$

So that

$$\begin{aligned} \|g_\sigma\|_{L^1(\mathbb{R}_+)} &= \int_0^\infty \sigma e^{-\sigma s} g(s) ds \\ &= 1 - \int_0^\infty e^{-\sigma s} \dot{g}(s) ds < 1, \end{aligned}$$

moreover, Young inequality entails for each $n \geq 1$

$$\|g_\sigma^{*n}\|_{L^1(\mathbb{R}_+)} \leq \|g_\sigma\|^n_{L^1(\mathbb{R}_+)}.$$

Therefore, the series converges normally in $L^1(\mathbb{R}_+)$ and thus $F_\sigma \in L^1(\mathbb{R}_+)$.

Before establishing the piecewise differentiability of F_σ, we prove the

Lemma 4.3. *For $f, g \in L^1(\mathbb{R}_+)$ such that f is bounded, continue at $t = 0$ and differentiable almost everywhere, we have*

$$\frac{d}{dt}\left(\int_0^t f(t-s)g(s)ds\right)(t) = f(0)g(t) + \int_0^t \left(\frac{d}{dt}f\right)(t-s)g(s)ds.$$

Proof. We have for each $h \neq 0$

$$\begin{aligned} \int_0^{t+h} f(t+h-s)g(s)ds - \int_0^t f(t-s)g(s)ds &= \int_t^{t+h} f(t+h-s)g(s)ds \\ &\quad + \int_0^t \left(f((t-s)+h) - f(t-s)\right)g(s)ds. \end{aligned} \tag{4.2}$$

As f is differentiable a.e., we have a.e. in $s \in (0, t]$

$$\frac{f((t-s)+h) - f(t-s)}{h} \to \frac{d}{dt}f(t-s) \text{ as } h \to 0,$$

besides, f is bounded, which implies that there exists $C > 0$ such that

$$\left|\left(f((t-s)+h) - f(t-s)\right)g(s)\right| \leq C\,|g(s)| \in L^1(\mathbb{R}_+).$$

Thus, one obtains by dominated convergence

$$\frac{1}{h}\int_0^t \left(f((t-s)+h) - f(t-s)\right)g(s)ds \to \int_0^t \left(\frac{d}{dt}f\right)(t-s)g(s)ds \text{ as } h \to 0.$$

We notice that for each $s \in [t, t+h]$, we have $t + h - s \in [0, h]$ so that we have

$$\mathbb{1}_{t\leq s\leq t+h} f(t+h-s) - f(0) \to 0 \text{ as } h \to 0$$

and we have

$$\left|\mathbb{1}_{t\leq s\leq t+h} f(t+h-s)-f(0)\right|\left|g(s)\right|\leq Cg(s)\in L^{1}\left(\mathbb{R}_{+}\right)$$

so that by dominated convergence

$$\frac{1}{h}\int_{t}^{t+h}\left(f(t+h-s)-f(0)\right)g(s)ds\to 0 \text{ as } h\to 0$$

and one has obviousely

$$\frac{1}{h}f(0)\int_{t}^{t+h}g(s)ds\to f(0)g(t) \text{ as } h\to 0.$$

Therefore

$$\frac{1}{h}\int_{t}^{t+h}f(t+h-s)g(s)ds\to f(0)g(t) \text{ as } h\to 0.$$

Consequently, equality (4.2) entails

$$\frac{\int_{0}^{t+h}f(t+h-s)g(s)ds-\int_{0}^{t}f(t-s)g(s)ds}{h}\to f(0)g(t)+\int_{0}^{t}\left(\frac{d}{dt}f\right)(t-s)g(s)ds$$

as $h\to 0$. □

That being established, we recall that F_σ verifies

$$F_\sigma = g_\sigma + g_\sigma * F_\sigma$$

The function g_σ is differentiable a.e. with $\dot{g}_\sigma$ piecwise continue. It is continue at $t=0$, bounded and F_σ is integrable, so that, by Lemma above, $g_\sigma * F_\sigma$ is differentiable and continue. So that F_σ is differentiable a.e. and $\dot{F}_\sigma$ is piecewise continue. □

4.3. **The characteristic exponent η_σ.**

4.3.1. *The abscissa of convergence of $\mathcal{L}[F_\sigma]$.*

Lemma 4.4. *For each $\sigma>0$, the equation*

$$\int_{0}^{\infty}e^{-(\sigma+\xi)t}g(t)dt=\frac{1}{\sigma}$$

with unknown ξ has a unique real root η_σ. This root η_σ satisfies

$$-\sigma<\eta_\sigma<0.$$

Therefore $\mathcal{L}\left[F\right](\xi)<+\infty$ if and only if $\xi>\eta_\sigma$.

Proof. Consider the function

$$\Upsilon(x):=\int_{0}^{\infty}e^{-xt}g(t)dt,\ x\geq 0.$$

As $0<g\leq 1$ and $g\notin L^{1}(\mathbb{R}_{+})$, Υ is of class C^{1} on $]0,+\infty[$ and

$$\dot{\Upsilon}(x)=-t\int_{0}^{\infty}e^{-xt}g(t)dt<0$$

since $g(t)>0$ for each $t\geq 0$. The function Υ is therefore decreasing. For each $t>0$, $e^{-xt}g(t)\to 0^{+}$ as $t\to+\infty$ and $e^{-xt}g(t)\leq e^{-t}$ for each $x\geq 1$, so that by dominated convergence,

$$\Upsilon(x)\to 0^{+} \text{ as } x\to+\infty.$$

For each $t>0$, $e^{-xt}g(t)$ converges increasly to g as $x\to 0^{+}$. Moreover, g is not integrable so that by monotone convergence,

$$\Upsilon(x)\to+\infty \text{ as } x\to 0^{+}.$$

Therefore, by the intermediate value theorem, there exists a unique $x_\sigma > 0$ such that

$$\Upsilon(x_\sigma) = \frac{1}{\sigma}. \tag{4.3}$$

Notice that

$$\begin{aligned}\Upsilon(\sigma) &= \int_0^\infty e^{-\sigma t} g(t)dt = \frac{1}{\sigma}\left(\sigma \int_0^\infty e^{-\sigma t} g(t)dt\right)\\ &= \frac{1}{\sigma}\left(1 + \int_0^\infty e^{-\sigma t}\dot{g}(t)dt\right) < \frac{1}{\sigma} = \Upsilon(x_\sigma)\end{aligned}$$

as g is nonincreasing, thus $0 < x_\sigma < \sigma$ since Υ is decreasing. We conclude by defining:

$$\eta_\sigma := x_\sigma - \sigma.$$

□

4.3.2. *The long-time behavior of F_σ.*

Lemma 4.5. *Under the assumptions and with the notations above, we have*

$$F_\sigma(t) \sim C e^{\eta_\sigma t} \text{ as } t \to +\infty$$

with

$$C = \frac{1}{\int_0^\infty s g_\sigma(s) e^{-\eta_\sigma s} ds}.$$

Proof. Notice that for each $\lambda \in \mathbb{R}_+$ and for each $f, g \in L^1(\mathbb{R}_+)$, we have

$$e^{\lambda t}(f * g)(t) = (f_\lambda * g_\lambda)(t)$$

where for each locally bounded measurable function f_λ denotes

$$f_\lambda(t) := e^{\lambda t} f(t) \text{ for each } t \in \mathbb{R}_+.$$

Hence, as F_σ verifies

$$F_\sigma = g_\sigma + g_\sigma * F_\sigma,$$

the function $\psi : t \mapsto e^{-\eta_\sigma t} F_\sigma(t)$ satisfies, with $\kappa_\sigma(t) := g_\sigma e^{-\eta_\sigma t}$,

$$\psi(t) = \kappa_\sigma + \kappa_\sigma * \psi$$

which is a renewal integral equation in the sense of [8]. Besides, by definition of η_σ, we have

$$\int_0^\infty e^{-\eta_\sigma s} g_\sigma(s) ds = 1$$

so that κ_σ is a decreasing probability density on $\mathbb{R}_+$ and in particular, it is directly Riemann integrable (see [8] pp. 348-349). Thus, by Theorem 2 on p. 349 in [8], one has

$$\psi(t) \to \frac{1}{\int_0^\infty s g_\sigma(s) e^{-\eta_\sigma s} ds} \text{ as } t \to +\infty.$$

One obtains therefore the asymptotic behavior of F_σ. □

4.3.3. *Proof of the first statement of Proposition* 4.1.

Proof. By Lemma 4.5, there exists a measurable h such that

$$h(t) \to 0 \text{ as } t \to +\infty$$

and for each $t \geq 0$

$$F_\sigma(t) = C e^{\eta_\sigma t} + h(t) e^{\eta_\sigma t}.$$

By Proposition 4.2, F_σ is continue and thus h is continue. Hence there exists $C > 0$ such that

$$F_\sigma(t) \leq C e^{\eta_\sigma t} \text{ for each } t \geq 0,$$

which is the conclusion of Proposition 4.1. □

4.4. **Asymptotic behavior of η_σ.** We conclude here our proof of Proposition 4.1 with a discussion of the asymptotic behavior of η_σ (statement 2 of Theorem 2.1) in collisionless regime ($\sigma \to 0^+$).

Proof. Recall that $x_\sigma = \sigma + \eta_\sigma$, where x_σ is defined in (4.3). Establishing that $\eta_\sigma \sim -\sigma$ as $\sigma \to 0^+$ amounts to proving that $\frac{x_\sigma}{\sigma} \to 0$ as $\sigma \to 0^+$. First, remark that since $-\sigma < \eta_\sigma$,

$$0 < x_\sigma < \sigma$$

so $x_\sigma \to 0^+$ as $\sigma \to 0^+$. Recall that

$$\int_0^\infty g(t) e^{-x_\sigma t} dt = \frac{1}{\sigma}.$$

or after substituting $z = x_\sigma t$ in the integral above, we obtain:

$$\frac{x_\sigma}{\sigma} = \int_0^\infty g\left(\frac{z}{x_\sigma}\right) e^{-z} dz.$$

Since $x_\sigma \to 0^+$ as $\sigma \to 0^+$ and $g(t) \to 0^+$ as $t \to +\infty$, one has $g(z/x_\sigma) \to 0^+$ as $\sigma \to 0^+$. Besides $0 \leq e^{-z} p_r(z/x_\sigma) \leq e^{-z}$ so that, by dominated convergence

$$\int_0^\infty g\left(\frac{z}{x_\sigma}\right) e^{-z} dz \to 0^+ \text{ as } \sigma \to 0^+,$$

consequently

$$\frac{x_\sigma}{\sigma} \to 0 \text{ as } \sigma \to 0^+.$$

□

5. Conclusion

We have proved that the total mass of a monokinetic system of point particles governed by a linear Boltzmann equation in a periodic distribution of spherical absorbers at the critical size decays exponentially fast in the long time limit. This behavior is at variance with the non collisional case, where the total mass decays like C/t as $t \to +\infty$.

As explained above, the alegraic decay in the noncollisional case is due to the presence of sufficiently many "channels", corresponding with arbitrary free particle trajectories. The collision operator in the linear Boltzmann equation destroys the purely geometric effect of these channels, even for very low collision frequencies.

It could be interesting to consider the analogous problem for a granular gas with inelastic collisions, as such collisions have a cooling effect that might destroy the exponential decay. For instance, if $f \equiv f(t, v)$ is a space homogeneous solution of the inelastic Boltzmann equation, then it is known that

$$f(t, v) \to \delta_{v=0} \text{ as } t \to +\infty$$

in $\mathcal{M}\left(\mathbb{R}^N\right)$ (see for instance [15, 16].) The slowing down of gas particles obviously downgrades the absorption effect of the holes.

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Chapitre 2

Homogenization of the linear Boltzmann equation in a perforated domain

CHAPTER II
HOMOGENIZATION OF THE LINEAR BOLTZMANN EQUATION IN A DOMAIN WITH A PERIODIC DISTRIBUTION OF HOLES

1. Introduction

The homogenization of a transport process describing the motion of particles in a system of fixed obstacles — such as scatterers, or holes — leads to very different results according to whether the distribution of obstacles is periodic or random. Before describing the specific problem analyzed in the present work, we recall a few results recently obtained on a more complicated, and yet related problem.

An important example of the phenomenon mentioned above is the Boltzmann-Grad limit of the Lorentz gas. The Lorentz gas is the dynamical system corresponding to the free motion of a single point particle in a system of fixed spherical obstacles, assuming that each collision of the particle with any one of the obstacles is purely elastic. Since the particle is not subject to any external force, we assume without loss of generality that its speed is 1. The Boltzmann-Grad limit is the scaling limit where the obstacle radius and the reciprocal number of obstacles per unit volume vanish in such a way that the average free path length of the particle between two consecutive collisions with the obstacles is of the order of unity.

Call $f(t,x,v)$ the particle distribution function in phase space in that scaling limit — in other words, the probability that the particle be located in an infinitesimal volume dx around the position x with direction in an infinitesimal element of solid angle dv around the direction v at time $t \geq 0$ is $f(t,x,v)dxdv$.

In the case of a random system of obstacles — more precisely, assuming that the obstacles centers are independent and distributed in the 3-dimensional Euclidian space under Poisson's law — Gallavotti proved in [15, 16] (see also [17] on pp. 48–55) that the average of f over obstacle configurations (i.e. the mathematical expectation of f) is a solution of the linear Boltzmann equation

$$(\partial_t + v\cdot\nabla_x + \sigma)f(t,x,v) = \frac{\sigma}{\pi}\int_{\substack{\omega\cdot v>0\\|\omega|=1}} f(t,x,v-2(\omega\cdot v)\omega)\omega\cdot v d\omega\,.$$

If, on the contrary, the obstacles are periodically distributed — specifically, if they are centered at the vertices of a cubic lattice — the limiting particle distribution function f cannot be the solution of any linear Boltzmann equation of the form

$$(\partial_t + v\cdot\nabla_x + \sigma)f(t,x,v) = \sigma\int_{|w|=1} p(v|w)f(t,x,w)dw\,,$$

where p is a continuous, symmetric transition probability density on the unit sphere: see [18] for a complete proof of this negative result, based on earlier estimates on the distribution of free path lengths for the periodic Lorentz gas [6, 19].

The correct limiting equation for the Boltzmann-Grad limit of the periodic Lorentz gas was found only very recently: see [8, 25]. In the 2-dimensional case, the most striking feature of the theory presented in these references, is that the limiting equation is set on an *extended phase space* involving not only the particle position x and direction v, as in all classical kinetic models, but also the (rescaled) distance τ to the next collision point with the obstacles and the impact parameter h at this next collision point.

The particle motion is described in terms of its distribution function in this extended phase space, $F \equiv F(t, x, v, \tau, h)$, which is governed by an equation of the form

$$(1)\quad \begin{aligned}(\partial_t + v \cdot \nabla_x - \partial_\tau)&F(t, x, v, \tau, h) \\ &= \int_{-1}^{1} P(\tau, h|h')F(t, x, R[\pi - 2\arcsin(h')]v, 0, h')dh'\end{aligned}$$

where $R[\theta]$ designates the rotation of an angle θ, and $P(\tau, h|h')$ is a nonnegative integral kernel whose explicit expression is given in [8] but is of little interest for the present discussion. The particle distribution function in the classical phase space of kinetic theory is recovered in terms of F by the following formula:

$$f(t, x, v) = \int_0^{+\infty} \int_{-1}^{1} F(t, x, v, \tau, h)dhd\tau\,.$$

However, the particle distribution function f itself does not satisfy a linear Boltzmann equation in closed form.

Loosely speaking, in the case of a periodic distribution of obstacles, the particle "feels" the correlations between the obstacles, since its trajectory consists of segments of maximal length avoiding the obstacles. This explains the need for an extended phase space in order to describe the Boltzmann-Grad limit of the Lorentz gas, in the periodic case. In the random case studied by Gallavotti, the obstacles centers are assumed to be independent, which reduces the complexity of the limiting dynamics.

In the present work, we shall study a much simpler homogenization problem, which can be formulated as follows:

Problem. Consider a system of point particles whose distribution function is governed by a linear Boltzmann equation. The particles are assumed to move in a periodic system of holes. Describe the asymptotic behavior of the total mass of the particle system in the long time limit, assuming that the radius of the holes and their reciprocal number per unit volume vanish so that the average distance between holes is of the order of 1.

This problem is the analogue in kinetic theory of the one studied in [23] and [11] for the diffusion equation, and in [2] for the Stokes equation.

Although the underlying dynamics in this problem is a lot simpler than that of the Lorentz gas, the homogenized equation is also set on an extended phase space, analogous to the one described above.

A we shall see, the mathematical derivation of the homogenized equation in the extended phase space for the problem above involves only very elementary arguments from functional analysis — at variance with the case of the Boltzmann-Grad limit of the Lorentz gas, which requires a fairly detailed knowledge of particle trajectories.

2. The model

We consider the monokinetic, linear Boltzmann equation

$$(2)\quad \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \sigma(f_\varepsilon - Kf_\varepsilon) = 0$$

in space dimension 2.

The unknown function $f(t,x,v)$ is the density at time $t\in\mathbb{R}_+$ of particles with velocity $v\in\mathbb{S}^1$, located at $x\in\mathbb{R}^2$. For each $\phi\in L^2(\mathbb{S}^1)$, we denote

$$K\phi(v):=\frac{1}{2\pi}\int_{\mathbb{S}^1}k(v,w)\phi(w)dw,$$

where dw is the uniform measure (arc length) on the unit circle $\mathbb{S}^1$. We henceforth assume that

$$\begin{aligned}&k\in L^2(\mathbb{S}^1\times\mathbb{S}^1)\,,\quad k(v,w)=k(w,v)\geq 0\ \text{ a.e. in } v,w\in\mathbb{S}^1\\ &\qquad\qquad\text{and } \tfrac{1}{2\pi}\int_{\mathbb{S}^1}k(v,w)dw=1\ \textit{a.e. in}\ v\in\mathbb{S}^1.\end{aligned}\tag{3}$$

The case of isotropic scattering, where k is a constant, is a classical model in the context of Radiative Transfer. Likewise, the case of Thomson scattering in Radiative Transfer involves the integral kernel

$$k(v,w)=\tfrac{3}{16}(1+(v\cdot w)^2)$$

— see for instance chapter I, §16 of [10]. Finally, the collision frequency is a constant $\sigma>0$.

The linear Boltzmann equation (2) is set on the spatial domain Z_ε, i.e. the space $\mathbb{R}^2$ with a periodic system of holes removed:

$$Z_\varepsilon:=\left\{x\in\mathbb{R}^2\,|\,\mathrm{dist}(x,\varepsilon\mathbb{Z}^2)>\varepsilon^2\right\}\,.$$

We assume an absorption boundary condition on ∂Z_ε:

$$f_\varepsilon=0\text{ for }(t,x,v)\in\mathbb{R}_+^*\times\partial Z_\varepsilon\times\mathbb{S}^1,\ \text{ whenever } v\cdot n_x>0\,,$$

where n_x denotes the inward unit normal vector to Z_ε at the point $x\in\partial Z_\varepsilon$. This condition means that a particle falling into any one of the holes remain there forever.

The same problem could of course be considered in any space dimension. Notice however that, in space dimension $N\geq 2$, the appropriate scaling, analogous to the one considered here, would be to consider holes of radius $\varepsilon^{N/(N-1)}$ centered at the points of the cubic lattice $\varepsilon\mathbb{Z}^N$ — see for instance [6, 19]. Most of the arguments considered in the present paper can be adapted without change to the higher dimensional case, except that the expression of one particular coefficient appearing in the homogenized equation is not yet known explicitly at the time of this writing.

The most natural question related to the dynamics of the system above is the asymptotic behavior of the total mass of the particle system in the small obstacle radius $\varepsilon\ll 1$ and long time limit.

Emanuele Caglioti and François Golse have considered in [7] the non-collisional case ($\sigma=0$) and proved that, in the limit as $\varepsilon\to 0^+$, the solution f_ε converges in $L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)$ weak-* to a solution f of the following non-autonomous equation:

$$\partial_t f+v\cdot\nabla_x f=\frac{\dot p(t)}{p(t)}f\,,\tag{4}$$

where p is a positive decreasing function defined below. In that case, the total mass of the particle system decays like Const.$/t$ as $t\to+\infty$.

Observe that, starting from the free transport equation, we obtain a non-autonomous (in time) equation in the small ε limit. In particular, the solution of equation (4) cannot be given by a semigroup in a function space such as $L^p(\mathbb{R}^2_x\times\mathbb{S}^1_v)$. As we shall see, the homogenization of the linear Boltzmann equation in the collisional case ($\sigma>0$) leads to an even more spectacular change of structure in the equivalent equation obtained in the limit.

The work of the last two authors [7] relies upon an explicit computation of the solution of the free transport equation, where the effect of the system of holes is handled with continued fraction techniques. In the present paper, we investigate the analogous homogenization problem in the collisional case ($\sigma > 0$). As we shall see, there is no explicit representation formula for the solution of the linear Boltzmann equation, other than the one based on the transport process, a particular stochastic process, defined for example in [26].

This representation formula was used in the previous chapter, who established a uniform in ε upper bound for the total mass of the particle system by a quantity of the form Const.$e^{-a_\sigma t}$ for some $a_\sigma > 0$. This exponential decay is quite remarkable: indeed, there is a "phase transition" between the collisionless case in which the total mass decays algebraically as $t \to +\infty$, and the collisional case in which the total mass decays at least exponentially fast in that same limit.

In the present paper, we further investigate this phenomenon and show that the exponential decay estimate found in the previous chapter is sharp, by giving an asymptotic equivalent of the total mass of the particle system in the small ε limit as $t \to +\infty$.

Instead of the semi-explicit representation formula by the transport process, our argument is based on the very special structure of the homogenized problem. The key observation in the present work is that this homogenized problem involves a renewal equation, for which exponential decay is a classical result that can be found in classical monographs such as [14].

3. The main results

First, we recall the definition of the free path length in the direction v for a particle starting from x in Z_ε:

$$\tau_\varepsilon(x,v) := \inf\{t > 0 \,|\, x - tv \in \partial Z_\varepsilon\} \,. \tag{5}$$

The distribution of free path length has been studied in [6, 19, 7, 4]. In particular, it is proved that, for each arc $I \subset \mathbb{S}^1$ and each $t \geq 0$, one has

$$\operatorname{meas}(\{(x,v) \in (Z_\varepsilon \cap [0,1]^2) \times I \,|\, \varepsilon\tau_\varepsilon(x,v) > t\}) \to p(t)|I| \tag{6}$$

as $\varepsilon \to 0^+$, where $|I|$ denotes the length of I and the measure considered in the statement above is the uniform measure on $[0,1]^2 \times \mathbb{S}^1$.

The following estimate for p can be found in [6]: there exist $C, C' > 0$ such that, for all $t \geq 1$:

$$\frac{C}{t} \leq \operatorname{meas}(\{(x,v) \in (Z_\varepsilon \cap [0,1]^2) \times I \,|\, \varepsilon\tau_\varepsilon(x,v) > t\}) \leq \frac{C'}{t} \tag{7}$$

uniformly as $\varepsilon \to 0^+$, so that

$$\frac{C}{t} \leq p(t) \leq \frac{C'}{t} \,. \tag{8}$$

In [4] F. Boca and A. Zaharescu have obtained an explicit formula for p:

$$p(t) = \int_t^{+\infty} (\tau - t)\Upsilon(\tau)d\tau \,, \tag{9}$$

where the function Υ is expressed as follows:

$$\Upsilon(t) = \frac{24}{\pi^2}\begin{cases} 1 & \text{if } t \in (0, \frac{1}{2}], \\ \frac{1}{2t} + 2(1 - \frac{1}{2t})^2 \ln(1 - \frac{1}{2t}) - \frac{1}{2}(1 - \frac{1}{t})^2 \ln|1 - \frac{1}{t}| & \text{if } t \in (\frac{1}{2}, +\infty) \,. \end{cases} \tag{10}$$

This is precisely at this point that the case of space dimension 2 differs from the higher dimensional case. Indeed, in space dimension higher than 2, the existence

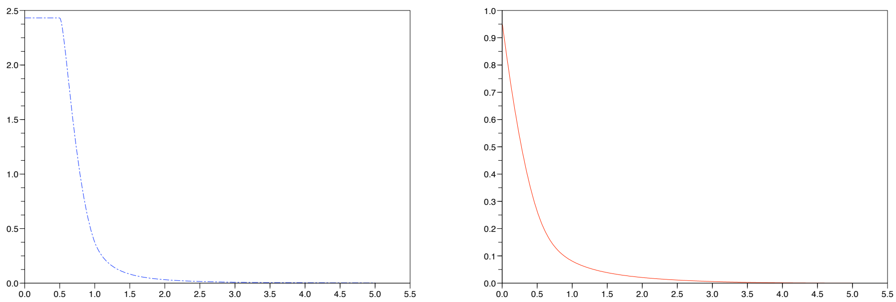

FIGURE 1. The graphs of Υ (left) and of p (right)

of the limit (6) has been proved in [24], while the uniform estimate analogous to (7) is to be found in [19]. However, no explicit formula analogous to (9) is known in that case, at least at the time of this writing. We have chosen to treat in the present paper only the case of the square lattice in space dimension 2 as it is the only case where the limit (6-9) is known completely.

Throughout this paper, we assume that the initial data of (Ξ_ε) satisfies the assumption

$$f^{in} \geq 0 \text{ on } \mathbb{R}^2 \times \mathbb{S}^1 \text{ and } \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv + \sup_{(x,v)\in\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v) < +\infty. \tag{11}$$

For each $0 < \varepsilon \ll 1$, let f_ε be the (mild) solution of the initial boundary value problem

$$(\Xi_\varepsilon)\begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \sigma(f_\varepsilon - Kf_\varepsilon) = 0, & (x,v)\in Z_\varepsilon\times\mathbb{S}^1, t>0,\\ f_\varepsilon = 0, \text{ if } v\cdot n_x > 0, & (x,v)\in\partial Z_\varepsilon\times\mathbb{S}^1,\\ f_\varepsilon(0,x,v) = f^{in}(x,v), & (x,v)\in Z_\varepsilon\times\mathbb{S}^1. \end{cases}$$

The classical theory of the linear Boltzmann equation guarantees the existence and uniqueness of a mild solution f_ε of the problem (Ξ_ε) satisfying

$$\begin{aligned} 0 \leq f_\varepsilon(t,x,v) \leq \sup_{(x,v)\in\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v) \quad \text{a.e. on } \mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1\,,\\ \iint_{Z_\varepsilon\times\mathbb{S}^1} f_\varepsilon(t,x,v)dxdv \leq \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv\,. \end{aligned} \tag{12}$$

Consider next $F := F(t,s,x,v)$ the solution of the Cauchy problem

$$(\Sigma)\begin{cases} \partial_t F + v\cdot\nabla_x F + \partial_s F = -\sigma F + \frac{\dot{p}}{p}(t\wedge s)F, & t,s>0,(x,v)\in\mathbb{R}^2\times\mathbb{S}^1\,,\\ F(t,0,x,v) = \sigma\displaystyle\int_0^{+\infty} KF(t,s,x,v)ds, & t>0,(x,v)\in\mathbb{R}^2\times\mathbb{S}^1\,,\\ F(0,s,x,v) = \sigma e^{-\sigma s} f^{in}(x,v), & s>0,(x,v)\in\mathbb{R}^2\times\mathbb{S}^1\,, \end{cases}$$

with the notation $t\wedge s := \min(t,s)$. Notice that F is a density defined on the extended phase space:

$$\big\{(s,x,v) | s\geq 0, x\in\mathbb{R}^2, v\in\mathbb{S}^1\big\}$$

involving the extra variable s, whose physical meaning is explained as follows.

Recall that the solution f_ε of the linear Boltzmann equation can be expressed in terms of the transport process (see [26]), a stochastic process involving a jump process in the v variable, perturbed by a drift in the x variable. The variable s is the "age" of the current velocity v in that process, i.e. the time since the last jump in the v variable.

Therefore, between jumps in the v variable, s increases with t, and this accounts for the sign of the additional term $+\partial_s F$ in the system (Σ).

On the contrary, in equation (1), the extra variable τ (the rescaled distance to the next collision point with one of the scatterers) decreases as t increases between collisions with the scatterers, which accounts for the minus sign in the additional term $-\partial_\tau F$ in that equation.

Henceforth, we shall frequently need to extend functions defined a.e. on Z_ε by 0 inside the holes (that is, in the complement of $\overline{Z_\varepsilon}$). We therefore introduce the following piece of notation.

Definition: For each function $\varphi \equiv \varphi(x)$ defined a.e. on Z_ε, we denote

$$\{\varphi\}(x) = \begin{cases} \varphi(x) & \text{if } x \in Z_\varepsilon, \\ 0 & \text{if } x \notin \overline{Z_\varepsilon}, \end{cases}$$

We use the same notation $\{f_\varepsilon\}$ or $\{F_\varepsilon\}$ to designate the same extension by 0 inside the holes for functions defined on cartesian products involving Z_ε as one of their factors, such as $\mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1$ in the case of f_ε, and $\mathbb{R}_+ \times \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1$ in the case of F_ε.

Our first main main result is

Theorem 1. *Under the assumptions above,*

$$\{f_\varepsilon\} \rightharpoonup \int_0^{+\infty} F ds$$

in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak-$*$ as $\varepsilon \to 0^+$, where F is the unique (mild) solution of (Σ).

Notice that the limit of the (extended) distribution function of the particle system is indeed defined in terms of the solution F of the homogenized integro-differential equation (Σ). However, it does not seem that the limit of $\{f_\varepsilon\}$ itself satisfies any natural equation.

Next we discuss the asymptotic decay as $t \to +\infty$ of the total mass of the particle system in the homogenization limit $\varepsilon \ll 1$. Obviously, the particle system loses mass due to particles falling into the holes.

In order to do so, we introduce the quantity:

$$m(t,s) := \tfrac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} F(t,s,x,v) dx dv \,.$$

A key observation in our work is that m is the solution of a renewal type PDE, as explained in the next proposition.

Proposition 1. *Denote*

$$B(t,s) = \sigma - \frac{\dot{p}}{p}(t \wedge s) \,,$$

and assume that f^{in} satisfies the condition (11).

Then the renewal PDE

$$\begin{cases} \partial_t \mu(t,s) + \partial_s \mu(t,s) + B(t,s)\mu(t,s) = 0, & t,s > 0\,, \\ \mu(t,0) = \sigma \displaystyle\int_0^{+\infty} \mu(t,s)ds, & t > 0\,, \\ \mu(0,s) = \sigma e^{-\sigma s}, & s > 0\,, \end{cases}$$

has a unique mild solution $\mu \in L^\infty([0,T]; L^1(\mathbb{R}_+))$ *for all* $T > 0$.

Moreover, one has

$$m(t,s) = \frac{\mu(t,s)}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv$$

a.e. in $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Renewal equations are frequently met in many different contexts. For instance they are used as a mathematical model in biology to study the dynamics of structured populations. The interested reader can consult [22] or [27] for more information on this subject.

Consider next the quantity:

$$M(t) := \tfrac{1}{2\pi} \int_0^{+\infty} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} F(t,s,x,v)dxdvds = \int_0^{+\infty} m(t,s)ds\,. \tag{13}$$

As explained in the theorem below, $M(t)$ is the total mass at time t of the particle system in the limit as $\varepsilon \to 0^+$; besides, the asymptotic behavior of M as $t \to +\infty$ is a consequence of the renewal PDE satisfied by the function $(t,s) \mapsto m(t,s)$.

Theorem 2. *Under the same assumptions as in theorem* 1*,*

(1) *the total mass*

$$\tfrac{1}{2\pi} \iint_{Z_\varepsilon \times \mathbb{S}^1} f_\varepsilon(t,x,v)dxdv \to M(t)$$

in $L^1_{loc}(\mathbb{R}_+)$ *as* $\varepsilon \to 0^+$, *and a.e. in* $t \geq 0$ *after extracting a subsequence of* $\varepsilon \to 0^+$*;*

(2) *the limiting total mass is given by the representation formula*

$$M(t) = \tfrac{1}{2\pi\sigma} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv \sum_{n \geq 1} \kappa^{*n}(t),\ t > 0$$

with

$$\kappa(t) := \sigma e^{-\sigma t} p(t) \mathbb{1}_{t \geq 0}\,, \quad \kappa^{*n} := \underbrace{\kappa * \cdots * \kappa}_{n \text{ factors}}$$

and $*$ *denoting as usual the convolution product on the real line;*

(3) *for each* $\sigma > 0$, *there exists* $\xi_\sigma \in (-\sigma, 0)$ *such that*

$$M(t) \sim C_\sigma e^{\xi_\sigma t} \text{ as } t \to +\infty$$

with

$$C_\sigma := \tfrac{1}{2\pi\sigma} \frac{\displaystyle\iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv}{\displaystyle\int_0^\infty tp(t)e^{-(\sigma+\xi_\sigma)t}dt}\,;$$

(4) *finally, the exponential mass loss rate* ξ_σ *satisfies*

$$\xi_\sigma \sim -\sigma \text{ as } \sigma \to 0^+, \text{ and } \xi_\sigma \to -2 \text{ as } \sigma \to +\infty\,.$$

Statement (1) above means that M is the limiting mass of the particle system at time t as $\varepsilon \to 0^+$. Statement (3) gives a precise asymptotic equivalent of $M(t)$ as $t \to +\infty$.

As recalled in the previous section, if $\sigma = 0$ in the linear Boltzmann equation (Ξ_ε), the total mass of the particle system in the vanishing ε limit is asymptotically equivalent to

$$\frac{\frac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv}{\pi^2 t}$$

as $t \to +\infty$. The reason for this slow, algebraic decay is the existence of channels — infinite open strips included in the spatial domain Z_ε, i.e. avoiding all the holes. Particles located in one such channel and moving in a direction close to the channel's direction will not fall into a hole before exiting the channel, and this can take an arbitrarily long time as the particles' direction approaches that of the channel. This construction based on channels leads to a sufficiently large fraction of the single-particle phase space and accounts for the algebraic lower bound in (8). The asymptotic equivalent mentioned above in the collisionless case $\sigma = 0$ is a consequence of a more refined analysis based on continued fractions given in [7].

When $\sigma > 0$, particles whose distribution function solves the linear Boltzmann equation in (Ξ_ε) travel on trajectories whose direction is discontinuous in time — more specifically, time discontinuities are distributed under an exponential law of parameter σ. Obviously, this circumstance destroys the channel structure that is responsible of the algebraic decay of the total mass of the particle system in the collisionless case, so that one expects that the total mass decay is faster than algebraic as $t \to +\infty$. That this decay is indeed exponential whenever $\sigma > 0$ is by no means obvious: see the argument in the previous chapter, leading to an upper bound for the total mass. Statement (3) above leads to an asymptotic equivalent of the total mass, thereby refining the conclusions of the previous chapter.

In section 4, we give the proof of theorem 1; the evolution of the total mass in the vanishing ε limit (governing equation and asymptotic behavior as $t \to +\infty$) is discussed in section 5.

4. The homogenized kinetic equation

Our argument for the proof of Theorem 1 is split into several steps.

4.1. A new formulation of the transport equation. Perhaps the most surprising feature in Theorem 1 is the introduction of the extended phase space involving the additional variable s.

As a matter of fact, this additional variable s can be used already at the level of the original linear Boltzmann equation — i.e. in the formulation of the problem (Ξ_ε).

Let us indeed return to the initial boundary value problem (Ξ_ε) for the linear Boltzmann equation.

As recalled above, the last two authors have obtained the homogenized equation corresponding to (Ξ_ε) in the noncollisional case ($\sigma = 0$) by explicitly computing the solution of the linear Boltzmann equation for each $0 < \varepsilon \ll 1$. In the collisionnal case ($\sigma > 0$), as recalled above, there is no such explicit formula giving the solution of the linear Boltzmann equation — except the semi-explicit formula involving the transport process defined in [26].

However, not all the information in that semi-explicit formula is needed for the proof of Theorem 1. The additional variable s is precisely the exact amount of information contained in that semi-explicit formula needed in the description of the homogenized process in the limit as $\varepsilon \to 0^+$.

Consider therefore the initial boundary value problem

$$(\Sigma_\varepsilon)\begin{cases} \partial_t F_\varepsilon + v\cdot\nabla_x F_\varepsilon + \partial_s F_\varepsilon + \sigma F_\varepsilon = 0, & t,s>0, (x,v)\in Z_\varepsilon\times\mathbb{S}^1, \\ F_\varepsilon(t,s,x,v)=0, \text{ if } v\cdot n_x>0, & t,s>0, (x,v)\in(\partial Z_\varepsilon\times\mathbb{S}^1), \\ F_\varepsilon(t,0,x,v)=\sigma\displaystyle\int_0^\infty KF_\varepsilon(t,s,x,v)ds, & t>0, (x,v)\in Z_\varepsilon\times\mathbb{S}^1, \\ F_\varepsilon(0,s,x,v)=\sigma e^{-\sigma s}f^{in}(x,v), & s>0, (x,v)\in Z_\varepsilon\times\mathbb{S}^1, \end{cases}$$

with unknown $F_\varepsilon := F_\varepsilon(t,s,x,v)$.

The relation between these two initial boundary value problems, (Ξ_ε) and (Σ_ε), is explained by the following proposition.

Proposition 2. *Assume that f^{in} satisfies the assumption (11). Then*
a) for each $\varepsilon>0$, the problem (Σ_ε) has a unique mild solution such that

$$(t,x,v)\mapsto\int_0^{+\infty}|F_\varepsilon(t,s,x,v)|ds \textit{ belongs to } L^\infty([0,T]\times Z_\varepsilon\times\mathbb{S}^1)$$

for each $T>0$;
b) moreover

$$0\le F_\varepsilon(t,s,x,v)\le\|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)}\sigma e^{-\sigma s}$$

a.e. in $t,s\ge 0$, $x\in Z_\varepsilon$ and $v\in\mathbb{S}^1$, and

$$\int_0^{+\infty}F_\varepsilon(t,s,x,v)ds=f_\varepsilon(t,x,v),$$

for a.e. $t\ge0, x\in Z_\varepsilon, v\in\mathbb{S}^1$, where f_ε is the solution of (Ξ_ε).

Proof. Applying the method of characteristics, we see that, should a mild solution F_ε of the problem (Σ_ε) exist, it must satisfy

$$F_\varepsilon(t,s,x,v)=F_{1,\varepsilon}(t,s,x,v)+F_{2,\varepsilon}(t,s,x,v), \tag{14}$$

with

$$\begin{aligned} F_{1,\varepsilon}(t,s,x,v) &= \mathbb{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbb{1}_{s<t}e^{-\sigma s}F_\varepsilon(t-s,0,x-vs,v) \\ &= \mathbb{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbb{1}_{s<t}\sigma e^{-\sigma s}\int_0^{+\infty}KF_\varepsilon(t-s,\tau,x-sv,v)d\tau \end{aligned} \tag{15}$$

and

$$\begin{aligned} F_{2,\varepsilon}(t,s,x,v) &= \mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbb{1}_{t<s}e^{-\sigma t}F_\varepsilon(0,s-t,x-vt,v) \\ &= \mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbb{1}_{t<s}\sigma e^{-\sigma s}f^{in}(x-tv,v) \end{aligned} \tag{16}$$

a.e. in $(t,s,x,v)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1$.

First, define $\mathcal{X}_T$ to be, for each $T>0$, the set of measurable functions G defined on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1$ such that

$$(t,x,v)\mapsto\int_0^{+\infty}|G(t,s,x,v)|ds \text{ belongs to } L^\infty([0,T]\times Z_\varepsilon\times\mathbb{S}^1)\,,$$

which is a Banach space for the norm

$$\|G\|_{\mathcal{X}_T}=\left\|\int_0^{+\infty}|G(\cdot,s,\cdot,\cdot)|ds\right\|_{L^\infty([0,T]\times Z_\varepsilon\times\mathbb{S}^1)}.$$

Next, for each $G \in \mathcal{X}_T$, we define

$$\mathcal{T}G(t,s,x,v) := \mathbb{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbb{1}_{s<t}\sigma e^{-\sigma s}\int_0^{+\infty} KG(t-s,\tau,x-sv,v)d\tau\,.$$

Obviously

$$\begin{aligned}\left\|\int_0^{+\infty}|\mathcal{T}^nG(t,s,\cdot,\cdot)|ds\right\|_{L^\infty(Z_\varepsilon\times\mathbb{S}^1)}&\\ \le\sigma\int_0^t\left\|\int_0^{+\infty}|\mathcal{T}^{n-1}G(t_1,\tau,\cdot,\cdot)|d\tau\right\|_{L^\infty(Z_\varepsilon\times\mathbb{S}^1)}&dt_1\\ \le\sigma^n\int_0^t\ldots\int_0^{t_{n-1}}\left\|\int_0^{+\infty}|G(t_n,s,\cdot,\cdot)|ds\right\|_{L^\infty(Z_\varepsilon\times\mathbb{S}^1)}&dt_n\ldots dt_1\,,\end{aligned}$$

so that

$$\|\mathcal{T}^nG\|_{\mathcal{X}_T}\le\frac{(\sigma T)^n}{n!}\|G\|_{\mathcal{X}_T}\,.$$

Now $F_{1,\varepsilon}=\mathcal{T}F_\varepsilon$, so that (14) can be recast as

$$F_\varepsilon=F_{2,\varepsilon}+\mathcal{T}F_\varepsilon\,.$$

This integral equation has a solution $F_\varepsilon\in\mathcal{X}_T$ for each $T>0$, given by the series

$$F_\varepsilon=\sum_{n\ge0}\mathcal{T}^nF_{2,\varepsilon}$$

which is normally convergent in the Banach space $\mathcal{X}_T$ since

$$\sum_{n\ge0}\|\mathcal{T}^nF_{2,\varepsilon}\|_{\mathcal{X}_T}\le\sum_{n\ge0}\frac{(\sigma T)^n}{n!}\|F_{2,\varepsilon}\|_{\mathcal{X}_T}<+\infty\,.$$

Assuming that the integral equation above has another solution $F'_\varepsilon\in\mathcal{X}_T$ would imply that

$$F_\varepsilon-F'_\varepsilon=\mathcal{T}(F_\varepsilon-F'_\varepsilon)=\ldots=\mathcal{T}^n(F_\varepsilon-F'_\varepsilon)\,,$$

so that

$$\|F_\varepsilon-F'_\varepsilon\|_{\mathcal{X}_T}=\|\mathcal{T}^n(F_\varepsilon-F'_\varepsilon)\|_{\mathcal{X}_T}\le\frac{(\sigma T)^n}{n!}\|F_\varepsilon-F'_\varepsilon\|_{\mathcal{X}_T}\to0$$

as $n\to+\infty$: hence $F'_\varepsilon=F_\varepsilon$. Thus we have proved statement a).

As for statement b), observe that $\mathcal{T}G\ge0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1$ if $G\ge0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1$. Hence, if $f^{in}\in L^\infty(\mathbb{R}^2\times\mathbb{S}^1)$ satisfies $f^{in}\ge0$ a.e. on $\mathbb{R}^2\times\mathbb{S}^1$, one has $F_{2,\varepsilon}\ge0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1$, so that $\mathcal{T}^nF_{2,\varepsilon}\ge0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1$ and the series defining F_ε is a.e. nonnegative on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{S}^1$.

Next, integrating both sides of (14) with respect to s, and setting

$$g_\varepsilon(t,x,v):=\int_0^{+\infty}F_\varepsilon(t,s,x,v)ds\,,$$

we arrive at

$$
\begin{aligned}
g_\varepsilon(t,x,v) &= \int_0^{+\infty} F_{2,\varepsilon}(t,s,x,v)ds + \int_0^{+\infty} F_{1,\varepsilon}(t,s,x,v)ds \\
&= \mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} f^{in}(x-tv,v) \int_0^{+\infty} \mathbb{1}_{t<s}\sigma e^{-\sigma s} ds \\
&+ \int_0^{+\infty} \mathbb{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \mathbb{1}_{s<t}\sigma e^{-\sigma s} \left(\int_0^{+\infty} KF_\varepsilon(t-s,\tau,x-sv,v)d\tau \right) ds \\
&= \mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} f^{in}(x-tv,v)e^{-\sigma t} \\
&+ \int_0^t e^{-\sigma s} \mathbb{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \sigma K g_\varepsilon(t-s,x-sv,v)ds
\end{aligned}
$$

in which we recognize the Duhamel formula giving the unique mild solution f_ε of (Ξ_ε). Hence

$$
f_\varepsilon(t,x,v) = \int_0^{+\infty} F_\varepsilon(t,s,x,v)ds \text{ a.e. in } (t,x,v) \in \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1 .
$$

Finally, since (Ξ_ε) satisfies the maximum principle, one has

$$
f_\varepsilon(t,x,v) \le \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)} \text{ a.e. in } (t,x,v) \in \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1 .
$$

Going back to (14), we recast it in the form

$$
\begin{aligned}
F_\varepsilon(t,s,x,v) &= \mathbb{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \mathbb{1}_{s<t}\sigma e^{-\sigma s} K f_\varepsilon(t-s,x-sv,v) \\
&+ \mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \mathbb{1}_{t<s}\sigma e^{-\sigma s} f^{in}(x-tv,v) \\
&\le \mathbb{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \mathbb{1}_{s<t}\sigma e^{-\sigma s} \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)} \\
&+ \mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \mathbb{1}_{t<s}\sigma e^{-\sigma s} \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)} \\
&\le \sigma e^{-\sigma s} \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)}
\end{aligned}
$$

a.e. in $(t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1$, which concludes the proof. □

Observe that if

$$
F_\varepsilon(0,s,x,v) = \sigma e^{-\sigma s} f^{in}(x,v)
$$

is replaced with

$$
F_\varepsilon(0,s,x,v) = \Pi(s) f^{in}(x,v)
$$

where Π is any probability density on $\mathbb{R}_+$ vanishing at ∞, the conclusion of the lemma above remains valid. In other words, the dependence of the solution F_ε of the problem (Σ) upon the choice of the initial probability density Π disappears after integration in s, so that the particle distribution function f_ε is indeed independent of the choice of Π.

The choice $\Pi(s) = \sigma e^{-\sigma s}$ corresponds with the situation where the gas molecules have been evolving under the linear Boltzmann equation for $t < 0$ and the holes are suddenly opened at $t = 0$.

Before giving the proof of Theorem 1, we need to establish a few technical lemmas.

4.2. **The distribution of free path lengths.** A straightforward consequence of the limit in (6) is the following lemma, which accounts eventually for the coefficient $\dot{p}(t\wedge s)/p(t\wedge s)$ in the limiting equation (Σ).

Lemma 1. *Let τ_ε be the free path length defined in* (5)*. Then for each $t > 0$*

$$
\{\mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\} \rightharpoonup p(t)
$$

in $L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$ weak-$*$ as $\varepsilon \to 0^+$.

(See the definition before Theorem 1 for the notation $\{\mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\}$.)

Proof. Since the linear span of functions $\phi \equiv \phi(x,v)$ of the form

$$\phi(x,v) = \chi(x)\mathbb{1}_I(v)\,, \quad \chi \in C_0^\infty(\mathbb{R}^2) \text{ and } I \text{ an arc of } \mathbb{S}^1$$

is dense in $L^1(\mathbb{R}^2 \times \mathbb{S}^1)$, and the family $\mathbb{1}_{\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)>t}$ is bounded in $L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$, it is enough to prove that

$$\iint_{Z_\varepsilon \times \mathbb{S}^1} \phi(x,v)\mathbb{1}_{\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)>t} dxdv \to p(t) \iint_{\mathbb{R}^2\times\mathbb{S}^1} \phi(x,v)dxdv \text{ as } \varepsilon \to 0\,.$$

Write

$$\begin{aligned}\iint_{Z_\varepsilon \times \mathbb{S}^1} \phi(x,v)\mathbb{1}_{\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)>t} dxdv &= \int_{Z_\varepsilon} \chi(x)\left(\int_I \mathbb{1}_{\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)>t} dv\right) dx \\ &= \int_{Z_\varepsilon} \chi(x) T_\varepsilon\left(\frac{x}{\varepsilon}\right) dx\end{aligned}$$

with

$$T_\varepsilon(y) := \int_I \mathbb{1}_{\varepsilon\tau_\varepsilon(y,v)>t} dv\,.$$

Obviously T_ε is 1-periodic in y_1 and y_2 and satisfies $0 \le T_\varepsilon \le |I|$. Hence

$$\mathbb{1}_{d(y,\mathbb{Z}^2)>\varepsilon} T_\varepsilon(y) = \sum_{k\in\mathbb{Z}^2} \hat{T}_\varepsilon(k) e^{2i\pi k\cdot y}$$

in $L^2(\mathbb{R}^2/\mathbb{Z}^2)$ with

$$\hat{T}_\varepsilon(k) := \int_{\substack{\max(|z_1|,|z_2|)<1/2 \\ |z|>\varepsilon}} T_\varepsilon(z)^{-2i\pi k\cdot z} dz$$

for each $k \in \mathbb{Z}^2$.

Then, by Parseval's identity,

$$\begin{aligned}\int_{Z_\varepsilon} \chi(x) T_\varepsilon\left(\frac{x}{\varepsilon}\right) dx &= \int_{\mathbb{R}^2} \chi(x) \left(\sum_{k\in\mathbb{Z}^2} \hat{T}_\varepsilon(k) e^{2i\pi\frac{k\cdot x}{\varepsilon}}\right) dx \\ &= \hat{\chi}(0)\hat{T}_\varepsilon(0) + \sum_{k\in\mathbb{Z}^2\setminus(0,0)} \hat{T}_\varepsilon(k)\hat{\chi}(-2\pi k/\varepsilon)\,,\end{aligned}$$

with

$$\hat{\chi}(\xi) := \int_{\mathbb{R}^2} \chi(x) e^{-i\xi\cdot x} dx\,.$$

Applying again Parseval's identity,

$$\sum_{k\in\mathbb{Z}^2} |\hat{T}_\varepsilon(k)|^2 = \int_{\substack{\max(|y_1|,|y_2|)<1/2 \\ |y|>\varepsilon}} |T_\varepsilon(y)|^2 dy \le |I|$$

while

$$|\hat{\chi}(\xi)| \le \frac{1}{|\xi|^2} \|\nabla^2\chi\|_{L^\infty}\,,$$

so that

$$|\hat{\chi}(-2\pi k/\varepsilon)| \le \frac{\varepsilon^2}{4\pi^2|\xi|^2} \|\nabla^2\chi\|_{L^\infty}\,.$$

Hence, by the Cauchy-Schwarz inequality,

$$\left|\sum_{k\in\mathbb{Z}^2\setminus(0,0)} \hat{T}_\varepsilon(k)\hat{\chi}(-2\pi k/\varepsilon)\right|^2 \le \sum_{k\in\mathbb{Z}^2\setminus(0,0)} |\hat{T}_\varepsilon(k)|^2 \sum_{k\in\mathbb{Z}^2\setminus(0,0)} \frac{\varepsilon^4\|\nabla^2\chi\|^2_{L^\infty}}{16\pi^4|k|^4} = O(\varepsilon^4)$$

and therefore

$$\int_{Z_\varepsilon} \chi(x) T_\varepsilon \left(\frac{x}{\varepsilon}\right) dx = \hat{\chi}(0)\hat{T}_\varepsilon(0) + O(\varepsilon^2)$$

as $\varepsilon \to 0^+$.

By (6)

$$\hat{T}_\varepsilon(0) = \int_{\substack{\max(|y_1|,|y_2|)<1/2 \\ |y|>\varepsilon}} T_\varepsilon(y)dy \to p(t)|I| \quad \text{as } \varepsilon \to 0^+ ,$$

so that

$$\hat{\chi}(0)\hat{T}_\varepsilon(0) \to p(t)|I| \int_{\mathbb{R}^2} \chi(x)dx = p(t) \iint_{\mathbb{R}^2\times\mathbb{S}^1} \phi(x,v)dxdv$$

as $\varepsilon \to 0^+$, and hence

$$\int_{Z_\varepsilon} \chi(x) T_\varepsilon \left(\frac{x}{\varepsilon}\right) dx = p(t) \iint_{\mathbb{R}^2\times\mathbb{S}^1} \phi(x,v)dxdv + o(1) + O(\varepsilon^2)$$

which entails the announced result. □

4.3. **Extending f_ε by 0 in the holes.** We begin with the equation satisfied by the (extension by 0 inside the holes of the) distribution function $\{f_\varepsilon\}$.

Lemma 2. *For each $\varepsilon > 0$, the function $\{f_\varepsilon\}$ satisfies*

$$(\partial_t + v\cdot\nabla_x)\{f_\varepsilon\} + \sigma(\{f_\varepsilon\} - K\{f_\varepsilon\}) = (v\cdot n_x) f_\varepsilon\big|_{\partial Z_\varepsilon\times\mathbb{S}^1}\delta_{\partial Z_\varepsilon}$$

in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^2\times\mathbb{S}^1)$, where $\delta_{\partial Z_\varepsilon}$ is the surface measure concentrated on the boundary of Z_ε, and n_x is the unit normal vector at $x\in\partial Z_\varepsilon$ pointing towards the interior of Z_ε.

Proof. One has

$$\partial_t\{f_\varepsilon\} = \{\partial_t f_\varepsilon\}$$

and

$$\nabla_x\{f_\varepsilon\} = \{\nabla_x f_\varepsilon\} + f_\varepsilon\,|_{\partial Z_\varepsilon\times\mathbb{S}^1}\,\delta_{\partial Z_\varepsilon} n_x$$

in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^2\times\mathbb{S}^1)$. Hence

$$\begin{aligned} 0 &= \{\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \sigma(f_\varepsilon - K f_\varepsilon)\} \\ &= \partial_t\{f_\varepsilon\} + v\cdot\nabla_x\{f_\varepsilon\} + (v\cdot n_x) f_\varepsilon\big|_{\partial Z_\varepsilon\times\mathbb{S}^1}\delta_{\partial Z_\varepsilon} + \sigma(\{f_\varepsilon\} - K\{f_\varepsilon\}) \end{aligned}$$

in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^2\times\mathbb{S}^1)$. □

A straightforward consequence of the scaling considered here is that the family of Radon measures

$$(v\cdot n_x) f_\varepsilon\big|_{\partial Z_\varepsilon\times\mathbb{S}^1}\delta_{\partial Z_\varepsilon}$$

is controlled uniformly as $\varepsilon \to 0^+$, in the following manner.

Lemma 3. *For each $R > 0$, the family of Radon measures*

$$(v\cdot n_x) f_\varepsilon\big|_{\partial Z_\varepsilon\times\mathbb{S}^1}\delta_{\partial Z_\varepsilon}\big|_{[-R,R]^2\times\mathbb{S}^1}$$

is bounded in[1] *$\mathcal{M}([-R,R]^2\times\mathbb{S}^1)$.*

[1]For each compact subset K of $\mathbb{R}^N$, we denote by $\mathcal{M}(K)$ the space of signed Radon measures on K, i.e. the set of all real-valued continuous linear functionals on $C(K)$ endowed with the topology of uniform convergence on K.

Proof. The total mass of the measure

$$(v \cdot n_x) f_\varepsilon\big|_{\partial Z_\varepsilon \times \mathbb{S}^1} \delta_{\partial Z_\varepsilon}\big|_{[-R,R]^2 \times \mathbb{S}^1}$$

is less than or equal to

$$2\pi \|f_\varepsilon\|_{L^\infty(\mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1)} \|\delta_{\partial Z_\varepsilon}\ |_{[-R,R]^2}\ \|_{\mathcal{M}([-R,R]^2)}$$

which is itself less than or equal to

$$2\pi \|f^{in}\|_{L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)} \left\|\delta_{\partial Z_\varepsilon}\ |_{[-R,R]^2}\right\|_{\mathcal{M}([-R,R]^2)}.$$

Since $\delta_{\partial Z_\varepsilon}\ |_{[-R,R]^2}$ is the union of $O\left(\left(\frac{2R}{\varepsilon}\right)^2\right)$ circles of radius ε^2,

$$\|\delta_{\partial Z_\varepsilon}\ |_{[-R,R]^2}\ \|_{\mathcal{M}([-R,R]^2)} = O\left(\left(\frac{2R}{\varepsilon}\right)^2\right) 2\pi\varepsilon^2 = O(1)R^2$$

as $\varepsilon \to 0^+$, whence the announced result. □

4.4. **The velocity averaging lemmas.** As is the case of all homogenization results, the proof of Theorem 1 is based on the strong L^1_{loc} convergence of certain quantities defined in terms of F_ε. In the case of kinetic models, strong L^1_{loc} compactness is usually obtained by velocity averaging — see for instance [1, 21, 20] for the first results in this direction. Below, we recall a classical result in velocity averaging that is a special case of theorem 1.8 in [5].

Proposition 3. *Let $p > 1$ and assume that $f_\varepsilon \equiv f_\varepsilon(t, x, v)$ is a bounded family in $L^p_{loc}(\mathbb{R}^+_t \times \mathbb{R}^d_x \times \mathbb{S}^{d-1}_v)$ such that*

$$\sup_\varepsilon \int_0^T \iint_{B(0,R)\times\mathbb{S}^{d-1}} |\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon| dx dv dt < +\infty$$

for each $T > 0$ and $R > 0$. Then, for each $\psi \in C(\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$, the family $\rho_\psi[f_\varepsilon]$, defined by

$$\rho_\psi[f_\varepsilon](t, x, v) = \int_{\mathbb{S}^{d-1}} f_\varepsilon(t, x, w)\psi(v, w) dw$$

is relatively compact in $L^1_{loc}(\mathbb{R}^+_t \times \mathbb{R}^d_x \times \mathbb{S}^{d-1}_v)$.

A straightforward consequence of Proposition 3 is the following compactness result in L^1_{loc} strong, which is the key argument in the proof of Theorem 1.

Lemma 4. *Let $f_\varepsilon \equiv f_\varepsilon(t, x, v)$ be the family of solutions of the initial boundary value problem (Ξ_ε). Then the families*

$$K\{f_\varepsilon\} = \{K f_\varepsilon\}$$

and

$$\int_{\mathbb{S}^1} \{f_\varepsilon\} dv$$

are relatively compact in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ strong.

Proof. We recall that, by the Maximum Principle for (Ξ_ε),

$$|f_\varepsilon(t, x, v)| \le \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)}$$

a.e. in $t \ge 0, x \in Z_\varepsilon$ and $v \in \mathbb{S}^1$, so that

$$\sup_\varepsilon \|\{f_\varepsilon\}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} \le \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)}. \tag{17}$$

By Lemma 2, $\{f_\varepsilon\}$ satisfies the equation

$$\partial_t \{f_\varepsilon\} + v \cdot \nabla_x \{f_\varepsilon\} = \sigma(K\{f_\varepsilon\} - \{f_\varepsilon\}) - \delta_{\partial Z_\varepsilon}(v.n_x) f_\varepsilon\ |_{\partial Z_\varepsilon \times \mathbb{S}^1}$$

in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{S}^1)$. Because of (17) and the fact that the scattering kernel k is a.e. nonnegative (see (3)), one has

$$\begin{aligned}\|\sigma(K\{f_\varepsilon\} - \{f_\varepsilon\})\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} &\le \sigma(1 + \|K1\|_{L^\infty(\mathbb{S}^1)})\|\{f_\varepsilon\}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} \\ &= 2\sigma\|\{f_\varepsilon\}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)}\end{aligned}$$

since $K1 = 1$ (see again (3).) Besides the family of Radon measures

$$\mu_\varepsilon = f_\varepsilon\,|_{\partial Z_\varepsilon\times\mathbb{S}^1}\,(v\cdot n_x)\delta_{\partial Z_\varepsilon}$$

satisfies

$$\sup_\varepsilon \int_{[0,T]\times\overline{B(0,R)}\times\mathbb{S}^1} |\mu_\varepsilon| < +\infty$$

for each $T > 0$ and $R > 0$ according to lemma 3.

Applying the Velocity Averaging result recalled above implies that the family

$$\int_{\mathbb{S}^1} g_\varepsilon dv$$

is relatively compact in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$.

By density of $C(\mathbb{S}^1 \times \mathbb{S}^1)$ in $L^2(\mathbb{S}^1 \times \mathbb{S}^1)$, replacing the integral kernel k with a continuous approximant and applying the Velocity Averaging Proposition 3 in the same way as above, we conclude that the family Kg_ε is also relatively compact in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$. □

4.5. **Uniqueness for the homogenized equation.** Consider the Cauchy problem with unknown $G \equiv G(t, s, x, v)$

$$\begin{cases} (\partial_t + v\cdot\nabla_x + \partial_s)G = -\sigma G + \dfrac{\dot p(t\wedge s)}{p(t\wedge s)}G, & t,s>0, x\in\mathbb{R}^2, v\in\mathbb{S}^1,\\[2ex] G(t,0,x,v) = S(t,x,v), & t>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1,\\[2ex] G(0,s,x,v) = G^{in}(s,x,v), & s>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1.\end{cases}$$

If, for a.e. $(t, s, x, v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$, the function $\tau \mapsto G(t+\tau, s+\tau, x+\tau v, v)$ is C^1 in $\tau > 0$, then, since the function $p \in C^1(\mathbb{R}_+)$ and $p > 0$ on $\mathbb{R}_+$, one has

$$\begin{aligned}&\left(\frac{d}{d\tau} + \sigma - \frac{\dot p(t\wedge s+\tau)}{p(t\wedge s+\tau)}\right)G(t+\tau, s+\tau, x+\tau v, v)\\ &\qquad = e^{-\sigma\tau}p(t\wedge s+\tau)\frac{d}{d\tau}\left(\frac{e^{\sigma\tau}G(t+\tau,s+\tau,x+\tau v,v)}{p(t\wedge s+\tau)}\right) = 0\,.\end{aligned}$$

Hence

$$\Gamma : \tau \mapsto \frac{e^{\sigma\tau}G(t+\tau,s+\tau,x+\tau v,v)}{p(t\wedge s+\tau)}$$

is a constant. Therefore

$$\Gamma(0) = \begin{cases}\Gamma(-t) & \text{if } t < s,\\ \Gamma(-s) & \text{if } s < t,\end{cases}$$

so that

$$G(t,s,x,v) = \mathbb{1}_{t<s}e^{-\sigma t}p(t)G^{in}(s-t, x-tv, v) + \mathbb{1}_{s<t}e^{-\sigma s}p(s)S(t-s, x-sv, v)\,.$$

Proposition 4. *Assume that $f^{in} \in L^\infty(\mathbb{R}^2 \times \mathbb{S}^1)$. Then the problem (Σ) has a unique mild solution F such that*

$$(t,x,v) \mapsto \int_0^{+\infty} |F(t,s,x,v)|ds \textit{ belongs to } L^\infty([0,T]\times\mathbb{R}^2\times\mathbb{S}^1)$$

for each $T > 0$*. This solution satisfies*

$$F(t,s,x,v) = \mathbb{1}_{t<s}\sigma e^{-\sigma t}p(t)f^{in}(x-tv,v) + \mathbb{1}_{s<t}\sigma e^{-\sigma s}p(s)\int_0^{+\infty} KF(t-s,\tau,x-sv,v)d\tau$$

for a.e. $(t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$.

Besides, $F \geq 0$ *a.e. on* $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$ *if* $f^{in} \geq 0$ *a.e. on* $\mathbb{R}^2 \times \mathbb{S}^1$.

Proof. That a mild solution of the problem (Σ), should it exist, satisfies the integral equation above follows from the computation presented before the proposition.

As above, let $\mathcal{Y}_T$ be, for each $T > 0$, the set of measurable functions G defined a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$ and such that

$$(t,x,v) \mapsto \int_0^{+\infty} |G(t,s,x,v)|ds \text{ belongs to } L^\infty([0,T]\times\mathbb{R}^2\times\mathbb{S}^1)\,,$$

which is a Banach space for the norm

$$\|G\|_{\mathcal{Y}_T} = \left\|\int_0^{+\infty} |G(\cdot,s,\cdot,\cdot)|ds\right\|_{L^\infty([0,T]\times Z_\varepsilon\times\mathbb{S}^1)}.$$

Next, for each $G \in \mathcal{Y}_T$, we define

$$\mathcal{Q}G(t,s,x,v) := \mathbb{1}_{s<t}\sigma e^{-\sigma s}p(s)\int_0^{+\infty} KG(t-s,\tau,x-sv,v)d\tau\,.$$

Since $0 < e^{-\sigma s}p(s) \leq 1$, the integral kernel $k \geq 0$ on $\mathbb{S}^1\times\mathbb{S}^1$ and $K1 = 1$ by (3), one has

$$\int_0^{+\infty} |\mathcal{Q}G(t,s,x,v)|ds \leq \sigma\int_0^t \left\|\int_0^{+\infty} |G(t-s,\tau,\cdot,\cdot)|d\tau\right\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)} ds$$

a.e. in $(t,x,v) \in [0,T]\times\mathbb{R}^2\times\mathbb{S}^1$, meaning that

$$\begin{aligned}\left\|\int_0^{+\infty} |\mathcal{Q}^n G(t,s,\cdot,\cdot)|ds\right\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)} &\\ \leq \sigma\int_0^t \left\|\int_0^{+\infty} |\mathcal{Q}^{n-1}G(t_1,s,\cdot,\cdot)|ds\right\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)} dt_1 &\\ \leq \sigma^n\int_0^t\dots\int_0^{t_{n-1}} \left\|\int_0^{+\infty} |G(t_n,s,\cdot,\cdot)|ds\right\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)} dt_n\dots dt_1\,.&\end{aligned}$$

In particular

$$\|\mathcal{Q}^n G\|_{\mathcal{Y}_T} \leq \frac{(\sigma T)^n}{n!}\|G\|_{\mathcal{Y}_T}\,.$$

The integral equation in the statement of the proposition is

$$F = F_2 + \mathcal{Q}F$$

where

$$F_2(t,s,x,v) = \mathbb{1}_{t<s}\sigma e^{-\sigma t}p(t)f^{in}(x-tv,v)\,.$$

Therefore, arguing as in the proof of Proposition 2, one obtains a mild solution of (Σ) as the sum of the series

$$F = \sum_{n\geq 0}\mathcal{Q}^n F_2\,,$$

which is normally convergent in the Banach space $\mathcal{Y}_T$ for each $T > 0$.

Should there exist another mild solution, say F', it would satisfy

$$(F - F') = \mathcal{Q}(F - F') = \dots = \mathcal{Q}^n(F - F')$$

for all $n \geq 0$, so that

$$\|F - F'\|_{\mathcal{Y}_T} = \|\mathcal{Q}^n(F - F')\|_{\mathcal{Y}_T} \leq \frac{(\sigma T)^n}{n!}\|F - F'\|_{\mathcal{Y}_T} \to 0$$

as $n \to +\infty$, which implies that $F = F'$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$.

Finally, $\mathcal{Q}F \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$ if $F \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$. Since F is given by the series above, one has $F \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$ whenever $f^{in} \geq 0$ a.e. on $\mathbb{R}^2 \times \mathbb{S}^1$. □

4.6. **Proof of the homogenization theorem.** Start from the decomposition (14) of F_ε. Passing to the limit as $\varepsilon \to 0^+$ in the term $F_{2,\varepsilon}$ is easy. Indeed, by Lemma 1

$$\{\mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\} \rightharpoonup p(t) \tag{18}$$

in $L^\infty(\mathbb{R}^2_x \times \mathbb{S}^1_v)$ weak-$*$ for each $t > 0$, as $\varepsilon \to 0^+$. Hence

$$\begin{aligned}\{F_{2,\varepsilon}\}(t,s,x,v) =& \mathbb{1}_{t<s}e^{-\sigma s}f^{in}(x-tv,v)\{\mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\}\\ &\rightharpoonup \mathbb{1}_{t<s}e^{-\sigma s}f^{in}(x-tv,v)p(t) =: F_2(t,s,x,v)\end{aligned} \tag{19}$$

in $L^\infty(\mathbb{R}^+_t \times \mathbb{R}^+_s \times \mathbb{R}^2_x \times \mathbb{S}^1_v)$ weak-$*$ as $\varepsilon \to 0^+$.

Next, we analyze the term $F_{1,\varepsilon}$; this is obviously more difficult as this term depends on the (unknown) solution F_ε itself.

We recall the uniform bound

$$\sup_\varepsilon \|\{f_\varepsilon\}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} \leq \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{S}^1)}$$

— see Proposition 2 b), so that, by the Banach-Alaoglu theorem

$$\{f_\varepsilon\} \rightharpoonup f \text{ in } L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1) \text{ weak-}* \tag{20}$$

for some $f \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$, possibly after extracting a subsequence of $\varepsilon \to 0^+$.

Thus, applying the strong compactness Lemma 4 shows that

$$K\{f_\varepsilon\} \to Kf \text{ in } L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1) \text{ strong}$$

as $\varepsilon \to 0^+$.

This and the weak-$*$ convergence in Lemma 1 imply that

$$\begin{aligned}\{F_{1,\varepsilon}\} =& \mathbb{1}_{s<t}\sigma e^{-\sigma s}K\{f_\varepsilon\}(t-s,x-sv,v)\mathbb{1}_{s<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\\ &\rightharpoonup \mathbb{1}_{s<t}\sigma e^{-\sigma s}Kf(t-s,x-sv,v)p(s)\end{aligned} \tag{21}$$

in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak as $\varepsilon \to 0^+$. Therefore

$$\begin{aligned}\{F_\varepsilon\}(t,s,x,v) \rightharpoonup& \mathbb{1}_{s<t}\sigma e^{-\sigma s}Kf(t-s,x-sv,v)p(s) + F_2(t,s,x,v)\\ &=: \tilde{F}(t,s,x,v)\end{aligned} \tag{22}$$

in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak as $\varepsilon \to 0^+$.

Fix $T > 0$; then, for $t \in [0,T]$, one has

$$\int_0^\infty F_\varepsilon(t,s,x,v)ds = \int_0^T F_{1,\varepsilon}(t,s,x,v)ds + e^{-\sigma t}f^{in}(x-tv,v)\mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}$$

since $F_{1,\varepsilon}$ is supported in $s \leq t \leq T$, so that

$$\begin{aligned}\int_0^\infty \{F_\varepsilon\}(t,s,x,v)ds \rightharpoonup& \int_0^T \mathbb{1}_{s\leq t}Kf(t-s,x-vs,v)\sigma e^{-\sigma s}p(s)ds\\ &+ f^{in}(x-tv,v)e^{-\sigma t}p(t)\\ &= \int_0^\infty \tilde{F}(t,s,x,v)ds\end{aligned} \tag{23}$$

in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weakly as $\varepsilon \to 0^+$, where $\tilde{F}$ is defined in (22).

On the other hand

$$\int_0^\infty \{F_\varepsilon\}(t,s,x,v)ds = \{f_\varepsilon\}(t,x,v) \rightharpoonup f(t,x,v)$$

in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak-$*$ as $\varepsilon \to 0^+$ — and therefore also in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak as $\varepsilon \to 0^+$. By uniqueness of the limit, we conclude that

$$f(t,x,v) = \int_0^\infty \tilde{F}(t,s,x,v)ds \text{ a.e. in } (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1 \tag{24}$$

so that $\tilde{F}$ satisfies

$$\begin{aligned}\tilde{F}(t,s,x,v) = \mathbb{1}_{s<t}\sigma e^{-\sigma s} K\left(\int_0^\infty \tilde{F}(t-s,u,x-sv,\cdot)du\right)(v)p(s)\\ + \mathbb{1}_{t<s}\sigma e^{-\sigma s} f^{in}(x-tv,v)p(t)\end{aligned}$$

a.e. in $(t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$. By Proposition 4, this means that $\tilde{F}$ is a solution of the Cauchy problem (Σ).

By uniqueness of the solution of (Σ), we conclude that $\tilde{F} = F$, and that the whole family

$$F_\varepsilon \rightharpoonup F \text{ in } L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$$

weakly as $\varepsilon \to 0^+$.

Finally, (20) and (24) imply that

$$\{f_\varepsilon\} \rightharpoonup f = \int_0^\infty F ds$$

in $L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak-$*$ as $\varepsilon \to 0^+$, which concludes the proof of Theorem 1. □

5. Asymptotic behavior of the total mass in the long time limit

The formulation of the homogenized equation (problem (Σ)) as an integro-differential equation set on the extended phase space involving the additional variable s is of considerable importance in understanding the asymptotic behavior of the total mass of the particle system as the time variable $t \to +\infty$. Indeed, this formulation implies that the total mass of the particle system satisfies a renewal equation, i.e. a class of integral equations for which a lot is known on the asymptotic behavior of the solutions in the long time limit — see for instance in [14] the basic results on renewal type integral equations.

5.1. The renewal PDE governing the mass.

The renewal PDE governing the mass. We begin with a proof of Proposition 1.

Proof. That μ is a mild solution of the renewal PDE means that, for a.e. $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$,

$$\begin{aligned}\mu(t,s) &= \mathbb{1}_{t<s}\sigma e^{-\sigma(s-t)}e^{-\sigma t}p(t) + \mathbb{1}_{s<t}e^{-\sigma s}p(s)\int_0^{+\infty}\mu(t-s,\tau)d\tau\\ &= \sigma e^{-\sigma s}p(t\wedge s)\left(\mathbb{1}_{t<s} + \mathbb{1}_{s<t}\int_0^{+\infty}\mu(t-s,\tau)d\tau\right).\end{aligned}$$

Let $T > 0$, and define

$$\mathcal{R}\mu(t,s) = \mathbb{1}_{s<t}\sigma e^{-\sigma s}p(s)\int_0^{+\infty}\mu(t-s,\tau)d\tau$$

a.e. in $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$. Obviously, for each $\phi \in L^\infty([0,T];L^1(\mathbb{R}_+))$ and a.e. $t \geq 0$,

$$\|\mathcal{R}\phi(t,\cdot)\|_{L^1(\mathbb{R}_+)} \leq \int_0^t \sigma e^{-\sigma(t-s)} p(t-s)\|\phi(s,\cdot)\|_{L^1(\mathbb{R}_+)} ds$$
$$\leq \sigma \int_0^t \|\phi(s,\cdot)\|_{L^1(\mathbb{R}_+)} ds\,,$$

so that, for each $n \geq 0$, one has

$$\|\mathcal{R}^n\phi(t,\cdot)\|_{L^1(\mathbb{R}_+)} \leq \int_0^t \int_0^{t_1} \cdots \int_0^{t_{n-1}} \|\phi(t_n,\cdot)\|_{L^1(\mathbb{R}_+)} dt_n \ldots dt_1$$
$$\leq \frac{(\sigma t)^n}{n!} \|\phi\|_{L^\infty([0,T];L^1(\mathbb{R}_+))}$$

a.e. in $t \in \mathbb{R}_+$.

Arguing as in the proof of Proposition 2, we see that the renewal PDE has a unique mild solution $\mu \in L^\infty([0,T];L^1(\mathbb{R}_+))$ for all $T > 0$, which is given by the series

$$\mu = \sum_{n \geq 0} \mathcal{R}^n(\mu^{in})$$

where

$$\mu^{in}(s) := \sigma e^{-\sigma s}\,.$$

Obviously $\mathcal{R}\phi \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+$ if $\phi \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+$, so that $\mu \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+$. Besides, for each $T > 0$,

$$\|\mu\|_{L^\infty([0,T];L^1(\mathbb{R}_+))} \leq \sum_{n \geq 0} \frac{(\sigma T)^n}{n!} \|\mu^{in}\|_{L^1(\mathbb{R}_+)} = e^{\sigma T}\,,$$

which implies in turn that

$$0 \leq \mu(t,s) \leq \sigma e^{-\sigma s} p(t \wedge s)\left(\mathbb{1}_{t<s} + \mathbb{1}_{s<t} e^{\sigma T}\right) \leq \sigma e^{\sigma T} e^{-\sigma s}$$

a.e. in $(t,s) \in [0,T] \times \mathbb{R}_+$.

Finally, let F be the mild solution of the problem (Σ) obtained in Proposition 2. Since $F \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$ is measurable, one can apply the Fubini theorem to show that

$$\begin{aligned}
m(t,s) :&= \tfrac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} F(t,s,x,v)dxdv \\
&= \mathbb{1}_{t<s} \sigma e^{-\sigma t} p(t) \tfrac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x-tv,v)dxdv \\
&+ \mathbb{1}_{t<s} \sigma e^{-\sigma t} p(s) \int_0^\infty \tfrac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} KF(t-s,\tau,x-sv,v)dxdvd\tau \\
&= \mathbb{1}_{t<s} \sigma e^{-\sigma t} p(t) \tfrac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(y,v)dydv \\
&+ \mathbb{1}_{t<s} \sigma e^{-\sigma t} p(s) \int_0^\infty \tfrac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} KF(t-s,\tau,y,v)dydvd\tau \\
&= \mathbb{1}_{t<s} \sigma e^{-\sigma t} p(t) \tfrac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(y,v)dydv \\
&+ \mathbb{1}_{t<s} \sigma e^{-\sigma t} p(s) \int_0^\infty \tfrac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} F(t-s,\tau,y,w)dydwd\tau \\
&= \mathbb{1}_{t<s} \sigma e^{-\sigma t} p(t) \tfrac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x-tv,v)dxdv \\
&+ \mathbb{1}_{t<s} \sigma e^{-\sigma t} p(s) \int_0^\infty m(t-s,\tau)d\tau\,,
\end{aligned}$$

where the second equality follows from the substitution $y = x - tv$ that leaves the Lebesgue measure invariant, while the third equality follows from the identity

$$\tfrac{1}{2\pi}\int_{\mathbb{S}^1} k(v,w)dv = 1\,,$$

which implies that

$$\tfrac{1}{2\pi}\int_{\mathbb{S}^1} KF(t-s,\tau,y,v)dv = \tfrac{1}{2\pi}\int_{\mathbb{S}^1} F(t-s,\tau,y,w)dw\,.$$

In other words,

$m(t,s)$ satisfies the same integral equation as $\dfrac{\mu(t,s)}{2\pi}\displaystyle\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(y,v)dydv.$

Now the solution f_ε of (Ξ_ε) satisfies

$$f_\varepsilon \geq 0 \text{ a.e. on } \mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1 \text{ and } \iint_{\mathbb{R}^2\times\mathbb{S}^1} f_\varepsilon(t,y,v)dydv \leq \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(y,v)dydv\,,$$

which implies by Theorem 1 that

$$\int_{|y|\leq R}\int_{\mathbb{S}^1} f_\varepsilon(t,y,v)dvdy \rightharpoonup \int_0^{+\infty}\int_{|y|\leq R}\int_{\mathbb{S}^1} F(t,s,y,v)dvdyds\,.$$

Hence, by Fatou's lemma

$$\begin{aligned}\int_0^{+\infty}\int_{|y|\leq R}\int_{\mathbb{S}^1} F(t,s,y,v)dvdyds &\leq \varliminf_{\varepsilon\to 0^+}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f_\varepsilon(t,x,v)dxdv\\ &\leq \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(y,v)dydv\,,\end{aligned}$$

a.e. in $t \geq 0$.

Letting $R \to +\infty$ in the inequality above, we see that $m \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}_+))$ and we have proved that the difference

$$\Lambda(t,s) = m(t,s) - \frac{\mu(t,s)}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(y,v)dydv$$

satisfies

$$\Lambda \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}_+)) \quad\text{ and }\quad \Lambda = \mathcal{R}\Lambda\,.$$

By the same uniqueness argument as in the proof of Proposition 4, we conclude that $\Lambda = 0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+$. $\square$

5.2. **The total mass in the vanishing ε limit.** By Theorem 1, the solution f_ε of (Ξ_ε) satisfies

$$\{f_\varepsilon\} \rightharpoonup \int_0^{+\infty} Fds \text{ in } L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1) \text{ weak-}*;$$

therefore, checking that

$$\iint_{\mathbb{R}^2\times\mathbb{S}^1}\{f_\varepsilon\}dxdv \rightharpoonup \int_0^{+\infty}\iint_{\mathbb{R}^2\times\mathbb{S}^1} Fdxdvds =: 2\pi M(t)$$

reduces to proving that there is no mass loss at infinity in the x variable.

Lemma 5. *Under the same assumptions as in Theorem 1*

$$\tfrac{1}{2\pi}\iint_{Z_\varepsilon\times\mathbb{S}^1} f_\varepsilon(t,x,v)dxdv = \tfrac{1}{2\pi}\iint_{\mathbb{R}^2\times\mathbb{S}^1}\{f_\varepsilon\}(t,x,v)dxdv \to M(t)$$

strongly in $L^1_{loc}(\mathbb{R}_+)$ as $\varepsilon\to 0^+$.

Proof. Going back to the proof of Proposition 2 (whose notations are kept in the present discussion), we have seen that

$$F_\varepsilon = \sum_{n\geq 0} \mathcal{T}^n F_{2,\varepsilon} \quad \text{on } \mathbb{R}_+ \times \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1\,,$$

with the notation

$$F_{2,\varepsilon}(t,s,x,v) = \mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)}\mathbb{1}_{t<s}\sigma e^{-\sigma s} f^{in}(x-tv,v)\,.$$

Since $\mathcal{T}\Phi \geq 0$ a.e. whenever $\Phi \geq 0$ a.e., the formula above implies that

$$F_\varepsilon \leq G := \sum_{n\geq 0} \mathcal{T}^n G_2 \text{ a.e. in } (t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1\,,$$

where

$$G_2(t,s,x,v) := \mathbb{1}_{t<s}\sigma e^{-\sigma s} f^{in}(x-tv,v)\,.$$

Thus, G satisfies the integral equation

$$G = G_2 + \mathcal{T}G$$

meaning that G is the mild solution of

$$\begin{cases} (\partial_t + v\cdot\nabla_x + \partial_s)G = -\sigma G\,, & t,s>0\,,\ x\in\mathbb{R}^2\,,\ |v|=1\,, \\ G(t,0,x,v) = \sigma\displaystyle\int_0^{+\infty} KG(t,s,x,v)ds\,, & t>0\,,\ x\in\mathbb{R}^2\,,\ |v|=1\,, \\ G(0,s,x,v) = f^{in}(x,v)\sigma e^{-\sigma s}\,, & s>0\,,\ x\in\mathbb{R}^2\,,\ |v|=1\,, \end{cases}$$

Reasoning as in Proposition 2 shows that

$$g(t,x,v) := \int_0^{+\infty} G(t,s,x,v)ds$$

is the solution of the linear Boltzmann equation

$$\begin{cases} (\partial_t + v\cdot\nabla_x)g + \sigma(g-Kg) = 0\,, & t>0\,,\ x\in\mathbb{R}^2\,,\ |v|=1\,, \\ g(0,x,v) = f^{in}(x,v)\,, & x\in\mathbb{R}^2\,,\ |v|=1\,. \end{cases}$$

In view of the assumption (11) bearing on f^{in}, we know that

$$G \geq 0 \text{ a.e. on } \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$$

and

$$\begin{aligned}\int_0^{+\infty}\iint_{\mathbb{R}^2\times\mathbb{S}^1} G(t,s,x,v)dxdvds &= \iint_{\mathbb{R}^2\times\mathbb{S}^1} g(t,x,v)dxdv \\ &= \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv\end{aligned}$$

for each $t \geq 0$.

Summarizing, we have

$$0 \leq \{F_\varepsilon\} \leq G$$

and

$$\iiint_{\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1} G(t,s,x,v)dsdxdv = \iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv < +\infty\,.$$

Then we conclude as follows: for each $R > 0$, one has

$$\begin{aligned}
&\iint_{Z_\varepsilon \times \mathbb{S}^1} f_\varepsilon(t,x,v)dxdv - \int_0^{+\infty} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} F(t,s,x,v)dxdvds \\
&\qquad = \int_0^{+\infty} \int_{|x|>R} \int_{\mathbb{S}^1} \{F_\varepsilon\}(t,s,x,v)dvdxds \\
&\qquad + \int_0^{+\infty} \int_{|x|\leq R} \int_{\mathbb{S}^1} \left(\{F_\varepsilon\} - F\right)(t,s,x,v)dvdxds \\
&\qquad - \int_0^{+\infty} \int_{|x|>R} \int_{\mathbb{S}^1} \{F\}(t,s,x,v)dvdxds = I_{R,\varepsilon}(t) + II_{R,\varepsilon}(t) + III_R(t)\,.
\end{aligned}$$

First, for a.e. $t > 0$, the term $I_{R,\varepsilon}(t) \to 0$ as $R \to +\infty$ uniformly in $\varepsilon > 0$ since $0 \leq \{F_\varepsilon\} \leq G$ and $G \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1))$.

Next, the term $II_{R,\varepsilon}(t) \to 0$ strongly in $L^1_{loc}(\mathbb{R}_+)$ as $\varepsilon \to 0^+$ for each $R > 0$ by Lemma 4.

Finally, since $\{F_\varepsilon\} \rightharpoonup F$ in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)$ weak as $\varepsilon \to 0^+$, one has $0 \leq \{F\} \leq G$, so that $F \in L^\infty(\mathbb{R}_+; L^1(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1))$. Hence the term $III_R(t) \to 0$ as $R \to +\infty$ for a.e. $t \geq 0$.

Thus we have proved that

$$\iint_{Z_\varepsilon \times \mathbb{S}^1} f_\varepsilon(t,x,v)dxdv \to \int_0^{+\infty} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} F(t,s,x,v)dxdvds$$

in $L^1_{loc}(\mathbb{R}_+)$ and therefore for a.e. $t \geq 0$, possibly after extraction of a subsequence of $\varepsilon \to 0^+$. $\square$

5.3. **An integral equation for M.** Given a function ψ defined (a.e.) on the half-line $\mathbb{R}_+$, we abuse the notation $\psi \mathbb{1}_{\mathbb{R}_+}$ to designate its extension by 0 on $\mathbb{R}^*_-$.

Henceforth we also denote

$$\kappa(t) := p(t)\sigma e^{-\sigma t} \mathbb{1}_{t \geq 0}.$$

Lemma 6. *The function M defined in (13) satisfies the integral equation*

$$M(t) = \kappa * (M \mathbb{1}_{\mathbb{R}_+})(t) + \tfrac{1}{2\pi\sigma}\kappa(t) \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv, \ t \geq 0$$

where $*$ denotes the convolution on the real line.

Proof. We apply the same method as for deriving the explicit representation formula for F starting from the equation in Corollary 1, in order to find an exact formula for m. Indeed, by the method of characteristics,

$$\begin{aligned}
m(t,s) &= \mathbb{1}_{s<t} p(s) e^{-\sigma s} m(t-s,0) + \mathbb{1}_{t<s} p(t) e^{-\sigma t} m(0, s-t) \\
&= \mathbb{1}_{s<t} p(s) \sigma e^{-\sigma s} \int_0^\infty m(t-s,u)du \\
&+ \mathbb{1}_{t<s} p(t) \sigma e^{-\sigma s} \tfrac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv\,.
\end{aligned}$$

The function m satisfies therefore

$$\begin{aligned}
m(t,s) &= \mathbb{1}_{s<t} p(s) \sigma e^{-\sigma s} M(t-s) \\
&\quad + \mathbb{1}_{t<s} p(t) \sigma e^{-\sigma s} \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv\,.
\end{aligned} \tag{25}$$

We next integrate both sides of (25) in $s \in \mathbb{R}_+$. By the definition (13) of M, we obtain

$$M(t) = \int_0^t \sigma p(s) e^{-\sigma s} M(t-s)ds + p(t) e^{-\sigma t} \tfrac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v)dxdv$$

a.e. in $t \geq 0$, which is precisely the desired integral equation for M:

$$M(t) = \int_0^t \kappa(s)M(t-s)ds + \tfrac{1}{2\pi\sigma}\kappa(t)\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv\,. \tag{26}$$

$\square$

5.4. **An explicit representation formula for M.**

Lemma 7. *Let M be the function defined in (13). Then*

$$M = \tfrac{1}{2\pi\sigma}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv \sum_{n\geq 1}\kappa^{*n}$$

with the notation

$$\kappa^{*n} = \underbrace{\kappa * \cdots * \kappa}_{n\ factors}\,.$$

Proof. Observe that

$$\begin{aligned}\int_0^{+\infty}\kappa(t)dt &= \sigma\int_0^{+\infty} e^{-\sigma t}p(t)dt \\ &= 1 + \int_0^{+\infty}\dot{p}(t)e^{-\sigma t}dt < 1\,,\end{aligned} \tag{27}$$

where the second equality results from integrating by parts the integral defining κ, and the final inequality is implied by the fact that p is a C^1 decreasing function.

By Lemma 5, $M \in L^1_{loc}(\mathbb{R}_+)$ and $M \geq 0$ a.e. on $\mathbb{R}_+$ since $f_\varepsilon \geq 0$ a.e. on $\mathbb{R}_+ \times Z_\varepsilon \times \mathbb{S}^1$ because $f^{in} \geq 0$ a.e. on $\mathbb{R}^2 \times \mathbb{S}^1$ — see the positivity assumption in (11). Applying the Fubini theorem shows that

$$\begin{aligned}\int_0^{+\infty} M(t)dt &= \int_0^{+\infty}\int_0^t \kappa(t-s)M(s)dsdt + \tfrac{1}{2\pi\sigma}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv\int_0^{+\infty}\kappa(t)dt \\ &= \int_0^{+\infty} M(s)\left(\int_s^{+\infty}\kappa(t-s)dt\right)ds + \tfrac{1}{2\pi\sigma}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv\int_0^{+\infty}\kappa(t)dt.\end{aligned}$$

In other words

$$\|M\|_{L^1(\mathbb{R}_+)} \leq \|M\|_{L^1(\mathbb{R}_+)}\|\kappa\|_{L^1(\mathbb{R}_+)} + \tfrac{1}{2\pi\sigma}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv\,,$$

so that $M \in L^1(\mathbb{R}_+)$ since $\|\kappa\|_{L^1(\mathbb{R}_+)} < 1$, and

$$\|M\|_{L^1(\mathbb{R}_+)} \leq \frac{1}{2\pi\sigma(1 - \|\kappa\|_{L^1(\mathbb{R}_+)})}\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv\,.$$

In particular, if

$$\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv = 0$$

then $M = 0$ a.e. on $\mathbb{R}_+$, so that the representation formula to be established obviously holds in this case.

Otherwise

$$\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv > 0\,;$$

define then

$$\psi(t) := 2\pi\sigma\left(\iint_{\mathbb{R}^2\times\mathbb{S}^1} f^{in}(x,v)dxdv\right)^{-1} M(t),\ t \geq 0\,.$$

According to Lemma 6, the function ψ verifies the integral equation

$$\psi(t) = (\kappa * (\psi\mathbb{1}_{\mathbb{R}_+}))(t) + \kappa(t)\,, \quad \text{a.e. in } t \geq 0\,. \tag{28}$$

Applying the Fubini theorem as above shows that the linear operator

$$\mathcal{A} : L^1(\mathbb{R}_+) \ni f \mapsto \kappa * (f\mathbb{1}_{\mathbb{R}_+}) \in L^1(\mathbb{R}_+)$$

satisfies

$$\|\mathcal{A}f\|_{L^1(\mathbb{R}_+)} \leq \|\mathcal{A}\|\|f\|_{L^1(\mathbb{R}_+)} \quad \text{with } \|\mathcal{A}\| = \int_0^{+\infty} \kappa(t)dt < 1\,.$$

Therefore $(1 - \mathcal{A})$ is invertible in the class of bounded operators on $L^1(\mathbb{R}_+)$ with inverse

$$(1 - \mathcal{A})^{-1} = \sum_{n\geq 0} \mathcal{A}^n\,.$$

In particular

$$\psi = (I - \mathcal{A})^{-1}\kappa = \sum_{n\geq 1} \kappa^{\star n}$$

is the unique solution of the integral equation (28) in $L^1(\mathbb{R}_+)$, which establishes the representation formula in the lemma. □

5.5. **Asymptotic behavior of M in the long time limit.**

5.5.1. *The characteristic exponent ξ_σ.*

Lemma 8. *For each $\sigma > 0$, the equation*

$$\int_0^{\infty} \sigma e^{-(\sigma+\xi)t} p(t)dt = 1$$

with unknown ξ has a unique real solution ξ_σ. This solution ξ_σ satisfies

$$-\sigma < \xi_\sigma < 0.$$

Proof. Consider the Laplace transform of the function κ defined above:

$$\mathcal{L}[\kappa](\xi) := \int_0^{\infty} \sigma e^{-(\sigma+\xi)t} p(t)dt.$$

As $0 < p \leq 1$, $\mathcal{L}[\kappa]$ is of class C^1 on $]-\sigma, +\infty[$, and

$$\dot{\mathcal{L}}[\kappa](\xi) = -\int_0^{\infty} \sigma e^{-(\sigma+\xi)t} tp(t)dt < 0$$

as $p(t) > 0$ for each $t \geq 0$. The function $\mathcal{L}[\kappa]$ is therefore decreasing on $]-\sigma, +\infty[$.

For each $t > 0$,

$$\kappa(t)e^{-\xi t} \to 0^+ \quad \text{as } \xi \to +\infty\,,$$

while

$$\kappa(t)e^{-\xi t} \leq \sigma e^{-\sigma t} \quad \text{for each } t \geq 0\,,$$

since $0 < p \leq 1$. By dominated convergence, one concludes that

$$\mathcal{L}[\kappa](\xi) \to 0^+ \text{ as } \xi \to +\infty.$$

Besides, for each $t > 0$,

$$\sigma p(t)e^{-(\sigma+\xi)t} \uparrow \sigma p(t)\,, \quad \text{as } \xi \downarrow -\sigma^+\,.$$

By monotone convergence,

$$\mathcal{L}[\kappa](\xi) \to \sigma \int_0^{+\infty} p(t)dt = +\infty\,, \quad \text{as } \xi \to -\sigma^+\,.$$

(Notice that the equality

$$\int_0^{+\infty} p(t)dt = +\infty$$

follows from the lower bound in (8).)

By the intermediate value theorem, there exists an unique $\xi_\sigma > -\sigma$ such that

$$\mathcal{L}[\kappa](\xi_\sigma) = 1.$$

Besides $\xi_\sigma < 0$ as $\mathcal{L}[\kappa]$ is decreasing and

$$\mathcal{L}[\kappa](0) = \int_0^\infty \kappa(t)dt < \int_0^{+\infty} \sigma e^{-\sigma t}dt = 1 = \mathcal{L}[\kappa](\xi_\sigma)\,,$$

which concludes the proof. □

In particular

$$t \mapsto \kappa(t)e^{-\xi_\sigma t}$$

is a decreasing probability density on $\mathbb{R}_+$.

5.5.2. *The Renewal Equation.* It remains to prove statement (3) in Theorem 2.

First, for each $\lambda \in \mathbb{R}$ and each locally bounded measurable function $f : \mathbb{R} \mapsto \mathbb{R}$ supported in $\mathbb{R}_+$, denote

$$f_\lambda(t) := e^{\lambda t} f(t) \text{ for each } t \in \mathbb{R}\,.$$

Notice that for each such f, g, we have

$$e^{\lambda t}(f * g)(t) = (f_\lambda * g_\lambda)(t) \text{ for each } t \in \mathbb{R}.$$

Hence, if ψ is a solution of the integral equation (28), the function $\psi_{-\xi_\sigma}$ satisfies

$$\psi_{-\xi_\sigma}(t) = (\kappa_{-\xi_\sigma} * \psi_{-\xi_\sigma})(t) + \kappa_{-\xi_\sigma}\,, \tag{29}$$

which is a renewal integral equation, in the sense of [14].

Moreover, as noticed above, $\kappa_{-\xi_\sigma}$ is a decreasing probability density on $\mathbb{R}_+$, so that in particular $\kappa_{-\xi_\sigma}$ is directly Riemann integrable (see [14] pp. 348-349). Thus, applying Theorem 2 on p. 349 in Feller's Introduction to Probability Theory [14] shows that

$$\psi(t)e^{-\xi_\sigma t} \to \frac{1}{\displaystyle\int_0^\infty t\kappa(t)e^{-\xi_\sigma t}dt} \quad \text{as } t \to +\infty. \tag{30}$$

By definition of ψ, this is precisely the asymptotic behavior of M in Theorem 2 (3).

5.6. **Two important limiting cases for ξ_σ.** We conclude our proof of Theorem 2 with a discussion of the asymptotic behavior of ξ_σ (statement (4) of Theorem 2) in the two following regimes:

(1) the collisionless regime $\sigma \to 0^+$, and
(2) the highly collisional regime $\sigma \to +\infty$.

End of the proof of Theorem 2. Denote for the sake of simplicity $\lambda_\sigma := \sigma + \xi_\sigma$. Establishing that $\xi_\sigma \sim -\sigma$ as $\sigma \to 0^+$ amounts to proving that $\lambda_\sigma = o(\sigma)$. First, notice that, since $-\sigma < \xi_\sigma$,

$$0 < \lambda_\sigma < \sigma$$

so $\lambda_\sigma \to 0^+$ as $\sigma \to 0^+$. Keeping this in mind, we have

$$\int_0^{+\infty} e^{-\lambda_\sigma t} p(t)dt = \frac{1}{\sigma} \tag{31}$$

by definition of ξ_σ. Substituting $z = \lambda_\sigma t$ in the integral above, we obtain:

$$0 < \frac{\lambda_\sigma}{\sigma} = \int_0^{+\infty} e^{-z} p(z/\lambda_\sigma)dz.$$

Since $\lambda_\sigma \to 0^+$ as $\sigma \to 0^+$ and $p(t) \to 0^+$ as $t \to +\infty$, one has $p(z/\lambda_\sigma) \to 0^+$ as $\sigma \to 0^+$. Besides $0 \le e^{-z}p(z/\lambda_\sigma) \le e^{-z}$ so that, by dominated convergence

$$\frac{\lambda_\sigma}{\sigma} \to 0 \text{ as } \sigma \to 0^+.$$

This establishes the asymptotic behavior of ξ_σ in the collisionless regime.

As for the highly collisional regime, we return to the equation (31) defining ξ_σ (written in terms of λ_σ):

$$\begin{aligned}1 &= \sigma \int_0^{+\infty} e^{-\lambda_\sigma t} p(t)dt \\ &= \lambda_\sigma \int_0^{\infty} e^{-\lambda_\sigma t} p(t)dt - \xi_\sigma \int_0^{\infty} e^{-\lambda_\sigma t} p(t)dt \\ &= 1 + \int_0^{\infty} e^{-\lambda_\sigma t} \dot{p}(t)dt - \xi_\sigma \int_0^{\infty} e^{-\lambda_\sigma t} p(t)dt\end{aligned}$$

where the last equality follows from integrating by parts the first integral on the left hand side. Therefore

$$\xi_\sigma = \frac{\displaystyle\int_0^{\infty} e^{-\lambda_\sigma t} \dot{p}(t)dt}{\displaystyle\int_0^{\infty} e^{-\lambda_\sigma t} p(t)dt},$$

or, after substituting $t' = \lambda_\sigma t$,

$$\xi_\sigma = \frac{\displaystyle\int_0^{\infty} e^{-t} \dot{p}(t/\lambda_\sigma)dt}{\displaystyle\int_0^{\infty} e^{-t} p(t/\lambda_\sigma)dt}. \tag{32}$$

Equation (31) shows that $\lambda_\sigma \to +\infty$ as $\sigma \to +\infty$. Passing to the limit in the right-hand side of (32), we find, by dominated convergence

$$\xi_\sigma \to \frac{\displaystyle\int_0^{\infty} e^{-t} \dot{p}(0)dt}{\displaystyle\int_0^{\infty} e^{-t} p(0)dt} = \dot{p}(0) \qquad \text{as } \sigma \to +\infty\,.$$

Indeed p is decreasing and convex, as can be verified for instance on the Boca-Zaharescu explicit formula[2] (10)-(9) for p, so that

$$0 \le -\dot{p}(t) \le -\dot{p}(0)\,, \qquad \text{for each } t \ge 0\,.$$

We conclude by observing that the same explicit formulas of Boca-Zaharescu [4] imply that

$$\dot{p(0)} = -2\,.$$

□

6. Final remarks and open problems

The present work provides a complete description of the homogenization of the linear Boltzmann equation for monokinetic particles in the periodic system of holes of radius ε^2 centered at the vertices of the square lattice $\varepsilon\mathbb{Z}^2$ (Theorem 1.) In particular, we have given an asymptotic equivalent of exponential type of the total mass of the particle system in the long time limit (Theorem 2.)

[2]In space dimension higher than 2, one can show that the analogue of p is also nonincreasing and convex, by using a variant of a formula due to L.A. Santalò established in [13], for want of an explicit formula giving the limiting distribution of free path lengths.

Since the discussion in the present paper is restricted to the two dimensional setting, it would be useful to extend the results above to the case of higher space dimensions, and to lattices other than the square or cubic lattice. Most of the arguments considered here can be adapted to these more general cases; however, the analogue of the distribution of free path lengths (the function $p(t)$) is not known explicitly so far. See [3] for these more general cases.

Otherwise, it would also be interesting to investigate other scalings than the Boltzmann-Grad type scaling considered here — holes of radius ε^2 centered at the vertices of a square lattice whose fundamental domain is a square of sise ε in the case of space dimension 2. Typically, one would like to mix the homogenization procedure considered in the present work with the assumption of a highly collisional regime $\sigma \gg 1$, so that the size of the holes and the distance between neighboring holes are scaled in a way that differs from the one considered here. We hope to return to this problem in a forthcoming publication.

Another problem of potential interest is the case where the periodically distributed holes considered in the present paper are replaced with scatterers, assuming that particles are specularly reflected on the surface of each scatterer. In other words, the problem (Ξ_ε) is replaced with

$$\begin{cases} \partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \sigma(f_\varepsilon - K f_\varepsilon) = 0, & (x,v) \in Z_\varepsilon \times \mathbb{S}^1, t > 0, \\ f_\varepsilon(t,x,v) = f_\varepsilon(t,x,v - 2(v \cdot n_x)n_x, v), & (x,v) \in \partial Z_\varepsilon \times \mathbb{S}^1, t > 0, \\ f_\varepsilon(0,x,v) = f^{in}(x,v), & (x,v) \in Z_\varepsilon \times \mathbb{S}^1. \end{cases}$$

Assume for simplicity that f_ε is periodic with period 1 in x_1, x_2, while ε designates the sequence of $1/n$, for each integer $n \geq 1$.

Most likely, the homogenized equation governing the vanishing ε limit of f_ε should involve an extended phase space, as in the case of the Boltzmann-Grad limit of the periodic Lorentz gas [8, 25]. The structure of this homogenized equation should be such that its solution converges to a constant state exponentially fast in the long time limit for each $\sigma > 0$. However, while the limiting constant state is fully determined by conservation of mass and is therefore independent of $\sigma > 0$, the exponential decay to that constant state is not expected to hold uniformly as $\sigma \to 0$. Indeed, the case $\sigma = 0$ is precisely the Boltzmann-Grad limit of the periodic Lorentz gas governed by the equation (1), and according to Theorem 3.5 in [9], this equation (1) does not have the spectral gap property.

Finally, the homogenization result considered in the present paper raises an interesting question, of quite general bearing. Usually, homogenization is a limiting process leading to a macroscopic description of some material that is known at the microscopic scale. In the problem considered here, it has been necessary to use a more detailed description of the particle system than that provided by the linear Boltzmann equation (problem (Ξ_ε) set in the extended phase space that involves the additional variable s.)

In other words, the formulation of the macroscopic homogenization limit for the linear Boltzmann equation considered here involves remnants of an *even more microscopic description* of the system than the linear Boltzmann equation itself — namely the extended phase space and the additional variable s.

We do not know whether this phenomenon (i.e. the need for a more microscopic description of a system to arrive at the formulation of a homogenized equation for that system) can be observed in homogenization problems other than the one considered here — for instance in the case of equations other than those found in context of kinetic theory.

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Chapitre 3

The Diffusion Approximation of the Homogenized Boltzmann Equation

CHAPTER III
THE DIFFUSION APPROXIMATION OF THE HOMOGENIZED BOLTZMANN EQUATION

Introduction

A classical result in kinetic theory is that solutions of the linear Boltzmann equation in the small mean free path limit are governed by a diffusion equation -see [2, 8] for instance. However, the structure of the linear Boltzmann equation can be deeply modified in the case of some homogenization limits involving perforated background media with a periodic distribution of holes : see [3]. Whether a diffusion limit can be established in such cases is not entirely obvious, as the homogenized equation may fail to have some of the crucial properties used in [2]. The purpose of the present work is to study this situation on an example.

We first recall more precisely the recent homogenization result evoked above and proved in the previous chapter and published in [3]. Consider $f_\eta \equiv f_\eta(t,x,v)$ the solution of the boundary-value problem

$$\begin{cases} \partial_t f_\eta + v\cdot\nabla_x f_\eta + \sigma(f_\eta - \overline{f}_\eta) = 0, & t>0, (x,v)\in Z_\eta\times\mathbb{S}^1,\\ f_\eta(t,x,v) = 0, & t>0, (x,v)\in \partial Z_\eta\times\mathbb{S}^1\ v\cdot n_x>0,\\ f_\varepsilon(0,x,v) = f^{in}(x), & (x,v)\in Z_\eta\times\mathbb{S}^{d-1}, \end{cases}$$

where

$$\overline{\phi} := \frac{1}{2\pi}\int_{\mathbb{S}^1}\phi(v)dv$$

and where Z_η designates the space $\mathbb{R}^2$ with a periodic system of holes removed

$$Z_\eta := \mathbb{R}^2 \setminus \bigcup_{k\in\mathbf{Z}^2} B\left(\eta k, \eta^2\right).$$

The authors proved in that

$$f_\eta \rightharpoonup f := \int_0^\infty F ds \text{ as } \eta\to 0^+$$

in $L^\infty\left(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1\right)$ −weak-*, where $F \equiv F(t,s,x,v)$ is the solution of the Cauchy problem

$$\text{(0.1)}\quad \begin{cases} \partial_t F + v\cdot\nabla_x F + \partial_s F = -\sigma F + \dfrac{\dot{p}(t\wedge s)}{p(t\wedge s)}F\,, & x\in\mathbb{R}^d\,,\ |v|=1\,,\ s,t>0\,,\\ F(t,0,x,v) = \sigma\displaystyle\int_0^\infty \overline{F}(t,s',x)ds',\\ F(0,s,x,v) = f^{in}(x)\sigma e^{-\sigma s}. \end{cases}$$

Here, p is a positive decreasing function which definition is given here. Let τ_η designate the free path length in the direction v for a particle starting from x in Z_η

$$\tau_\eta(x,v) := \{t>0|x-tv\in\partial Z_\eta\}.$$

The distribution of free path lengths has been studied in [4, 5, 6, 7]. In particular, we have established in Lemma 1 in [3] that for each $t \geq 0$

$$\mathbb{1}_{t<\tau_\eta\left(\frac{x}{\eta},v\right)} \to p(t) \text{ in } L^\infty\left(\mathbb{R}^2 \times \mathbb{S}^1\right) - \text{weak* as } \eta \to 0^+.$$

F. Boca and A. Zaharescu have given an explicit representation formula for p

$$p(t) = \int_t^\infty (s-t)\Upsilon(s)ds,$$

where

$$\Upsilon(t) = \frac{24}{\pi^2}\begin{cases} 1 & \text{if } t \in \left(0, \frac{1}{2}\right], \\ \frac{1}{2t} + 2\left(1-\frac{1}{2t}\right)^2 ln\left(1-\frac{1}{2t}\right) - \frac{1}{2}ln\left|1-\frac{1}{t}\right| & \text{if } t > \frac{1}{2}. \end{cases}$$

In particular, $t \mapsto \frac{\dot{p}}{p}$ is not a constant. That remark will be crucial in the sequel.

In the present work, we shall study a diffusion approximation of (0.1) More precisely, we scale σ and x in (0.1) so that

$$\sigma = \frac{\hat{\sigma}}{\varepsilon^2}$$

and

$$\hat{x} = \frac{x}{\varepsilon}$$

with $\varepsilon \ll 1$, and define

$$\hat{F}_\varepsilon\left(t, s, \hat{x}, v\right) := F(t, s, x, v).$$

Since F is a solution of (0.1), $\hat{F}_\varepsilon$ satisfies

$$\begin{cases} \partial_t \hat{F}_\varepsilon + \frac{v}{\varepsilon}\cdot\nabla_{\hat{x}}\hat{F}_\varepsilon + \partial_s\hat{F}_\varepsilon = -\frac{\hat{\sigma}}{\varepsilon^2}\hat{F}_\varepsilon + \frac{\dot{p}(t\wedge s)}{p(t\wedge s)}\hat{F}_\varepsilon\,, & \hat{x}\in\mathbb{R}^d\,,\ |v|=1\,,\ s,t>0\,, \\ \hat{F}_\varepsilon(t,0,\hat{x},v) = \frac{\hat{\sigma}}{\varepsilon^2}\int_0^\infty \overline{\hat{F}}_\varepsilon(t,s',\hat{x})ds', \\ \hat{F}_\varepsilon(0,s,\hat{x},v) = f^{in}(\varepsilon\hat{x})\frac{\hat{\sigma}}{\varepsilon^2}e^{-\frac{\hat{\sigma}}{\varepsilon^2}s}. \end{cases}$$

For the sake of simplicity, we henceforth drop the hat in $\hat{\sigma}, \hat{x}$ and $\hat{F}_\varepsilon$. In other words, we consider $F_\varepsilon \equiv F_\varepsilon(t,s,x,v)$ the solution of the Cauchy problem

(0.2)
$$\begin{cases} \partial_t F_\varepsilon + \frac{v}{\varepsilon}\cdot\nabla_x F_\varepsilon + \partial_s F_\varepsilon = -\frac{\sigma}{\varepsilon^2}F_\varepsilon + \frac{\dot{p}(t\wedge s)}{p(t\wedge s)}F_\varepsilon\,, & x\in\mathbb{R}^2\,,\ |v|=1\,,\ s,t>0\,, \\ F_\varepsilon(t,0,x,v) = \frac{\sigma}{\varepsilon^2}\int_0^\infty \overline{F}_\varepsilon(t,s',x)ds', \\ F_\varepsilon(0,s,x,v) = \rho^{in}(x)\frac{\sigma}{\varepsilon^2}e^{-\frac{\sigma}{\varepsilon^2}s}. \end{cases}$$

The problem studied in this paper is to find the correct equation for F_ε in the vanishing ε limit.

Observe that by integrating equation (0.2) in $s \in \mathbb{R}_+$, one obtains

$$\begin{cases} \left(\partial_t + \frac{v}{\varepsilon}.\nabla_x + \frac{\sigma}{\varepsilon^2}\right)f_\varepsilon - \frac{\sigma}{\varepsilon^2}\overline{f}_\varepsilon = \int_0^\infty \frac{\dot{p}(t\wedge s)}{p(t\wedge s)}F_\varepsilon ds, \\ \\ f_\varepsilon(0,x,v) = \rho^{in}(x). \end{cases} \tag{0.3}$$

This is not a closed equation for $f_\varepsilon := \int_0^\infty F_\varepsilon ds$, since $(t,s) \mapsto \frac{\dot{p}(t\wedge s)}{p(t\wedge s)}$ is not a constant as mentioned above. That being said, we observe that

$$F_\varepsilon\big|_{t=0} \to \rho^{in}\delta_{s=0} \text{ in } \mathcal{M}\left(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1\right),\ \text{as } \varepsilon\to 0^+,$$

which suggests that

$$F_\varepsilon \to f\delta_{s=0} \text{ in } \mathcal{M}\left(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1\right) \text{ as } \varepsilon \to 0^+, \tag{0.4}$$

where $f \equiv f(t,x,v)$.
In this case, we expect that, in the limit as ε tends to 0, the solution f_ε of (0.3) behaves like the solution g_ε of the Cauchy problem

$$\begin{cases} \left(\partial_t + \dfrac{v}{\varepsilon}.\nabla_x + \dfrac{\sigma}{\varepsilon^2}\right) g_\varepsilon - \dfrac{\sigma}{\varepsilon^2}\overline{g}_\varepsilon = \dfrac{\dot{p}(0)}{p(0)} g_\varepsilon ds, \\ \\ g_\varepsilon(0,x,v) = \rho^{in}(x). \end{cases} \tag{0.5}$$

This a classical linear Boltzmann equation for which the diffusion approximation is well-known, i.e.

$$g_\varepsilon \to \rho \text{ in } L^2\left(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1\right)\text{-strong as } \varepsilon \to 0^+$$

where $\rho \equiv \rho(t,x)$ is the solution of the Cauchy problem

$$\begin{cases} \left(\partial_t - \dfrac{1}{2\sigma}\Delta_x\right)\rho = \dfrac{\dot{p}}{p}(0)\rho(t,x),\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \qquad\qquad \rho(0,x) = \rho^{in}(x), \qquad x \in \mathbb{R}^2. \end{cases} \tag{0.6}$$

We now give a argument in support of (0.4).
We define $\phi(s) := s \wedge 1$ for each $s \geq 0$, and

$$m_\phi(t) := \iiint_{\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1} F_\varepsilon(t,s,x,v)\phi(s)dsdxdv \text{ for each } t \geq 0.$$

Multiplying both sides of (0.2) by ϕ and integrating in $(s,x,v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$, one obtains

$$\begin{aligned} \dot{m}_\phi + \frac{\sigma}{\varepsilon^2} m_\phi - m_{\dot{\phi}} &= \iiint \frac{\dot{p}}{p}(s \wedge t) F_\varepsilon \phi dsdxdv \\ &\leq \left\|\frac{\dot{p}}{p}\right\|_{L^\infty(\mathbb{R}_+)} \iiint F_\varepsilon \phi dsdxdv = \left\|\frac{\dot{p}}{p}\right\|_{L^\infty(\mathbb{R}_+)} m_\phi \end{aligned}$$

or equivalently

$$\dot{m}_\phi + \frac{\sigma}{\varepsilon^2} m_\phi - \int_0^1 \iint F_\varepsilon dxdvds \leq \left\|\frac{\dot{p}}{p}\right\|_{L^\infty(\mathbb{R}_+)} m_\phi.$$

Thus

$$\dot{m}_\phi + \frac{\sigma}{\varepsilon^2} m_\phi \leq \left\|\frac{\dot{p}}{p}\right\|_{L^\infty(\mathbb{R}_+)} m_\phi + \int_0^\infty F_\varepsilon dxdvds.$$

By maximum principle, we have

$$\int_0^\infty F_\varepsilon dxdvds \leq \int_{\mathbb{R}^2} \rho^{in}(x)dx \text{ for each } \varepsilon > 0,$$

so that

$$\dot{m}_\phi + \frac{\sigma}{\varepsilon^2} m_\phi \leq \left\|\frac{\dot{p}}{p}\right\|_{L^\infty(\mathbb{R}_+)} m_\phi + \int_{\mathbb{R}^2} \rho^{in}(x)dx,$$

or equivalently

$$\dot{m}_\phi + C_\varepsilon m_\phi \leq C^{in}$$

with the notation

$$C_\varepsilon := \frac{\sigma}{\varepsilon^2} - \left\|\frac{\dot{p}}{p}\right\|_{L^\infty(\mathbb{R}_+)},$$

and

$$C^{in} := \int_{\mathbb{R}^2} \rho^{in}(x)dx.$$

That implies in particular

$$\frac{d}{dt}\left(e^{C_\varepsilon t} m_\phi\right) \leq C^{in} e^{C_\varepsilon t}$$

so that for each $t \geq 0$

$$0 \leq m_\phi(t) \leq e^{-C_\varepsilon t} m_\phi(0) + \frac{C^{in}}{C_\varepsilon} = O\left(C_\varepsilon^{-1}\right).$$

Since $C_\varepsilon^1 \to 0$ as $\varepsilon \to 0^+$

$$F_\varepsilon \to 0 \text{ as } \varepsilon \to 0^+.$$

in $\mathcal{D}'_s\left(]0,\infty[\right)$. We can therefore expect that

$$F_\varepsilon \to \rho\delta_{s=0} \text{ as } \varepsilon \to 0^+ \tag{0.7}$$

in $\mathcal{M}\left(\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1\right)$ and where ρ is the solution of (0.6).

We have evoked above the diffusion approximation for some classical linear Boltzmann equation, that is established either by the Hilbert decomposition -see [2] for instance, or compactness in L^2 see for instance [8]. Both methods do not fit to a convergence such that (0.7) so that our argument is based on a dual formulation and Fourier transform.

0.1. **Main result.** We give here the diffusion approximation for the homogenized equation. Consider $F_\varepsilon \equiv F_\varepsilon(t,s,x,v)$ the solution of the Cauchy problem

$$(\Sigma_\varepsilon)\begin{cases} \partial_t F_\varepsilon + \frac{v}{\varepsilon}.\nabla_x F_\varepsilon + \partial_s F_\varepsilon = -\frac{\sigma}{\varepsilon^2} F_\varepsilon + \frac{\dot{p}}{p}(t\wedge s) F_\varepsilon, & t,s>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1 \\ F_\varepsilon(t,0,x,v) = \frac{\sigma}{\varepsilon^2}\int_0^{+\infty} \overline{F}_\varepsilon(t,s,x)ds, & t>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1 \\ F_\varepsilon(0,s,x,v) = \frac{\sigma}{\varepsilon^2} e^{-\sigma s/\varepsilon^2} f^{in}(x), & s>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1 \end{cases}$$

where $p \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ and is decreasing. We assume besides that

$$f^{in} \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2)$$

and there exists $C > 0$ such that

$$\sup_{x\in\mathbb{R}^2} \left|f^{in}(x)\right| \leq C.$$

The main result is then

Theorem 0.1. *Under the assumptions and with the notations above,*

$$\int_0^\infty F_\varepsilon(s)ds \to u \text{ in } L^2_{loc}\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2\times\mathbb{S}^1\right)\right) - \text{weak as } \varepsilon \to 0^+$$

where u is a solution in the sense of distributions of the Cauchy problem

$$\begin{cases} \partial_t u - \frac{1}{2\sigma}\Delta u = \dot{p}(0)u & t>0, x\in\mathbb{R}^2, \\ u(0,x) = f^{in}(x) & x\in\mathbb{R}^2. \end{cases}$$

Moreover, we have in fact a strong convergence

Theorem 0.2. *Under the assumptions and with the notations above,*

$$\int_0^\infty \int_{\mathbb{S}^1} F_\varepsilon dv ds \to u \text{ in } L^{\frac{3}{2}}\left([0,T]; L^{\frac{2}{3}}_{loc}\left(\mathbb{R}^2_{loc}\right)\right) - \text{strong as } \varepsilon \to 0^+.$$

Theorem 0.2 is established in section 4 while Theorem 0.1 is proved in the sections preceding.

1. A priori bounds

In the present section, we establish some a priori estimates.

We recall that by Proposition 4 in [2], we have

$$F_\varepsilon \geq 0 \text{ a.e. on } \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1$$

so that

$$f_\varepsilon \geq 0 \text{ a.e. on } \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1. \tag{1.1}$$

That being said, we recall that f_ε verifies the equation

$$\partial_t f_\varepsilon + \frac{v}{\varepsilon}.\nabla_x f_\varepsilon + \frac{\sigma}{\varepsilon^2}\left(f_\varepsilon - \overline{f}_\varepsilon\right) = \int_0^\infty \frac{\dot{p}}{p}(t \wedge s) F_\varepsilon ds.$$

Since $p \geq 0$ and is C^1 and is nonicreasing, this implies that

$$\partial_t f_\varepsilon + \frac{v}{\varepsilon}.\nabla_x f_\varepsilon + \frac{\sigma}{\varepsilon^2}\left(f_\varepsilon - \overline{f}_\varepsilon\right) \leq 0.$$

Mutiplying both sides of the inequality above by f_ε and integrating in $x, v \in \mathbb{R}^2 \times \mathbb{S}^1$ gives

$$\frac{1}{2}\partial_t \iint_{\mathbb{R}^2\times\mathbb{S}^1} f_\varepsilon^2 dxdv + \frac{1}{2\varepsilon}\iint_{\mathbb{R}^2\times\mathbb{S}^1} v.\nabla_x \left(f_\varepsilon\right)^2 dxdv + \frac{\sigma}{\varepsilon^2}\iint_{\mathbb{R}^2\times\mathbb{S}^1}\left(f_\varepsilon - \overline{f}_\varepsilon\right) f_\varepsilon dxdv \leq 0$$

As

$$\frac{1}{2\varepsilon}\iint_{\mathbb{R}^2\times\mathbb{S}^1} v.\nabla_x \left(f_\varepsilon\right)^2 dxdv = 0$$

and

$$\begin{aligned}
\frac{\sigma}{\varepsilon^2}\iint_{\mathbb{R}^2\times\mathbb{S}^1}\left(f_\varepsilon - \overline{f}_\varepsilon\right) f_\varepsilon dxdv &= \frac{\sigma}{\varepsilon^2}\iint_{\mathbb{R}^2\times\mathbb{S}^1}\left(f_\varepsilon - \overline{f}_\varepsilon\right)^2 dxdv \\
&\quad + \frac{\sigma}{\varepsilon^2}\iint_{\mathbb{R}^2\times\mathbb{S}^1}\left(f_\varepsilon - \overline{f}_\varepsilon\right)\overline{f}_\varepsilon dxdv, \\
&= \frac{\sigma}{\varepsilon^2}\iint_{\mathbb{R}^2\times\mathbb{S}^1}\left(f_\varepsilon - \overline{f}_\varepsilon\right)^2 dxdv,
\end{aligned}$$

the inequality above is equivalent to

$$\frac{1}{2}\partial_t \iint_{\mathbb{R}^2\times\mathbb{S}^1} f_\varepsilon^2 dxdv + \frac{\sigma}{\varepsilon^2}\iint_{\mathbb{R}^2\times\mathbb{S}^1}\left(f_\varepsilon - \overline{f}_\varepsilon\right)^2 dxdv \leq 0.$$

Integrating both sides of this inequality further in $t \in [0, T]$ gives

$$\|f_\varepsilon\|^2_{L^2(\mathbb{R}^2\times\mathbb{S}^1)}(T) + \frac{2\sigma}{\varepsilon^2}\|f_\varepsilon - \overline{f}_\varepsilon\|^2_{L^2([0,T]\times\mathbb{R}^2\times\mathbb{S}^1)} \leq \|f^{in}\|^2_{L^2(\mathbb{R}^2\times\mathbb{S}^1)}.$$

We notice that by Jensen inequality, for each $\varepsilon > 0$ and for each $t \geq 0$

$$\begin{aligned}
\int_{\mathbb{R}^2} \overline{f}_\varepsilon^2(t,x) dx &= \int_{\mathbb{R}^2}\left(\int_{\mathbb{S}^1} f_\varepsilon(t,x,v) dv\right)^2 dx \\
&\leq \int_{\mathbb{R}^2}\left(f_\varepsilon(t,x,v)\right)^2 dvdx = \|f_\varepsilon\|^2_{L^2(\mathbb{R}^2\times\mathbb{S}^1)}(t).
\end{aligned}$$

As a result, we obtain the following bounds.

Proposition 1.1. *Under the notations and assumptions above, we have*

- $\left(f_\varepsilon\right)_{\varepsilon>0} = O(1)$ *in* $L^\infty\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2 \times \mathbb{S}^1\right)\right)$.
- $\left(\overline{f}_\varepsilon\right)_{\varepsilon>0} = O(1)$ *in* $L^\infty\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2\right)\right)$.
- *for each* $T > 0$, $\left(\frac{1}{\varepsilon}\left(f_\varepsilon - \overline{f}_\varepsilon\right)\right)_{\varepsilon>0} = O(1)$ *in* $L^2\left([0,T] \times \mathbb{R}^2 \times \mathbb{S}^1\right)$.

The proposition above immediately entails the following

Corollary 1.2. *Under the assumptions and notations above, the family $(\hat{u}_\varepsilon)_{\varepsilon>0}$ is relatively compact in $L^\infty\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2\right)\right)$ weak-*.*
In particular, there exists $u \in L^\infty\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2\right)\right)$ and $\varepsilon_k \to 0^+$ as $k \to +\infty$ such that

$$\hat{u}_\varepsilon \to \hat{u} \textit{ in } L^\infty\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2\right)\right) - \textit{weak-*}.$$

Now we give an integral equation for u_ε.

2. Integral Formulation of the Homogenized Equation

2.1. An Integral Equation for u_ε. First we recall some notations. Henceforth, we denote $F_\varepsilon \equiv F_\varepsilon(t,s,x,v)$ the solution of the Cauchy problem :

$$(\Sigma_\varepsilon)\begin{cases} \partial_t F_\varepsilon + \frac{v}{\varepsilon}.\nabla_x F_\varepsilon + \partial_s F_\varepsilon = -\frac{\sigma}{\varepsilon^2}F_\varepsilon + \frac{\dot{p}}{p}(t\wedge s)F_\varepsilon, & t,s>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1 \\ F_\varepsilon(t,0,x,v) = \frac{\sigma}{\varepsilon^2}\int_0^{+\infty}\overline{F}_\varepsilon(t,s,x)ds, & t>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1 \\ F_\varepsilon(0,s,x,v) = \frac{\sigma}{\varepsilon^2}e^{-\sigma s/\varepsilon^2}f^{in}(x), & s>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1 \end{cases}$$

where $p \in C^1(\mathbb{R}_+;\mathbb{R}_+)$ and is decreasing. We assume besides that

$$f^{in} \in L^1(\mathbb{R}^2)\cap L^2(\mathbb{R}^2)$$

and there exists $C>0$ such that

$$\sup_{x\in\mathbb{R}^2}\left|f^{in}(x)\right| \le C.$$

We denote also

$$f_\varepsilon(t,x,v) := \int_0^\infty F_\varepsilon(t,s,x,v)ds,\ t>0, (x,v)\in\mathbb{R}^2\times\mathbb{S}^1,$$

and

$$u_\varepsilon(t,x) := \int_{\mathbb{S}^1} f_\varepsilon(t,x,v)dv$$

where dv is the uniform probability measure in the unit sphere $\mathbb{S}^2$. We now give some a priori estimates on f_ε and u_ε that will give the compactness of $(u_\varepsilon)_{\varepsilon>0}$ in $L^\infty\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2\right)\right)$ −weak.

Let $F_\varepsilon \equiv F_\varepsilon(t,s,x,v)$ a generalized solution of (Σ_ε). For a.e. $(t,s,x,v)\in\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1$, the function $\tau\mapsto F_\varepsilon\left(t+\tau, s+\tau, x+\tau\frac{v}{\varepsilon}, v\right)$ is C^1 in $\tau>0$ and since $p\in C^1\left(\mathbb{R}_0\right)$ and does not vanish one has

$$\begin{aligned}&\left(\frac{d}{d\tau}+\frac{\sigma}{\varepsilon^2}-\frac{\dot{p}}{p}(t\wedge s+\tau)\right)F_\varepsilon\left(t+\tau, s+\tau, x+\tau\frac{v}{\varepsilon}, v\right)\\ &= e^{-\frac{\sigma}{\varepsilon^2}\tau}p(t\wedge s+\tau)\frac{d}{d\tau}\left(\frac{e^{\frac{\sigma}{\varepsilon^2}\tau}F_\varepsilon(t+\tau,s+\tau,x+\tau\frac{v}{\varepsilon},v)}{p(t\wedge s+\tau)}\right)=0.\end{aligned}$$

Therefore the function

$$\Gamma:\tau\mapsto\frac{e^{\frac{\sigma}{\varepsilon^2}\tau}F_\varepsilon(t+\tau,s+\tau,x+\tau\frac{v}{\varepsilon},v)}{p(t\wedge s+\tau)}$$

is constant. In particular

$$\Gamma(0)=\begin{cases}\Gamma(-t) & \text{if } t<s,\\ \Gamma(-s) & \text{if } s<t.\end{cases}$$

We arrive therefore at the following expression for F_ε :

$$\begin{aligned}F_\varepsilon\left(t,s,x,v\right) = &\mathbb{1}_{t<s}\frac{\sigma}{\varepsilon^2}e^{-\frac{\sigma}{\varepsilon^2}s}p(t)f^{in}\left(x-\frac{tv}{\varepsilon}\right)\\ &+\mathbb{1}_{s<t}\frac{\sigma}{\varepsilon^2}e^{-\frac{\sigma}{\varepsilon^2}s}p(s)\int_0^\infty \bar{F}_\varepsilon\left(t-s,\tau,x-\frac{sv}{\varepsilon}\right)d\tau.\end{aligned}$$

The uniqueness of the solution of (Σ_ε) can be found in the proof of Proposition 4 in [3]. Integrating the equality above in $s \in \mathbb{R}_+$ leads to

$$\begin{aligned}f_\varepsilon(t,x,v) = &p(t)f^{in}\left(x-\frac{tv}{\varepsilon}\right)\int_t^\infty \frac{\sigma}{\varepsilon^2}e^{-\frac{\sigma}{\varepsilon^2}}ds\\ &+\int_0^t \frac{\sigma}{\varepsilon^2}e^{-\frac{\sigma}{\varepsilon^2}s}p(s)\overline{f}_\varepsilon\left(t-s,x-\frac{sv}{\varepsilon}\right)ds,\end{aligned}$$

or equivalently

$$\begin{aligned}f_\varepsilon(t,x,v) = &p(t)f^{in}\left(x-\frac{tv}{\varepsilon}\right)e^{-\frac{\sigma}{\varepsilon^2}t}\\ &+\int_0^t \frac{\sigma}{\varepsilon^2}e^{-\frac{\sigma}{\varepsilon^2}s}p(s)\overline{f}_\varepsilon\left(t-s,x-\frac{sv}{\varepsilon}\right)ds.\end{aligned}\tag{2.1}$$

Integrating the equality above further in $v \in \mathbb{S}^1$ leads to the following integral equation for u_ε :

$$\begin{aligned}u_\varepsilon(t,x) = &p(t)e^{-\frac{\sigma}{\varepsilon^2}t}\int_{\mathbb{S}^1} f^{in}\left(x-\frac{tv}{\varepsilon}\right)dv\\ &+\int_{\mathbb{S}^1}\int_0^t \frac{\sigma}{\varepsilon^2}e^{-\frac{\sigma}{\varepsilon^2}s}p(s)u_\varepsilon\left(t-s,x-\frac{sv}{\varepsilon}\right)dsdv.\end{aligned}\tag{2.2}$$

We will apply the Fourier Transform to the equation above and give an integral equation for the Fourier Transform of u_ε.

2.2. **An Integral Equation in Fourier Variables.** Recall that

$$\hat{u}_\varepsilon(t,\xi) := \int_{\mathbb{R}^2} e^{-i\xi.x}u(t,x)dx,\ (t,\xi)\in\mathbb{R}_+\times\mathbb{R}^2.$$

Applying the Fourier transform to equality (2.2) leads to

$$\begin{aligned}\hat{u}_\varepsilon(t,\xi) &= \int_{\mathbb{R}^2} e^{-ix.\xi}p(t)e^{-\frac{\sigma}{\varepsilon^2}t}\int_{\mathbb{S}^1} f^{in}\left(x-\frac{tv}{\varepsilon}\right)dvdx\\ &+\int_{\mathbb{R}^2} e^{-ix.\xi}\int_{\mathbb{S}^1}\int_0^t \frac{\sigma}{\varepsilon^2}e^{-\frac{\sigma}{\varepsilon^2}s}p(s)u_\varepsilon\left(t-s,x-\frac{sv}{\varepsilon}\right)dsdvdx.\\ &= \int_{\mathbb{S}^1} e^{-ivt.\xi/\varepsilon}p(t)e^{-\sigma t/\varepsilon^2}\hat{f}^{in}(\xi)dv\\ &+\int_0^t e^{-ivs.\xi/\varepsilon}\frac{\sigma}{\varepsilon^2}e^{-\frac{\sigma}{\varepsilon^2}s}p(s)\hat{u}_\varepsilon(t-s,\xi)dsdv\\ &= \left(\int_{\mathbb{S}^1} e^{-ivt.\xi/\varepsilon}dv\right)p(t)e^{-\sigma t/\varepsilon^2}\hat{f}^{in}(\xi)\\ &+\int_0^t p(s)\left(\int_{\mathbb{S}^1} e^{-ivs.\xi/\varepsilon}dv\right)\frac{\sigma}{\varepsilon^2}e^{-\sigma s/\varepsilon^2}\hat{u}_\varepsilon(t-s,\xi)ds.\end{aligned}\tag{2.3}$$

Therefore, the function $\hat{u}_\varepsilon$ verifies

$$
\begin{aligned}
\hat{u}_\varepsilon(t,\xi) &= J\left(\frac{t\xi}{\varepsilon}\right)p(t)e^{-\sigma t/\varepsilon^2}\hat{f}^{in}(\xi) \\
&\quad + \int_0^t p(s)J\left(\frac{s\xi}{\varepsilon}\right)\frac{\sigma}{\varepsilon^2}e^{-\sigma s/\varepsilon^2}\hat{u}_\varepsilon(t-s,\xi)ds \ (t,\xi)\in\mathbb{R}_+\times\mathbb{R}^2,
\end{aligned}
\tag{2.4}
$$

where

$$J(\omega) := \int_{\mathbb{S}^1} e^{-i\omega.v}dv.$$

2.3. **A dual formulation.** Let $\phi \in C^\infty\left(\mathbb{R}_+\times\mathbb{R}^2\right)$ be real-valued, so that

$$\hat{\phi}(t,\xi)^* := \int_{\mathbb{R}^2} e^{ix\cdot\xi}\phi(t,x)dx,\ \ (t,\xi)\in\mathbb{R}_+\times\mathbb{R}^2,$$

(denoting z^* the complex conjugate of z.) Multiplying both sides of equation (2.4) by $\hat{\phi}(t,\xi)^*$ and integrating in $(t,\xi)\in\mathbb{R}_+\times\mathbb{R}^2$ leads to

$$
\begin{aligned}
\iint_{\mathbb{R}_+\times\mathbb{R}^2}\hat{u}_\varepsilon(t,\xi)\hat{\phi}(t,\xi)^*dtd\xi &= \iint_{\mathbb{R}_+\times\mathbb{R}^2} J\left(t\frac{\xi}{\varepsilon}\right)e^{-\sigma t/\varepsilon^2}p(t)\hat{f}^{in}(\xi)\hat{\phi}(t,\xi)^*dtd\xi \\
&\quad + \iint_{\mathbb{R}_+\times\mathbb{R}^2}\int_0^t J\left(\frac{s\xi}{\varepsilon}\right)p(s)\frac{\sigma}{\varepsilon^2}e^{-\sigma s/\varepsilon^2}\hat{u}_\varepsilon(t-s,\xi)\hat{\phi}(t,\xi)^*dsdtd\xi.
\end{aligned}
$$

Rewriting this equatlity as

$$
\begin{aligned}
\iint_{\mathbb{R}_+\times\mathbb{R}^2} J\left(t\frac{\xi}{\varepsilon}\right)e^{-\sigma t/\varepsilon^2}p(t)\hat{f}^{in}(\xi)\hat{\phi}(t,\xi)^*dtd\xi &= \iint_{\mathbb{R}_+\times\mathbb{R}^2}\hat{u}_\varepsilon(t,\xi)\hat{\phi}(t,\xi)^*dtd\xi \\
&\quad - \iint_{\mathbb{R}_+\times\mathbb{R}^2}\int_0^t J\left(\frac{s\xi}{\varepsilon}\right)p(s)\frac{\sigma}{\varepsilon^2}e^{-\sigma s/\varepsilon^2}\hat{u}_\varepsilon(t-s,\xi)\hat{\phi}(t,\xi)^*dsdtd\xi,
\end{aligned}
\tag{2.5}
$$

we put it in the form

$$S_\varepsilon = K_\varepsilon,\ \forall\varepsilon>0$$

with the notation

$$S_\varepsilon := \iint_{\mathbb{R}_+\times\mathbb{R}^2} J\left(t\frac{\xi}{\varepsilon}\right)e^{-\sigma t/\varepsilon^2}p(t)\hat{f}^{in}(\xi)\hat{\phi}(t,\xi)^*dtd\xi,$$

and

$$
\begin{aligned}
K_\varepsilon := &\iint_{\mathbb{R}_+\times\mathbb{R}^2}\hat{u}_\varepsilon(t,\xi)\hat{\phi}(t,\xi)^*dtd\xi \\
&- \iint_{\mathbb{R}_+\times\mathbb{R}^2}\int_0^t J\left(\frac{s\xi}{\varepsilon}\right)p(s)\frac{\sigma}{\varepsilon^2}e^{-\sigma s/\varepsilon^2}\hat{u}_\varepsilon(t-s,\xi)\hat{\phi}(t,\xi)^*dsdtd\xi.
\end{aligned}
$$

We are going to compute the limits of both S_ε and K_ε as $\varepsilon\to 0^+$.

3. Passing to the vanishing ε limit in the integral equation

3.1. **The Source term.** Consider the left-hand side of equality (2.5):

$$S_\varepsilon := \iint_{\mathbb{R}_+\times\mathbb{R}^2} J\left(t\frac{\xi}{\varepsilon}\right)e^{-\sigma t/\varepsilon^2}p(t)\hat{f}^{in}(\xi)\hat{\phi}(t,\xi)^*dtd\xi.$$

Substituting $T=\frac{t}{\varepsilon^2}$

$$S_\varepsilon = \varepsilon^2\int_0^\infty\int_{\mathbb{R}^2} e^{-\sigma T}\hat{f}^{in}(\xi)p\left(\varepsilon^2T\right)J\left(\varepsilon T\xi\right)\hat{\phi}(\varepsilon^2T,\xi)^*d\xi dT,$$

or equivalently

$$\frac{\sigma}{\varepsilon^2}S_\varepsilon = \int_0^\infty \int_{\mathbb{R}^2} \sigma e^{-\sigma T} \hat{f}^{in}(\xi) p\left(\varepsilon^2 T\right) J\left(\varepsilon T \xi\right) \hat{\phi}(\varepsilon^2 T, \xi)^* d\xi dT.$$

Since $J(0) = p(0) = 1$,

$$\sigma e^{-\sigma T} \hat{f}^{in}(\xi) p\left(\varepsilon^2 T\right) J\left(\varepsilon T \xi\right) \hat{\phi}(\varepsilon^2 T, \xi)^* \to \sigma e^{-\sigma T} \hat{f}^{in}(\xi) \hat{\phi}(0, \xi)^* \tag{3.1}$$

as $\varepsilon \to 0$ a.e. in $\mathbb{R}_+ \times \mathbb{R}^2$. Moreover, for each $(T, \xi) \in \mathbb{R}_+ \times \mathbb{R}^2$, and each $\varepsilon \in (0, 1)$

$$\left|\sigma e^{-\sigma T} \hat{f}^{in}(\xi) J\left(\varepsilon T \xi\right) \phi(\varepsilon^2 T, \xi)\right| \le \sigma e^{-\sigma T} \left|\hat{f}^{in}(\xi)\right| \sup_{T \ge 0} \left|\hat{\phi}(T, \xi)^*\right| \tag{3.2}$$

since $|J(t)| \le 1$ and $|p(t)| \le 1$ for each $t \ge 0$ and $\phi \in C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^2)$. Besides, since $\phi \in \mathcal{S}\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$, we have

$$\left|\hat{\phi}(t, \xi)^*\right| \le \frac{C}{1 + |\xi|^2}, \ (t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^2,$$

and therefore

$$\xi \mapsto \sup_{t \ge 0} \left|\hat{\phi}(t, \xi)^*\right| \in L^2\left(\mathbb{R}^2\right).$$

Since $f^{in} \in L^2\left(\mathbb{R}^2\right)$, we have $\hat{f}^{in} \in L^2\left(\mathbb{R}^2\right)$ so that

$$\xi \mapsto \left|\hat{f}^{in}(\xi)\right| \sup_{t \ge 0} \left|\hat{\phi}(t, \xi)^*\right| \in L^1\left(\mathbb{R}^2\right);$$

Together with

$$t \mapsto \sigma e^{-\sigma t} \in L^1\left(\mathbb{R}_+\right),$$

it implies that

$$(t, \xi) \mapsto \sigma e^{-\sigma t} \left|\hat{f}^{in}(\xi)\right| \sup_{t \ge 0} \left|\hat{\phi}(t, \xi)^*\right| \in L^1\left(\mathbb{R}_+ \times \mathbb{R}^2\right).$$

We conclude from (3.1) and (3.2) that

$$\frac{\sigma}{\varepsilon^2}S_\varepsilon \to \int_0^\infty \sigma e^{-\sigma t} dt \int_{\mathbb{R}^2} \hat{f}^{in}(\xi) \hat{\phi}(0, \xi)^* d\xi$$

by dominated convergence, or equivalently that

$$\frac{\sigma}{\varepsilon^2}S_\varepsilon \to \int_{\mathbb{R}^2} \hat{f}^{in}(\xi) \hat{\phi}(0, \xi)^* d\xi$$

as $\varepsilon \to 0^+$.

3.2. **The Diffusion term.** We now consider the right-hand side of equality (2.5)

$$\begin{aligned} K_\varepsilon :=& \iint_{\mathbb{R}_+ \times \mathbb{R}^2} \hat{u}_\varepsilon(t, \xi) \hat{\phi}(t, \xi)^* dt d\xi \\ &- \iint_{\mathbb{R}_+ \times \mathbb{R}^2} \int_0^t J\left(\frac{s\xi}{\varepsilon}\right) p(s) \frac{\sigma}{\varepsilon^2} e^{-\sigma s/\varepsilon^2} \hat{u}_\varepsilon(t - s, \xi) \hat{\phi}(t, \xi)^* ds dt d\xi \\ &= K_\varepsilon^{(1)} - K_\varepsilon^{(2)}, \end{aligned} \tag{3.3}$$

with

$$K_\varepsilon^{(1)} := \iint_{\mathbb{R}_+ \times \mathbb{R}^2} \hat{u}_\varepsilon(t, \xi) \hat{\phi}(t, \xi)^* dt d\xi$$

and

$$K_\varepsilon^{(2)} := \iint_{\mathbb{R}_+ \times \mathbb{R}^2} \int_0^t J\left(\frac{s\xi}{\varepsilon}\right) p(s) \frac{\sigma}{\varepsilon^2} e^{-\sigma s/\varepsilon^2} \hat{u}_\varepsilon(t - s, \xi) \hat{\phi}(t, \xi)^* ds dt d\xi.$$

3.2.1. *A reformulation of the Diffusion Term.* Let

$$K_\varepsilon^{(2)} := \iint_{\mathbb{R}_+\times\mathbb{R}^2} \int_0^t J\left(\frac{s\xi}{\varepsilon}\right) p(s)\frac{\sigma}{\varepsilon^2}e^{-\sigma s/\varepsilon^2}\hat{u}_\varepsilon(t-s,\xi)\hat{\phi}(t,\xi)^* dsdtd\xi.$$

Substituting $T = t-s$ into the expression above,

$$\begin{aligned} K_\varepsilon^{(2)} &= \iint_{\mathbb{R}_+\times\mathbb{R}^2} \int_0^\infty \mathbb{1}_{t-s\geq 0} J\left(\frac{s\xi}{\varepsilon}\right) p(s)\frac{\sigma}{\varepsilon^2}e^{-\sigma s/\varepsilon^2}\hat{u}_\varepsilon(t-s,\xi)\hat{\phi}(t,\xi)^* dsdtd\xi \\ &= \iint_{\mathbb{R}_+\times\mathbb{R}^2} \int_0^\infty \mathbb{1}_{T\geq 0} J\left(\frac{s\xi}{\varepsilon}\right) p(s)\frac{\sigma}{\varepsilon^2}e^{-\sigma s/\varepsilon^2}\hat{u}_\varepsilon(T,\xi)\hat{\phi}(T+s,\xi)^* dsdTd\xi. \end{aligned}$$

Substituting $S = \frac{s}{\varepsilon^2}$ in the expression above, we find

$$K_\varepsilon^{(2)} = \iint_{\mathbb{R}_+\times\mathbb{R}^2} \int_0^\infty J\left(\varepsilon S\xi\right) p(\varepsilon^2 S)\sigma e^{-\sigma S}\hat{u}_\varepsilon(T,\xi)\hat{\phi}(T+\varepsilon^2 S,\xi)^* dSdTd\xi. \tag{3.4}$$

Besides, for each $\varepsilon > 0$

$$\begin{aligned} K_\varepsilon^{(1)} &= \iint_{\mathbb{R}_+\times\mathbb{R}^2} \hat{u}_\varepsilon(t,\xi)\hat{\phi}(t,\xi)^* dtd\xi \\ &= \int_0^\infty \sigma e^{-\sigma S}\left(\iint_{\mathbb{R}_+\times\mathbb{R}^2} \hat{u}_\varepsilon(T,\xi)\hat{\phi}(T,\xi)^* dTd\xi\right) dS \end{aligned} \tag{3.5}$$

Consequently, from (3.3) and (3.4) and (3.5), we conclude that
(3.6)

$$\begin{aligned} K_\varepsilon &= \int_0^\infty \sigma e^{-\sigma s}\left(\iint_{\mathbb{R}_+\times\mathbb{R}^2} \hat{u}_\varepsilon(t,\xi)\left(\hat{\phi}(t,\xi)^* - J\left(\varepsilon s\xi\right) p(\varepsilon^2 s)\hat{\phi}(t+\varepsilon^2 s,\xi)^*\right) dtd\xi\right) ds \\ &= \int_0^\infty \iint_{\mathbb{R}_+\times\mathbb{R}^2} \hat{u}_\varepsilon(t,\xi)\left(\Psi(t,s,0,\xi) - \Psi(t,s,\varepsilon^2,\xi)\right) dtd\xi ds, \end{aligned}$$

where

$$\Psi : (t,s,h,\xi) \in \mathbb{R}_+^3 \times \mathbb{R}^2 \mapsto \sigma e^{-\sigma s} J\left(\sqrt{h}s\xi\right) p\left(hs\right)\hat{\phi}(t+hs,\xi)^*.$$

3.2.2. *The Function* Ψ. Before going further, we consider first the function $J : \mathbb{R}^2 \mapsto \mathbb{R}^+$ whose definition is recalled below :

$$J(\omega) := \frac{1}{2\pi}\int_{\mathbb{S}^1} e^{-iv.\omega}dv, \ \omega \in \mathbb{R}^2.$$

Obviously, by the symmetry $v \mapsto -v$,

$$J(\omega) = \frac{1}{2\pi}\int_{\mathbb{S}^1} \cos(v.\omega)dv.$$

Since

$$\cos(x) = \sum_{n=0}^\infty \frac{(-1)^n}{(2n)!}x^{2n}, \ x \in \mathbb{R}.$$

one has

$$J(\omega) = 1 - \frac{1}{4}|\omega|^2 + o(|\omega|^2) \text{ as } |\omega| \to 0^+.$$

and let the function $C : \mathbb{R}_+ \mapsto \mathbb{R}_+$ be the analytic function defined by

$$J(x) = C(|x|^2), \ x \in \mathbb{R}^2,$$

where

$$C(r) = \sum_{n\geq 0} \frac{(-1)^n}{2n!}r^n.$$

Obviously $C(0) = 1$ and $\dot{C}(0) = -\frac{1}{4}$. Thus

$\Psi(t,s,h,\xi) = \sigma e^{-\sigma s} C\left(hs^2|\xi|^2\right) p(hs)\hat{\phi}(t+hs,\xi)^*,\ (t,s,h,\xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2,$

Since $C, p, \hat{\phi}^* \in C^1(\mathbb{R}_+)$, for each $(t,s,\xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2$

$$h \in \mathbb{R}_+ \mapsto \Psi(t,s,h,\xi) \in C^1(\mathbb{R}_+),$$

with

$$\begin{aligned}
\partial_h \Psi(t,s,\xi,h) &= \sigma e^{-\sigma s} \partial_h \left(C\left(hs^2|\xi|^2\right)\right) p(hs)\hat{\phi}(t+hs,\xi)^* \\
&\quad + \sigma e^{-\sigma s} C\left(hs^2|\xi|^2\right) p(hs) \partial_h \left(\hat{\phi}(t+hs,\xi)^*\right) \\
&\quad + \sigma e^{-\sigma s} C\left(hs^2|\xi|^2\right) \partial_h \left(p(hs)\right) \hat{\phi}(t+hs,\xi)^* \\
&= \sigma e^{-\sigma s} s^2 |\xi|^2 \dot{C}\left(hs^2|\xi|^2\right) p(hs)\hat{\phi}(t+hs,\xi)^* \\
&\quad + \sigma e^{-\sigma s} C\left(hs^2|\xi|^2\right) p(hs) s \partial_t \hat{\phi}(t+hs,\xi)^* \\
&\quad + \sigma e^{-\sigma s} s \dot{p}(hs) C\left(hs^2|\xi|^2\right) \hat{\phi}(t+hs,\xi)^*.
\end{aligned} \tag{3.7}$$

In particular,

$$\begin{aligned}
\partial_h \Psi(t,s,0,\xi) &= \sigma e^{-\sigma s} s^2 |\xi|^2 \dot{C}(0)\, \hat{\phi}(t,\xi)^* \\
&\quad + \sigma e^{-\sigma s} s \partial_t \hat{\phi}(t,\xi)^* + \sigma e^{-\sigma s} s \dot{p}(0) \hat{\phi}(t,\xi)^*,
\end{aligned} \tag{3.8}$$

for each $(t,s,\xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2$.

Lemma 3.1. *Under the notations and assumptions above,*

$$\frac{\Psi(t,s,0,\xi) - \Psi(t,s,h,\xi)}{h} \to -\partial_h \Psi(t,s,0,\xi) \text{ as } h \to 0^+$$

in $L^1_t\left(\mathbb{R}_+; L^2\left(\mathbb{R}_+ \times \mathbb{R}^2\right)\right)$ *−strong.*

Proof. We easily check that

$$\frac{\Psi(t,s,0,\xi) - \Psi(t,s,h,\xi)}{h} \to -\partial_h \Psi(t,s,0,\xi) \text{ as } h \to 0^+$$

a.e. in $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2$. Since $\phi \in \mathcal{S}\left(\mathbb{R}_+ \times \mathbb{R}^2\right)$ there exists C, C' such that

$$\begin{aligned}
\left|\hat{\phi}(t+hs,\xi)^*\right| &\le C \frac{1}{1+(t+hs)^2} \frac{1}{1+|\xi|^4} \\
&\le C \frac{1}{1+t^2} \frac{1}{1+|\xi|^4},\ \forall (t,s,h,\xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2,
\end{aligned}$$

and

$$\begin{aligned}
\left|\partial_h \hat{\phi}(t+hs,\xi)^*\right| &\le C' \frac{1}{1+(t+hs)^2} \frac{1}{1+|\xi|^2} \\
&\le C' \frac{1}{1+t^2} \frac{1}{1+|\xi|^4},\ \forall (t,s,h,\xi) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{R}^2.
\end{aligned}$$

Consequently by 3.7

$$\begin{aligned}
\sup_{h>0} |\partial_h \Psi(t,s,\xi,h)| &\le C\sigma e^{-\sigma s} s^2 |\xi|^2 \frac{1}{1+t^2} \frac{1}{1+|\xi|^4} \\
&\quad + C'\sigma e^{-\sigma s} s \frac{1}{1+t^2} \frac{1}{1+|\xi|^4} \\
&\quad + C\sigma e^{-\sigma s} s \frac{1}{1+t^2} \frac{1}{1+|\xi|^4}
\end{aligned}$$

so that

$$(t,s,\xi)\mapsto \sup_{h>0}|\partial_h\Psi(t,s,\xi,h)|\in L^1_t\left(\mathbb{R}_+;L^2\left(\mathbb{R}_+\times\mathbb{R}^2\right)\right).$$

By the Mean Value Theorem, for each $h\in(0,1)$

$$\left|\frac{\Psi(t,s,0,\xi)-\Psi(t,s,h,\xi)}{h}\right|\le\sup_{h>0}|\partial_h\Psi(t,s,\xi,h)|$$

and we conclude by Dominated Convergence. $\square$

3.2.3. *Passing to the limit in the Diffusion Term.* We start from equality (3.6) in the form

$$K_\varepsilon=\int_0^\infty\iint_{\mathbb{R}_+\times\mathbb{R}^2}\hat{u}_\varepsilon(t,\xi)\left(\Psi(t,s,0,\xi)-\Psi(t,s,\varepsilon^2,\xi)\right)dtd\xi ds$$

or equivalently

$$\frac{\sigma}{\varepsilon^2}K_\varepsilon=\sigma\int_0^\infty\iint_{\mathbb{R}_+\times\mathbb{R}^2}\hat{u}_\varepsilon(t,\xi)\frac{\Psi(t,s,0,\xi)-\Psi(t,s,\varepsilon^2,\xi)}{\varepsilon^2}dtd\xi ds.$$

By Lemma (3.1), for each $T>0$

$$\frac{\Psi(t,s,0,\xi)-\Psi(t,s,\varepsilon^2,\xi)}{\varepsilon^2}\to-\partial_h\Psi(t,s,0,\xi)\text{ as }\varepsilon\to0^+$$

in $L^1\left(\mathbb{R}_+;L^2\left(\mathbb{R}_+\times\mathbb{R}^2\right)\right)$ −strong while

$$\hat{u}_\varepsilon\to\hat{u}\text{ as }\varepsilon\to0^+$$

in $L^\infty\left(\mathbb{R}_+;L^2\left(\mathbb{R}^2\right)\right)$-weak-*, so that by weak-strong convergence,

$$\frac{\sigma}{\varepsilon^2}K_\varepsilon\to\sigma\iint_{\mathbb{R}_+\times\mathbb{R}^2}\hat{u}(t,\xi)\left(\int_0^\infty-\partial_h\Psi(t,s,0,\xi)ds\right)dtd\xi$$

as $\varepsilon\to0^+$. As for each $(t,\xi)\in\mathbb{R}_+\times\mathbb{R}^2$, by (3.8)

$$\begin{aligned}
&\int_0^\infty\partial_h\Psi(t,s,0,\xi)ds\\
=&\int_0^\infty\left(\sigma e^{-\sigma s}s^2|\xi|^2\dot{C}\left(0\right)\hat{\phi}(t,\xi)^*+\sigma e^{-\sigma s}s\partial_t\hat{\phi}(t,\xi)^*+\sigma e^{-\sigma s}s\dot{p}(0)\hat{\phi}(t,\xi)^*\right)ds.\\
=&-\frac{|\xi|^2}{4}\hat{\phi}(t,\xi)^*\int_0^\infty\sigma e^{-\sigma s}s^2ds+\partial_t\hat{\phi}(t,\xi)^*\int_0^\infty\sigma e^{-\sigma s}sds\\
&+\dot{p}(0)\hat{\phi}(t,\xi)^*\int_0^\infty\sigma e^{-\sigma s}sds\\
=&\frac{1}{\sigma}\partial_t\hat{\phi}(t,\xi)^*+\frac{1}{\sigma}\dot{p}(0)\hat{\phi}(t,\xi)^*-\frac{1}{\sigma^2}\frac{|\xi|^2}{4}\hat{\phi}(t,\xi)^*2.
\end{aligned}$$

Thus

$$\frac{\sigma}{\varepsilon^2}K_\varepsilon\to\iint_{\mathbb{R}_+\times\mathbb{R}^2}\left(-\partial_t\hat{\phi}(t,\xi)^*+\frac{|\xi|^2}{2\sigma}\hat{\phi}(t,\xi)^*-\dot{p}(0)\hat{\phi}(t,\xi)^*\right)\hat{u}(t,\xi)dtd\xi$$

as $\varepsilon\to0^+$.

3.3. **Summary.** Starting from equation (2.5) in the form

$$K_\varepsilon = S_\varepsilon$$

where K_ε is the Diffusion term and S_ε the Source term, we have

$$\frac{\sigma}{\varepsilon^2} K_\varepsilon = \frac{\sigma}{\varepsilon^2} S_\varepsilon.$$

According to the result obtained in section 4.1 and 4.2, we pass to the limit as $\varepsilon \to 0^+$ in both sides of the equality

$$\iint_{\mathbb{R}_+\times\mathbb{R}^2} \left(-\partial_t \hat{\phi}(t,\xi)^* + \frac{|\xi|^2}{2\sigma}\hat{\phi}(t,\xi)^* - \dot{p}(0)\hat{\phi}(t,\xi)^*\right)\hat{u}(t,\xi)dtd\xi$$
$$= \int_{\mathbb{R}^2} \hat{f}^{in}(\xi)\hat{\phi}(0,\xi)^* d\xi$$

which means that u is the solution in the sense of distributions of the Cauchy problem for the damped diffusion equation

$$\begin{cases} \partial_t u - \frac{1}{2\sigma}\Delta u = \dot{p}(0)u, \\ u(0,x) = f^{in}(x). \end{cases}$$

Notice that $\dot{p}(0) < 0$ so that the right-hand side is a damping term indeed.

4. Strong convergence in $L^1_{loc}\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2\right)\right)$

In the present subsection, we show that the convergence $u_\varepsilon \to u$ holds in $L^1_{loc}\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2\right)\right)$ −strong, and not only in $L^\infty\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2\right)\right)$ weak-*. The two key arguments are

(1) a velocity averaging result (to gain regularity in x),
(2) and a Aubin type lemma, to handle the time dependence.

4.1. **Gaining regularity in x.** The first key point of the argument is a result in Velocity Averaging that is a special case of Averaging Lemma 2.1 in [9]

Proposition 4.1. *Let $f \equiv f(t,x,v) \in L^2_{loc}(\mathbb{R}_t \times \mathbb{R}^2_x \times \mathbb{S}^1)$ and let g be a locally bounded measure, $g \in \mathcal{M}_{loc}(\mathbb{R}_t \times \mathbb{R}^2_x \times \mathbb{S}^1)$. Assume that*

$$\partial_t f + v.\nabla_x f = g;$$

then for each $s \in (0, \frac{1}{3})$ $\overline{f} := \int_{\mathbb{S}^1} f dv$ belongs to $W^{s,\frac{3}{2}}_{loc}(\mathbb{R}_t \times \mathbb{R}^2_x)$ and

$$\|\overline{f}\|_{W^{s,\frac{3}{2}}_{loc}(\mathbb{R}_t\times\mathbb{R}^2_x)} \lesssim \|g\|^{\frac{1}{3}}_{\mathcal{M}_{loc}(\mathbb{R}_t\times\mathbb{R}^2_x\times\mathbb{S}^1)} \|f\|^{\frac{2}{3}}_{L^2_{loc}(\mathbb{R}_t\times\mathbb{R}^2_x\times\mathbb{S}^1)}.$$

With the help of Proposition 4.1, we obtain the following control on u_ε.

Lemma 4.2. *Under the definitions and with the assumptions above, then for each $T > 0$ and $s \in (0, \frac{1}{3})$,*

$$(u_\varepsilon)_{\varepsilon>0} \text{ is unformly bounded in } L^{\frac{3}{2}}([0,T]; W^{s,\frac{3}{2}}_{loc}(\mathbb{R}^2))$$

Proof. We split the proof into several steps.
We recall that

$$u_\varepsilon(t,x) = \int_0^\infty \int_{\mathbb{S}^1} F_\varepsilon(t,s,x,v)dvds, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2$$

where F_ε is the solution of the Cauchy problem

$$\begin{cases} \partial_t F_\varepsilon + \frac{v}{\varepsilon}.\nabla_x F_\varepsilon + \partial_s F_\varepsilon = -\frac{\sigma}{\varepsilon^2} F_\varepsilon + \frac{\dot{p}}{p}(t \wedge s) F_\varepsilon, & t, s > 0, (x,v) \in \mathbb{R}^2 \times \mathbb{S}^1 \\ F_\varepsilon(t,0,x,v) = \frac{\sigma}{\varepsilon^2} \int_0^{+\infty} \bar{F}_\varepsilon(t,s,x) ds, & t > 0, (x,v) \in \mathbb{R}^2 \times \mathbb{S}^1 \\ F_\varepsilon(0,s,x,v) = \frac{\sigma}{\varepsilon^2} e^{-\sigma s/\varepsilon^2} f^{in}(x), & s > 0, (x,v) \in \mathbb{R}^2 \times \mathbb{S}^1 \end{cases}$$

Therefore $f_\varepsilon := \int_0^\infty F_\varepsilon ds$ verifies

$$\begin{cases} \varepsilon \partial_t f_\varepsilon + v.\nabla_x f_\varepsilon = \frac{\sigma}{\varepsilon}\left(\overline{f}_\varepsilon - f_\varepsilon\right) + \varepsilon \int_0^\infty \frac{\dot{p}}{p}(t \wedge s) F_\varepsilon(t,s,x,v) ds, & t > 0, (x,v) \in \mathbb{R}^2 \times \mathbb{S}^1 \\ f_\varepsilon(0,x,v) = f^{in}(x), & (x,v) \in \mathbb{R}^2 \times \mathbb{S}^1. \end{cases}$$

For each $\varepsilon \in (0,1)$, we define $g_\varepsilon \equiv g_\varepsilon(t,x,v)$ and $G_\varepsilon \equiv G_\varepsilon(t,s,x,v)$ by

$$g_\varepsilon\left(\frac{t}{\varepsilon}, x, v\right) = f_\varepsilon(t,x,v), \quad (t,x,v) \in \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1,$$

and

$$G_\varepsilon\left(\frac{t}{\varepsilon}, s, x, v\right) = F_\varepsilon(t,s,x,v).$$

Denoting $\tau := \frac{t}{\varepsilon}$, we easily check that the function g_ε is the solution of the Cauchy problem

$$\begin{cases} \partial_\tau g_\varepsilon + v.\nabla_x g_\varepsilon = h_\varepsilon, & (x,v) \in \mathbb{R}^2 \times \mathbb{S}^1,\ \tau > 0, \\ g_\varepsilon(0,x,v) = f^{in}(x), & (x,v) \in \mathbb{R}^2 \times \mathbb{S}^1, \end{cases} \tag{4.1}$$

where

$$h_\varepsilon(\varepsilon\tau, x, v) := \frac{\sigma}{\varepsilon}\left(g_\varepsilon - \overline{g}_\varepsilon\right) + \varepsilon \int_0^\infty \frac{\dot{p}}{p}(\varepsilon\tau \wedge s) G_\varepsilon(\tau,s,x,v) ds. \tag{4.2}$$

Applying Proposition 4.1 to equation (4.1) gives the estimate

$$\|\overline{g}_\varepsilon\|_{W_{loc}^{s,\frac{3}{2}}(\mathbb{R}_t \times \mathbb{R}^2_x)} \lesssim \|h_\varepsilon\|^{\frac{1}{3}}_{\mathcal{M}_{loc}(\mathbb{R}_t \times \mathbb{R}^2_x \times \mathbb{S}^1)} \|g_\varepsilon\|^{\frac{2}{3}}_{L^2_{loc}(\mathbb{R}_t \times \mathbb{R}^2_x \times \mathbb{S}^1)}.$$

As

$$W_{loc}^{s,\frac{3}{2}}(\mathbb{R}_+ \times \mathbb{R}^2) \hookrightarrow L^{\frac{3}{2}}_{loc}(\mathbb{R}_+; W_{loc}^{s,\frac{3}{2}}(\mathbb{R}^2)),$$

the inequality above leads to

$$\|\overline{g}_\varepsilon\|_{L^{\frac{3}{2}}_{loc}(\mathbb{R}_+; W_{loc}^{s,\frac{3}{2}}(\mathbb{R}^2))} \lesssim \|h_\varepsilon\|^{\frac{1}{3}}_{\mathcal{M}_{loc}(\mathbb{R}_t \times \mathbb{R}^2_x \times \mathbb{S}^1)} \|g_\varepsilon\|^{\frac{2}{3}}_{L^2_{loc}(\mathbb{R}_t \times \mathbb{R}^2_x \times \mathbb{S}^1)}. \tag{4.3}$$

Next, we easily verify that for each $\varepsilon \in (0,1)$

$$\begin{aligned} \|h_\varepsilon\|_{\mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)} &= \frac{1}{\varepsilon}\left\| \frac{\sigma}{\varepsilon}\left(f_\varepsilon - \overline{f}_\varepsilon\right) + \varepsilon \int_0^\infty \frac{\dot{p}}{p}(t \wedge s) F_\varepsilon ds \right\|_{\mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)}, \\ &\leq \frac{1}{\varepsilon}\left\| \frac{\sigma}{\varepsilon}\left(f_\varepsilon - \overline{f}_\varepsilon\right)\right\|_{\mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)} \\ &\quad + \frac{1}{\varepsilon}\left\| \varepsilon \int_0^\infty \frac{\dot{p}}{p}(t \wedge s) F_\varepsilon ds \right\|_{\mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)} \\ &\leq \frac{1}{\varepsilon}\left\| \frac{\sigma}{\varepsilon}\left(f_\varepsilon - \overline{f}_\varepsilon\right)\right\|_{\mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)} \\ &\quad + \left\| \int_0^\infty \frac{\dot{p}}{p}(t \wedge s) F_\varepsilon ds \right\|_{\mathcal{M}_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1)}, \end{aligned}$$

and thus

$$\|h_\varepsilon)\|^{1/3}_{\mathcal{M}_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} \leq \frac{1}{\varepsilon^{1/3}} \left\| \frac{\sigma}{\varepsilon}\left(f_\varepsilon - \overline{f}_\varepsilon\right)\right\|^{1/3}_{\mathcal{M}_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} + \left\| \int_0^\infty \frac{\dot p}{p}(t\wedge s)F_\varepsilon ds\right\|^{1/3}_{\mathcal{M}_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)}.$$

and for each $q\geq 1$

$$\|g_\varepsilon\|_{L^q(\mathbb{R}_+;B)} = \frac{1}{\varepsilon^{1/q}}\|f_\varepsilon\|_{L^q(\mathbb{R}_t^+;B)},$$

and

$$\|\overline{g}_\varepsilon\|_{L^q(\mathbb{R}_+;B)} = \frac{1}{\varepsilon^{1/q}}\|\overline{f}_\varepsilon\|_{L^q(\mathbb{R}_t^+;B)}.$$

Therefore inequality (4.3) implies that

$$\begin{aligned}
&\left(\frac{1}{\varepsilon}\right)^{2/3}\|\overline{f}_\varepsilon\|_{L^{\frac32}_{loc}(\mathbb{R}_+;W^{s,\frac32}_{loc}(\mathbb{R}^2))} \\
&\quad\lesssim \frac{1}{\varepsilon^{1/3}}\left\|\frac{\sigma}{\varepsilon}\left(f_\varepsilon-\overline{f}_\varepsilon\right)\right\|^{1/3}_{\mathcal{M}_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} \frac{1}{\varepsilon^{1/3}}\|f_\varepsilon\|^{\frac23}_{L^2_{loc}(\mathbb{R}_t\times\mathbb{R}^2_x\times\mathbb{S}^1)} \\
&\quad\quad+\left\|\int_0^\infty \frac{\dot p}{p}(t\wedge s)F_\varepsilon ds\right\|^{1/3}_{\mathcal{M}_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)} \frac{1}{\varepsilon^{1/3}}\|f_\varepsilon\|^{\frac23}_{L^2_{loc}(\mathbb{R}_t\times\mathbb{R}^2_x\times\mathbb{S}^1)}.
\end{aligned}$$

Consequently, the family $(f_\varepsilon)_{\varepsilon>0}$ verifies

(4.4)

$$\begin{aligned}
\|\overline{f}_\varepsilon\|_{L^{\frac32}_{loc}(\mathbb{R}_+;W^{s,\frac32}_{loc}(\mathbb{R}^2))} &\lesssim \left\|\frac{\sigma}{\varepsilon}\left(f_\varepsilon-\overline{f}_\varepsilon\right)\right\|^{1/3}_{\mathcal{M}_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)}\|f_\varepsilon\|^{\frac23}_{L^2_{loc}(\mathbb{R}_t\times\mathbb{R}^2_x\times\mathbb{S}^1)} \\
&\quad+\varepsilon^{\frac13}\left\|\int_0^\infty \frac{\dot p}{p}(t\wedge s)F_\varepsilon ds\right\|^{1/3}_{\mathcal{M}_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1)}\|f_\varepsilon\|^{\frac23}_{L^2_{loc}(\mathbb{R}_t\times\mathbb{R}^2_x\times\mathbb{S}^1)}.
\end{aligned}$$

Next we show that $\left(\overline{f}_\varepsilon\right)_{\varepsilon>0}$ is uniformly bounded in $L^{\frac32}_{loc}(\mathbb{R}_+;W^{s,\frac32}_{loc}(\mathbb{R}^2))$. Since there exists $C>0$ such that

$$\left|\frac{\dot p(t)}{p(t)}\right| \leq C,\ \text{for each } t\geq 0,$$

indeed, one has

$$\left|\int_0^\infty \frac{\dot p}{p}(t\wedge s)F_\varepsilon(t,s,x,v)ds\right| \leq Cf_\varepsilon(t,x,v) \text{ for each } t,s\geq 0.$$

Therefore by Proposition 1.1

$$\left(\int_0^\infty \frac{\dot p}{p}(t\wedge s)F_\varepsilon ds\right)_{\varepsilon>0} = O(1) \text{ in } L^\infty\left(\mathbb{R}_+;L^2\left(\mathbb{R}^2\times\mathbb{S}^1\right)\right),$$

and in particular

$$\left(\int_0^\infty \frac{\dot p}{p}(t\wedge s)F_\varepsilon ds\right)_{\varepsilon>0} = O(1) \text{ in } \mathcal{M}_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1). \tag{4.5}$$

By Proposition 1.1

$$\left(\frac{\sigma}{\varepsilon}\left(f_\varepsilon-\overline{f}_\varepsilon\right)\right)_{\varepsilon>0} = O(1) \text{ in } L^2_{loc}\left(\mathbb{R}_+;L^2\left(\mathbb{R}^2\times\mathbb{S}^1\right)\right)$$

so that

$$\left(\frac{\sigma}{\varepsilon}\left(f_\varepsilon-\overline{f}_\varepsilon\right)\right)_{\varepsilon>0} = O(1) \text{ in } \mathcal{M}_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{S}^1). \tag{4.6}$$

Besides Proposition 1.1 also implies that

$$(f_\varepsilon)_{\varepsilon>0} = O(1) \text{ in } L^\infty\left(\mathbb{R}_+;L^2\left(\mathbb{R}^2\times\mathbb{S}^1\right)\right). \tag{4.7}$$

By (4.5) and (4.6) and (4.7), inequality (4.4) implies that $\left(\overline{f}_\varepsilon\right)_{\varepsilon>0}$ is uniformly bounded in $L^{\frac{3}{2}}_{loc}(\mathbb{R}_+; W^{s,\frac{3}{2}}_{loc}(\mathbb{R}^2))$. □

4.2. **Controlling time derivatives.**

Lemma 4.3. *Under the notations and with the definitions above, for each $T > 0$*

the family $(\partial_t u_\varepsilon)_{\varepsilon<1}$ is uniformly bounded in $L^2([0,T]; H^{-2}(\mathbb{R}^2))$.

Proof. We recall that

$$\partial_t f_\varepsilon + \frac{v}{\varepsilon}.\nabla_x f_\varepsilon + \frac{\sigma}{\varepsilon^2}\left(f_\varepsilon - \overline{f}_\varepsilon\right) = \int_0^\infty \frac{\dot{p}}{p}(t \wedge s) F_\varepsilon ds.$$

Integrating the equation above in $v \in \mathbb{S}^1$ gives

$$\partial_t u_\varepsilon = -\frac{1}{\varepsilon}\nabla_x . \int_{\mathbb{S}^1} v f_\varepsilon dv + \int_{\mathbb{S}^1}\int_0^\infty \frac{\dot{p}}{p}(t \wedge s) F_\varepsilon ds dv. \tag{4.8}$$

Notice that for each $\varepsilon > 0$

$$\frac{1}{\varepsilon}\nabla_x . \int_{\mathbb{S}^1} v f_\varepsilon dv = \nabla_x . \int_{\mathbb{S}^1} v \frac{1}{\varepsilon}\left(f_\varepsilon - \overline{f}_\varepsilon\right) dv.$$

By Proposition 1.1

$$\left(\frac{1}{\varepsilon}\left(f_\varepsilon - \overline{f}_\varepsilon\right)\right)_{\varepsilon>0} = O(1) \text{ in } L^2\left([0,T] \times \mathbb{R}^2 \times \mathbb{S}^1\right)$$

so that the family $\left(\frac{1}{\varepsilon}\nabla_x . \int_{\mathbb{S}^1} v f_\varepsilon dv\right)_{\varepsilon>0}$ is uniformly bounded in $L^2([0,T]; H^{-2}(\mathbb{R}^2))$. Besides, since for each $t \geq 0$

$$\left|\frac{\dot{p}}{p}(t)\right| \leq C < +\infty,$$

one has

$$\left|\int_{\mathbb{S}^1}\int_0^\infty \frac{\dot{p}}{p}(t \wedge s) F_\varepsilon ds dv\right| \leq C \int_{\mathbb{S}^1}\int_0^\infty F_\varepsilon ds dv \leq C \int_{\mathbb{S}^1} f_\varepsilon dv \lesssim u_\varepsilon. \tag{4.9}$$

By Proposition 1.1

$$\left(u_\varepsilon\right)_{\varepsilon>0} = O(1) \text{ in } L^\infty\left(\mathbb{R}_+; L^2\left(\mathbb{R}^2 \times \mathbb{S}^1\right)\right)$$

so that by (4.9), for each $T > 0$:

$$\left(\int_{\mathbb{S}^1}\int_0^\infty \frac{\dot{p}}{p}(t \wedge s) F_\varepsilon ds dv\right)_{\varepsilon>0} = O(1) \text{ in } L^2\left([0,T]; H^{-2}\left(\mathbb{R}^2\right)\right).$$

Therefore, by (4.8), the family $(\partial_t u_\varepsilon)_{\varepsilon<1}$ is uniformly bounded in $L^2\left([0,T]; H^{-2}\left(\mathbb{R}^2\right)\right)$ □

4.3. **Compactness in $L^{\frac{3}{2}}\left([0,T]; L^{\frac{2}{3}}_{loc}\left(\mathbb{R}^2\right)\right)$ −strong.** By Aubin's lemma [1], we conclude that $(u_\varepsilon)_{\varepsilon>0}$ is relatively compact in $L^{\frac{3}{2}}\left([0,T]; L^{\frac{2}{3}}_{loc}\left(\mathbb{R}^2\right)\right)$ strong. So that we obtain the following lemma

Lemma 4.4. *Under the assumptions and with the notations above, up to an extraction*

$$u_\varepsilon \to u \textit{ in } L^{\frac{3}{2}}\left([0,T]; L^{\frac{2}{3}}_{loc}\left(\mathbb{R}^2\right)\right) \textit{-strong as } \varepsilon \to 0^+.$$

Hence Theorem 0.2 is established.

5. Conclusion

We have established a diffusion approximation for a linear Boltzmann equation in extended phase space of the type (0.1). Our method applies to the case of isotropic scattering, since it is based on studying the integral equation satisfied by the macroscopic density (2.1). If we consider the same problem for scattering operators as

$$Kf := \int_{\mathbb{S}^1} k(v,w)f(w)dw, \quad \text{for each } f \in L^1\left(\mathbb{S}^1\right),$$

we would obtain, by a similar argument as in section 2,

$$\begin{aligned} F_\varepsilon &= \mathbb{1}_{t<s} p(t) \frac{\sigma}{\varepsilon^2} e^{-\frac{\sigma}{\varepsilon^2}s} f^{in}\left(x - \frac{tv}{\varepsilon}\right) \\ &\quad + \mathbb{1}_{s<t} \frac{\sigma}{\varepsilon^2} e^{-\frac{\sigma}{\varepsilon^2}s} p(s) \int_0^\infty (KF_\varepsilon)\left(t-s, \tau, x - \frac{sv}{\varepsilon}, v\right) d\tau. \end{aligned}$$

Applying K to both sides of the equality above leads to

$$\begin{aligned} KF_\varepsilon &= \mathbb{1}_{t<s} \frac{\sigma}{\varepsilon^2} e^{-\frac{\sigma}{\varepsilon^2}s} p(t) \int_{\mathbb{S}^1} k(v,w) f^{in}\left(x - \frac{tw}{\varepsilon}\right) dw \\ &\quad + \mathbb{1}_{s<t} \frac{\sigma}{\varepsilon^2} e^{-\frac{\sigma}{\varepsilon^2}s} p(s) \int_{\mathbb{S}^1} \int_0^\infty k(v,w) (KF_\varepsilon)\left(t-s, \tau, x - \frac{sw}{\varepsilon}, w\right) d\tau dw. \end{aligned}$$

If we apply the Fourier transform to both sides of the equation above, the second term on the right-hand side becomes, after substituting $y = x - \frac{sw}{\varepsilon}$

$$\mathbb{1}_{s<t} \frac{\sigma}{\varepsilon^2} p(s) \int_{\mathbb{S}^1 \times \mathbb{R}^2} \int_0^\infty k(v,w) e^{-i\xi \cdot \left(y + \frac{sw}{\varepsilon}\right)} KF_\varepsilon (t-s, \tau, y, w)\, d\tau dw dy.$$

Notice that the velocity variable cannot be integrated out, as in the case studied in the present paper, so that our method does not apply to general scattering kernels. We hope to return to this question in a forthcoming publication.

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Chapitre 4

Homogenization of the nonmonokinetic linear Boltzmann equation in a perforated domain

CHAPTER IV
HOMOGENIZATION OF
THE NONMONOKINETIC LINEAR BOLTZMANN EQUATION
IN A DOMAIN WITH A PERIODIC DISTRIBUTION OF HOLES

Introduction

We pursue here our analysis, started in the second chapter [1], of the homogenization problem for the linear Boltzmann equation in a domain with a periodic distribution of holes. Specifically, we remove here the monokinetic assumption. Before describing the specific problem analyzed here, we recall the results obtained in the monokinetic case. For each $\varepsilon \in \left(0, \frac{1}{2}\right)$ we define

$$Z_\varepsilon := \{x \in \mathbb{R}^d \,|\, \mathrm{dist}(x, \varepsilon\mathbb{Z}^2) > \varepsilon^2\} = \mathbb{R}^d \setminus \bigcup_{k\in\mathbb{Z}^2} \overline{B(\varepsilon k, \varepsilon^2)}\,.$$

Consider next the linear Boltzmann equation for a monokinetic system of particles

$$\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \sigma(f_\varepsilon - T_k\left(f_\varepsilon\right)) = 0 \tag{0.1}$$

in Z_ε. The unknown function $f \equiv f(t,x,v)$ is the density of particles located at the position $x \in \mathbb{R}^2$ with velocity $v \in \mathbb{S}^1$, which amounts to assuming that the particles moves with fixed speed, only direction can change, at time $t \geq 0$. The frequency collision is a constant $\sigma > 0$ and the operator $T_k \in \mathcal{L}(L^1(\mathbb{S}^1))$ governing the change of direction is defined by:

$$T_k\left(f\right)(t,x,v) = \int_{\mathbb{S}^1} k(v,w) f(t,x,w)dw \tag{0.2}$$

where dw is the uniform probability measure on the unit sphere $\mathbb{S}^1$, while $k \in C(\mathbb{S}^1\times\mathbb{S}^1)$ is a scattering kernel satisfying

$$k(v,w) = k(w,v) > 0\,, \quad \text{and} \int_{\mathbb{S}^1} k(v,w)dw = 1\,. \tag{0.3}$$

Henceforth we supplement the equation (0.1) with the absorption condition

$$f(t, \varepsilon k + \varepsilon^2\omega, v) = 0\,, \quad \text{for each } k \in \mathbb{Z}^2\,,\ \omega, v \in \mathbb{S}^1\,,\ \text{whenever } v\cdot\omega > 0\,. \tag{0.4}$$

Finally, as initial data, we have:

$$f_\varepsilon(0,x,v) = f^{in}(x,v)\,, \quad (x,v) \in Z_\varepsilon \times \mathbb{S}^1 \tag{0.5}$$

where $0 \leq f^{in} \in C_c(\mathbb{R}^2\times\mathbb{S}^1)$. Henceforth, for each measurable function f on Z_ε, we denote its extension by 0 in the holes

$$\{f\}(x) = \begin{cases} f(x) \text{ if } x \in Z_\varepsilon, \\ 0 \text{ otherwise.} \end{cases}$$

Consider next $F \equiv F(t,s,x,v)$ the solution of the Cauchy problem

$$\begin{cases} \partial_t F + v\cdot\nabla_x F + \partial_s F = -\sigma F + \dfrac{\dot{p}(t\wedge s)}{p(t\wedge s)}F\,, \quad x \in \mathbb{R}^N\,,\ |v| = 1\,,\ s,t > 0\,, \\ F(t,0,x,v) = \sigma \displaystyle\int_0^\infty T_k\left(F\right)(t,s',x,v)ds' \\ F(0,s,x,v) = f^{in}(x,v)\sigma e^{-\sigma s} \end{cases}$$

with the notation $t \wedge s = \min(t,s)$ and where p is a decreasing function on $\mathbb{R}_+$ whose definition will be recalled in the sequel.

Theorem 0.1. *Under the assumptions and with the notations above,*

$$\{f_\varepsilon\} \to \int_0^\infty F ds \quad \text{in } L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{S}^1) \text{ weak-* as } \varepsilon \to 0.$$

Next, we recall the results about the total mass of the particle system in the homogenization limit $\varepsilon \ll 1$. We introduce the quantity

$$M(t,s) := \frac{1}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} F(t,s,x,v) dx dv$$

and we denote

$$B(t,s) := \sigma - \frac{\dot{p}}{p}(t \wedge s).$$

The first result is that M is the solution of a Renewal Equation :

Proposition 0.2. *Under the notations above, and with the notations above, the Renewal equation*

$$\begin{cases} \partial_t \mu + \partial_s \mu + B(t,s)\mu = 0 \qquad s,t > 0, \\ \mu(t,0) = \sigma \displaystyle\int_0^\infty \mu(t,\tau) d\tau \\ \mu(0,s) = \sigma e^{-\sigma s} \end{cases}$$

has a unique mild solution $\mu \in L^\infty\left([0,T]; L^1(\mathbb{R}_+)\right)$ for all $T > 0$. Moreover one has

$$M(t,s) = \frac{\mu(t,s)}{2\pi} \iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v) dx dv$$

a.e. in $(t,s) \in \mathbb{R}_+ \times \mathbb{R}_+$.

Consider next the quantity

$$m(t) := \int_0^\infty M(t,s) ds$$

it is the total mass of the particle system in the homogenization limit $\varepsilon \ll 1$ and the asymptotic behavior of M as $t \to +\infty$ is a consequence of the renewal equation above.

Theorem 0.3. *Under the assumptions, and with the notations above,*

(1) *The total mass*

$$\frac{1}{2\pi} \iint_{Z_\varepsilon \times \mathbb{S}^1} f_\varepsilon(t,x,v) dx dv \to m(t)$$

in $L^1_{loc}(\mathbb{R}_+)$ as $\varepsilon \to 0^+$, and a.e. in t 0 after extracting a subsequence of $\varepsilon \to 0^+$;

(2) *there exists $\xi_\sigma \in (-\sigma, 0)$ such that:*

$$m(t) \sim C_\sigma e^{\xi_\sigma t} \text{ as } t \to +\infty,$$

with

$$C_\sigma := \frac{1}{2\pi\sigma} \frac{\iint_{\mathbb{R}^2 \times \mathbb{S}^1} f^{in}(x,v) dx dv}{\int_0^\infty t p(t) e^{-(\sigma + \xi_\sigma)t}};$$

(3) *finally the exponential mass loss rate ξ_σ satisfies*

$$\xi_\sigma \sim -\sigma \text{ as } \sigma \to 0^+ \text{ and } \xi_\sigma \to -2 \text{ as } \sigma \to +\infty.$$

In other words, statement (1) claims that m is the total mass in the ε-vanishing limit and statement (2) claims that the mass has a exponential decay rate whenever $\sigma > 0$. Retrieving these results without the monokinetic assumption is the present question.

1. The model

1.1. **The nonmonokinetic problem.** In the present paper, we remove the monokinetic assumption treated in [1], which means that the speed is no longer fixed. More precisely, we consider the linear Boltzman equation

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \sigma(f_\varepsilon - T_k\,(f)_\varepsilon) = 0. \tag{1.1}$$

The quantity $f(t,x,v)dxdv$ represents again the density of particles at time $t \in \mathbb{R}_+$, located at $x \in \mathbb{R}^2$ and with velocity $v \in \mathbb{B}^2$, the unit disk of $\mathbb{R}^2$. The operator $T_k \in \mathcal{L}(L^1(\mathbb{B}^2))$ is defined by

$$(T_k\phi)\,(v) := \int_{\mathbb{B}^2} k(v,w)\phi(w)dw,\ \ \phi \in L^1(\mathbb{B}^2),$$

where dw is the uniform probability measure on the unit ball $\mathbb{B}^2$ and the kernel k is such that

$$\begin{aligned} k \in C(\mathbb{B}^2_v \times \mathbb{B}^2_w)\,,\quad & k(w,v) = k(v,w) \geq 0 \ \text{ a.e. in } v,w \in \mathbb{B}^2 \\ & \text{and } \int_{\mathbb{B}^2} k(v,w)dw = 1 \ \textit{a.e. in } v \in \mathbb{B}^2. \end{aligned} \tag{1.2}$$

The linear Boltzman equation (1.1) is set on the spatial domain, i.e. the plane $\mathbb{R}^2$ with a periodic system of holes removed:

$$Z_\varepsilon := \left\{x \in \mathbb{R}^2 \,|\, \mathrm{dist}(x, \varepsilon\mathbb{Z}^2) > \varepsilon^2\right\}$$

and we still assume the absorption condition

$$f_\varepsilon(t,x,v) = 0,\ n_x \cdot v > 0$$

that means that every particle falling into a hole remains here for ever. To sum-up, the function f_ε is the solution of the Initial Bounadry-value problem

$$(\Xi_\varepsilon)\begin{cases} \partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon + \sigma(f_\varepsilon - T_k\,(f)_\varepsilon) = 0, & (x,v) \in Z_\varepsilon \times \mathbb{B}^2, t > 0, \\ f_\varepsilon(t,x,v) = 0,\ \text{if } v\cdot n_x > 0, & (x,v) \in \partial Z_\varepsilon \times \mathbb{B}^2, \\ f_\varepsilon(0,x,v) = f^{in}(x,v), & (x,v) \in Z_\varepsilon \times \mathbb{B}^2. \end{cases}$$

Eventually, we assume that the initial data satisfies the assumption

$$f^{in} \geq 0 \text{ on } \mathbb{R}^2 \times \mathbb{B}^2 \text{ and } \iint_{\mathbb{R}^2\times\mathbb{B}^2} f^{in}(x,v)dxdv + \sup_{(x,v)\in\mathbb{R}^2\times\mathbb{B}^2} f^{in}(x,v) < +\infty. \tag{1.3}$$

2. The main results

First, we recall the definition of the free path length in the direction ω for a particle starting from x in Z_ε:

$$\tau_\varepsilon(x,\omega) := \inf\left\{t > 0 \,|\, x - t\omega \in \partial Z_\varepsilon\right\}. \tag{2.1}$$

The distribution of free path length has been studied in [4, 12, 6, 2]. In particular, it is proved that, for each arc $I \subset \mathbb{S}^1$ and each $t \geq 0$, one has

$$\mathrm{meas}(\{(x,\omega) \in (Z_\varepsilon \cap [0,1]^2) \times I \,|\, \varepsilon\tau_\varepsilon(x,\omega) > t\}) \to p(t)|I| \tag{2.2}$$

as $\varepsilon \to 0^+$, where $|I|$ denotes the length of I and the measure considered in the statement above is the uniform measure on $[0,1]^2 \times \mathbb{S}^1$. That implies (see Lemma 1 in [1])

Proposition 2.1. *Under the assumptions and with the notations above,*

$$\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},\omega\right)}* \rightharpoonup p(t) \text{ in } L^\infty(\mathbb{R}^2\times\mathbb{S}^1) \text{ weak-* as } \varepsilon\to 0^+.$$

The following estimate for p can be found in [4, 12]: there exist $C, C' > 0$ such that, for all $t \geq$:

$$\frac{C}{t} \leq p(t) \leq \frac{C'}{t}\,. \tag{2.3}$$

In [2] F. Boca and A. Zaharescu have obtained an explicit formula for p:

$$p(t) = \int_t^{+\infty} (\tau - t)\Upsilon(\tau)d\tau\,, \tag{2.4}$$

where the function Υ is expressed as follows:
(2.5)

$$\Upsilon(t) = \frac{24}{\pi^2}\begin{cases} 1 & \text{if } t\in(0,\frac{1}{2}], \\ \frac{1}{2t} + 2(1-\frac{1}{2t})^2\ln(1-\frac{1}{2t}) - \frac{1}{2}(1-\frac{1}{t})^2\ln|1-\frac{1}{t}| & \text{if } t\in(\frac{1}{2},+\infty)\,. \end{cases}$$

2.1. **The homogenized equation.** Consider next $F := F(t,s,x,v)$ the solution of the Cauchy problem

$$(\Sigma)\begin{cases} \partial_t F + v\cdot\nabla_x F + \partial_s F = -\sigma F + |v|\frac{\dot{p}}{p}(|v|(t\wedge s))F, & t,s>0, (x,v)\in\mathbb{R}^2\times\mathbb{B}^2 \\ F(t,0,x,v) = \sigma\int_0^{+\infty} T_k\left(F\right)(t,s,x,v)ds, & t>0, (x,v)\in\mathbb{R}^2\times\mathbb{B}^2 \\ F(0,s,x,v) = \sigma e^{-\sigma s}f^{in}(x,v), & s>0, (x,v)\in\mathbb{R}^2\times\mathbb{B}^2 \end{cases}$$

with the notation $t\wedge s := \min(t,s)$. Notice that F is a density defined on the extended phase space:

$$\left\{(s,x,v)|s\geq 0, x\in\mathbb{R}^2, v\in\mathbb{B}^2\right\}.$$

Our first main result is

Theorem 2.2. *Under the assumptions and with the notations above,*

$$\{f_\varepsilon\} \rightharpoonup \int_0^\infty F ds$$

in $L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2)$ weak- as $\varepsilon\to 0^+$, where F is the unique (mild) solution of* (Σ).

2.2. **The mass and its asymptotic behavior in the long time limit.** Consider then the following equation

$$(R)\begin{cases} \partial_t\mu(t,s,v) + \partial_s\mu(t,s,v) + B(t,s,v)\mu(t,s,v) = 0, & t,s>0, v\in\mathbb{B}^2, \\ \mu(t,0,v) = \sigma\int_0^\infty T_k\left(\mu\right)(t,s,v)ds & t>0, v\in\mathbb{B}^2, \\ \mu(0,s,v) = \sigma e^{-\sigma s}\int_{\mathbb{R}^2} f^{in}(x,v)dx, & s>0, v\in\mathbb{B}^2, \end{cases}$$

with

$$B(t,s,v) := \sigma - |v|\frac{\dot{p}}{p}(|v|(t\wedge s)),\ t,s\geq 0, v\in\mathbb{B}^2.$$

The first corollary of Theorem 2.2 is

Corollary 2.3. *For all $T > 0$, the PDE (R) has a unique mild solution μ in $L^\infty\left([0,T];L^1\left(\mathbb{R}_+\times\mathbb{B}^2\right)\right)$. Moreover,*

$$\iint_{\mathbb{R}_+\times\mathbb{B}^2}\mu(t,s,v)=\int_{\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2}F(t,s,x,v)dsdxd.$$

a.e. in $t\in\mathbb{R}_+$.

Next we discuss the asymptotic decay as $t\to+\infty$ of the total mass of the particle system in the homogenization limit $\varepsilon\ll 1$. As in the monokinetic case, the asymptotic behavior in the long time limit of the total mass of the solution of the homogenized equation is a consequence of the renewal PDE sarisfied.

Theorem 2.4. *Under the assumptions and with the notations above,*

(1) *the total mass*

$$\iiint_{\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2}F_\varepsilon dsdxdv\to\int_{\mathbb{R}_+\times\mathbb{B}^2}\mu(t,s,v)dsdv \text{ as } \varepsilon\to 0^+,$$

in $L^1_{loc}\left(\mathbb{R}_+\right)$ and a.e. in $t\geq 0$ after extracting a subsequence of $\varepsilon\to 0^+$;

(2) *assume moreover for the sake of simplicity that T_k has finite rank, then there exists $C_\sigma>0$, $n_\sigma\in\mathbf{N}$ and $\xi_\sigma\in(-\sigma,0)$ such that*

$$m(t)\sim C_\sigma t^n e^{\xi_\sigma t} \text{ as } t\to+\infty,$$

with C_σ and ξ_σ depending on T_k and σ and n depending only on T_k;

(3) *and the exponential mass loss rate ξ_σ satisfies*

$$\xi_\sigma\sim-\sigma \text{ as } \sigma\to 0^+.$$

Statement (1) above means that $t\mapsto\int_{\mathbb{R}_+\times\mathbb{B}^2}\mu(t,s,v)dsdv$ is the limiting total mass of the particle system at time t as $\varepsilon\to 0^+$. Statement (2) gives a precise asymptotic equivalent of $t\mapsto\int_{\mathbb{R}_+\times\mathbb{B}^2}\mu(t,s,v)dsdv$ as $t\to+\infty$. The main difference with the monokinetic case is the total mass decay. We recall that if $\sigma=0$ in the linear Boltzmann equation (Ξ_ε), the total mass of the particle system in the vanishing ε limit is asymptotically equivalent to

$$\frac{\iint_{\mathbb{R}^2\times\mathbb{B}^2}f^{in}(x,v)dxdv}{\pi^2 t}$$

as $t\to+\infty$. This algebraic decay is due to the existence of infinite open strips included in the spatial domain Z_ε avoiding all the holes. A particle located in one such channel and moving in a direction close to the channel's direction will not fall into a hole before exiting the channel and thus in a long time, longer as the particle's direction is closer to the channel's one.

In the monokinetic case, the collision operator in the linear Boltzmann equation destroys the influence of the channels, which entails a exponential decay — see [1]. If we keep $\sigma>0$ whithout the monokinetic assumption, that means that the speed can change. More precisely, as a particle slows down, it will obviously fall into a hole in a longer time. Therefore, the decay of the total mass will be downgraded.

Indeed, as we will see in section 5 and section 6, the particle system is split into subpopulations organized as a nest by the collision operator. That phenomenon is illustrated by the graph in Figure 1

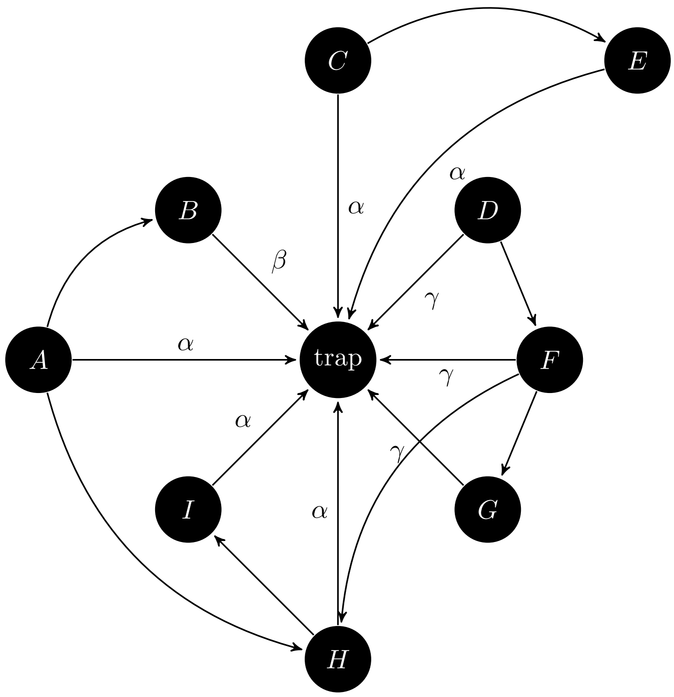

Figure 1 : a graph

$A, \ldots, I$ designate the subpopulations. The arrow between two supopulations means that the collision operator send one into the other, and the arrows with greek letters mean that each subpopulation has a exponential mass decay rate designated by that greek letter that depends upon the spectral properties of the collision operator and σ. Obviously, the exponential decay rate ξ_σ of the total mass is the lowest exponential decay rate among all the subpopulations. The integer n is the length of the longest chain connecting the subpopulations with the same exponential decay rate — see section 6.

It is worth noticing that it is similar to a random walk on graph with at least one trap. When a graph is finite, the asymptotic decay has an exponential rate. ON the contrary, in the case of a infinite graph, the asymptotic behavior may be a power law — see the review [5]. Therefore, we may think that statement (2) of Theorem 2.4 holds for each compact collision operator. And some noncompact collision operators — as for the inelastic linear Boltzmann equation — will have a cooling effect that destroys the exponential decay.

In section 3, we give the proof of Theorem 2.2; in section 4, we study the governing equation of the total mass in the vanishing ε limit; while its asymptotic behavior as $\varepsilon \to 0^+$ is discussed in section 5, section 6 and section 7.

3. The homogenized kinetic equation

As in [1], our argument for Theorem 2.2 is splitted into the same steps.

3.1. **A new formulation of the transport equation.**

As in the monokinetic case, we introduce here a extended phase space involving the additional variable s. Consider the initial boundary value problem

$$(\Sigma_\varepsilon)\begin{cases} \partial_t F_\varepsilon + \partial_s F_\varepsilon + v.\nabla_x F_\varepsilon + \sigma F_\varepsilon = 0, & t,s>0,(x,v)\in Z_\varepsilon\times\mathbb{B}^2,\\ F_\varepsilon(t,s,x,v)=0, \text{ if } v\cdot n_x>0, & t,s>0,(x,v)\in \partial Z_\varepsilon\times\mathbb{B}^2,\\ F_\varepsilon(t,0,x,v)=\sigma\int_0^\infty T_k\left(F_\varepsilon\right)(t,s,x,v)ds, & t>0,(x,v)\in Z_\varepsilon\times\mathbb{B}^2,\\ F_\varepsilon(0,s,x,v)=\sigma e^{-\sigma s}f^{in}(x,v), & s>0,(x,v)\in Z_\varepsilon\times\mathbb{B}^2,\end{cases}$$

with unknown $F_\varepsilon := F_\varepsilon(t,s,x,v)$. We here establish the relation between theses two initial boundary value problems, (Ξ_ε) and (Σ_ε)

Proposition 3.1. *Assume that f^{in} satisfies the assumptions above. Then*

(1) *for each $\varepsilon>0$, the problem (Σ_ε) has a unique mild solution such that*

$$(t,x,v)\mapsto\int_0^\infty |F_\varepsilon(t,s,x,v)|ds \text{ belongs to } L^\infty([0,T]\times Z_\varepsilon\times\mathbb{B}^2)$$

for each $T>0$;

(2) *moreover*

$$0\le F_\varepsilon(t,s,x,v)\le \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{B}^2)}\sigma e^{-\sigma s}$$

a.e. in $t,s\ge 0$, $x\in Z_\varepsilon$ and $v\in\mathbb{B}^2$, and

$$\int_0^\infty F_\varepsilon(t,s,x,v)ds = f_\varepsilon(t,x,v),$$

for a.e. $t\ge0$, $x\in Z_\varepsilon$ and $v\in\mathbb{B}^2$, where f_ε is the solution of (Ξ_ε).

Proof. (1) By the methods of characteristics, we see that if a mild solution F_ε of (Σ_ε) exists then it must satisfy

$$F_\varepsilon = F_{1,\varepsilon}+F_{2,\varepsilon} \tag{3.1}$$

with

$$\begin{aligned}F_{1,\varepsilon} &:= \mathbb{1}_{s<t}\mathbb{1}_{s<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}e^{-\sigma s}F_\varepsilon(t-s,0,x-sv,v)\\ &= \mathbb{1}_{s<t}\mathbb{1}_{s<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}\sigma e^{-\sigma s}\int_0^\infty T_k\left(F_\varepsilon\right)(t-s,\tau,x-sv,v)d\tau,\end{aligned}$$

and

$$\begin{aligned}F_{2,\varepsilon} &:= \mathbb{1}_{t<s}\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}e^{-\sigma t}F_\varepsilon(0,s-t,x-tv,v)\\ &= \mathbb{1}_{t<s}\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}\sigma e^{-\sigma s}f^{in}(x-tv,v).\end{aligned}$$

a.e. in $(t,s,x,v)\in\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2$. We define then for all $T>0$ $\mathcal{X}_T$ the set of measurable functions G defined on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2$ such that

$$(t,x,v)\mapsto\int_0^\infty |G(t,s,x,v)|ds\in L^\infty\left(\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2\right).$$

It is obviously a Banach space for the norm

$$\|G\|_{\mathcal{X}_T} := \left\|\int_0^\infty |G(t,s,x,v)|ds\right\|_{L^\infty(\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2)}.$$

Next, let $\mathcal{T} : \mathcal{X}_T \mapsto \mathcal{X}_T$ defined by

$$\mathcal{T}(G) := \mathbb{1}_{s<t}\mathbb{1}_{s<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}\sigma e^{-\sigma s}\int_0^\infty T_k(G)(t-s,\tau,x-sv,v)d\tau$$

for each $G \in \mathcal{X}_T$. Notice that since $k \in C\left(\mathbb{B}^2 \times \mathbb{B}^2\right)$, we have $k \in L^\infty\left(\mathbb{B}^2 \times \mathbb{B}^2\right)$. That being said, observe that for each $n \geq 0$

$$\begin{aligned}
\|\mathcal{T}^n G\|_{\mathcal{X}_T} &= \left\|\int_0^\infty (\mathcal{T}^n G)(t,s,x,v)ds\right\|_{L^\infty(\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2)} \\
&= \left\|\int_0^t \sigma e^{-\sigma s}\mathbb{1}_{s<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}\int_0^\infty T_k\left(\mathcal{T}^{n-1}G\right)(t-s,\tau,x-sv,v)d\tau\right\|_{L^\infty(\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2)} \\
&\leq \sigma\left\|\int_0^t\int_0^\infty\int_{\mathbb{B}^2} k(v,w)\left(\mathcal{T}^{n-1}G\right)(t-s,\tau,x-sv,w)dvd\tau\right\|_{L^\infty(\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2)} \\
&\leq \sigma\|k\|_{L^\infty(\mathbb{B}^2\times\mathbb{B}^2)}\left\|\int_0^\infty\left(\mathcal{T}^{n-1}G\right)(t,\tau,x,w)d\tau\right\|_{L^\infty(\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2)} \\
&\leq \sigma\|k\|_{L^\infty(\mathbb{B}^2\times\mathbb{B}^2)}\int_0^t \|\mathcal{T}^{n-1}G\|_{\mathcal{X}_T}ds.
\end{aligned}$$

By induction

$$\|\mathcal{T}^n G\|_{\mathcal{X}_T} \leq \frac{\left(\sigma\|k\|_{L^\infty(\mathbb{B}^2\times\mathbb{B}^2)}\right)^n}{n!}\|G\|_{\mathcal{X}_T}.$$

We have

$$F_{1,\varepsilon} = \mathcal{T}F_\varepsilon$$

so that

$$F_\varepsilon = \mathcal{T}F_\varepsilon + F_{2,\varepsilon}.$$

This integral equation has a solution $F_\varepsilon \in \mathcal{X}_T$ for each $T > 0$ given by the serie

$$F_\varepsilon = \sum_{n\geq 0}\mathcal{T}^n F_{2,\varepsilon}$$

that converges normally in the Banach $\mathcal{X}_T$ as

$$\sum_{n\geq 0}\|\mathcal{T}^n F_{2,\varepsilon}\|_{\mathcal{X}_T} \leq \sum_{n\geq 0}\frac{\left(\sigma\|k\|_{L^\infty(\mathbb{B}^2\times\mathbb{B}^2)}\right)^n}{n!}\|F_{2,\varepsilon}\|_{\mathcal{X}_T} < +\infty.$$

Moreover, this solution is unique in $\mathcal{X}_T$ since if G_ε is another solution in $\mathcal{X}_T$ we have

$$\begin{aligned}
\|F_\varepsilon - G_\varepsilon\|_{\mathcal{X}_T} &= \|\mathcal{T}^n(F_\varepsilon - G_\varepsilon)\|_{\mathcal{X}_T} \\
&\leq \frac{\left(\sigma\|k\|_{L^\infty(\mathbb{B}^2\times\mathbb{B}^2)}\right)^n}{n!}\|F_\varepsilon - G_\varepsilon\|_{\mathcal{X}_T} \\
&\to 0 \text{ as } n \to +\infty.
\end{aligned}$$

(2) We observe that if $G \in \mathcal{X}_T$ is nonnegative a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2$ then $\mathcal{T}G \geq 0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2$. Since $f^{in} \geq 0$ a.e. on $\mathbb{R}^2\times\mathbb{B}^2$, we have $F_{2,\varepsilon} \geq 0$ on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2$ and consequently, the serie defined above is nonnegative on $\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2$. Next, integrating both sides

of (3.1) with respect to s leads to

$$\begin{aligned}\int_0^\infty F_\varepsilon(t,s,x,v)ds &= \int_0^\infty F_{1,\varepsilon}(t,s,x,v)ds + \int_0^\infty F_{2,\varepsilon}(t,s,x,v)ds\\ &= \mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)} f^{in}(x-tv,v)e^{-\sigma t}\\ &\quad + \sigma\int_0^t e^{-\sigma s}\mathbb{1}_{s<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}\left(T_k\int_0^\infty F_\varepsilon d\tau\right)(t-s,x-sv,v)ds\end{aligned}$$

in which we recognize the Duhamel formula giving the unique mild solution f_ε of (Ξ_ε). Hence

$$f_\varepsilon(t,x,v) = \int_0^\infty F_\varepsilon(t,s,x,v)ds \text{ a.e. in } (t,x,v)\in\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2.$$

As (Ξ_ε) satisfies the maximum principle, we have

$$0\le f_\varepsilon(t,x,v)\le f^{in}(x,v) \text{ a.e. in } (t,x,v)\in\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2.$$

And (3.1) can be recast in the form

$$\begin{aligned}F_\varepsilon(t,s,x,v) &= \mathbb{1}_{s<t}\mathbb{1}_{s<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}\sigma e^{-\sigma s}T_k(f_\varepsilon)(t-s,x-sv,v)\\ &\quad + \mathbb{1}_{t<s}\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}\sigma e^{-\sigma s}f^{in}(x-tv,v)\\ &\le \sigma e^{-\sigma s}\|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{B}^2)}\end{aligned}$$

a.e. in $(t,s,x,v)\in\mathbb{R}_+\times\mathbb{R}_+\times Z_\varepsilon\times\mathbb{B}^2$ which concludes the proof.

□

3.2. **The distribution of free path lengths.** We extend the $\varepsilon-$vanishing limit of $\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}$ in the nonmonokinetic case.

Lemma 3.2. *Let τ_ε be the free path length defined in. Then for each $t>0$*

$$\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)} \to p(|v|t) \tag{3.2}$$

in $L^\infty(\mathbb{R}^2\times\mathbb{B}^2)$ weak- as ε vanishes.*

Proof. Recall that for each $t>0$

$$\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},\omega\right)} \to p(t) \tag{3.3}$$

in $L^\infty(\mathbb{R}^2\times\mathbb{S}^1)$ weak-* as ε vanishes — see Lemma 1 in chapter II ([1]). Since the linear span of functions $\chi\equiv\chi(x,v)\in C_0^\infty(\mathbb{R}^2\times\mathbb{B}^2)$ is dense in $L^1(\mathbb{R}^2\times\mathbb{B}^2)$, and the family $\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}$ is bounded in $L^\infty(\mathbb{R}^2\times\mathbb{B}^2)$, it is enough to prove that

$$\iint_{Z_\varepsilon\times\mathbb{B}^2}\chi(x,v)\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}dxdv \to \iint_{\mathbb{R}^2\times\mathbb{B}^2}p(|v|t)\chi(x,v)dxdv \text{ as } \varepsilon\to 0^+,$$

for each $\chi\in C_0^\infty(\mathbb{R}^2\times\mathbb{B}^2)$. Notice that

$$\begin{aligned}\iint_{Z_\varepsilon\times\mathbb{B}^2}\chi(x,v)\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}dxdv &= \int_{Z_\varepsilon}\int_0^1\int_{\mathbb{S}^1}\chi(x,r\omega)\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},r\omega\right)}rdrd\omega dx\\ &= \int_{Z_\varepsilon}\int_0^1\left(\int_{\mathbb{S}^1}\chi(x,r\omega)\mathbb{1}_{rt<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},\omega\right)}d\omega\right)rdrdx.\end{aligned}$$

As $\chi\in C_0^\infty(\mathbb{R}^2\times\mathbb{B}^2)$, for each $r>0$, $(x,\omega)\mapsto\chi(x,r\omega)\in L^\infty(\mathbb{R}^2_x\times\mathbb{S}^1_\omega)$. That implies by (3.3)

$$\int_{\mathbb{S}^1}\chi(x,r\omega)\mathbb{1}_{rt<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},\omega\right)}d\omega \to p(rt)\int_{\mathbb{S}^1}\chi(x,r\omega)d\omega \text{ as } \varepsilon\to 0^+.$$

Thus

$$\int_{Z_\varepsilon}\int_0^1\left(\int_{\mathbb{S}^1}\chi(x,r\omega)\mathbb{1}_{rt<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},\omega\right)}d\omega\right)rdrdx \to \int_{\mathbb{R}^2}\int_0^1 p(rt)\int_{\mathbb{S}^1}\chi(x,r\omega)d\omega rdrdx$$
$$= \int_{\mathbb{R}^2}\int_{\mathbb{B}^2} p\left(|v|t\right)\chi(x,v)dxdv.$$

which is the announced result. □

3.3. **Extending f_ε by 0 in the holes.** We recall (see Lemma 2 in chapter II ([1])) that for each $\varepsilon > 0$ the function $\{f_\varepsilon\}$ satisfies

$$\left(\partial_t + v.\nabla_x\right)\{f_\varepsilon\} + \sigma(\{f_\varepsilon\} - T_k\left(\{f_\varepsilon\}\right)) = (v.n_x)f_\varepsilon\mid_{\partial Z_\varepsilon\times\mathbb{B}^2}\delta_{\partial Z_\varepsilon}$$

in $\mathcal{D}'(\mathbb{R}_+^*\times\mathbb{R}^2\times\mathbb{B}^2)$, where $\delta_{\partial Z_\varepsilon}$ is the surface measure concentrated on the boundary of Z_ε, and n_x is the unit normal vector at $x \in \partial Z_\varepsilon$ pointing towards the interior of Z_ε. Moreover (see lemma 3) for each $R > 0$ the family of Radon measures

$$(v.n_x)f_\varepsilon\mid_{\partial Z_\varepsilon\times\mathbb{B}^2}\delta_{\partial Z_\varepsilon}\mid_{[-R,R]^2\times\mathbb{B}^2}$$

is bounded in $\mathcal{M}([-R,R]^2\times\mathbb{B}^2)$.

3.4. **The velocity averaging lemma.** We recall first a classical result averaging that is a special case of Theorem 1.8 in [3].

Proposition 3.3. *Let $p > 1$ and assume that $f_\varepsilon \equiv f_\varepsilon(t,x,v)$ is a bounded family in $L^p_{loc}(\mathbb{R}_t^+\times\mathbb{R}_x^d\times\mathbb{B}_v^{d-1})$ such that*

$$\sup_{\varepsilon>0}\int_0^T\iint_{B(0,R)\times\mathbb{B}^{d-1}}|\partial_t + v\cdot\nabla_x f_\varepsilon|dxdvdt < +\infty$$

for each $T > 0$ and each $R > 0$. Then, for each $\psi \in C(\mathbb{B}^{d-1}\times\mathbb{B}^{d-1})$, the family $\rho_\psi\left[f_\varepsilon\right]$, defined by

$$\rho_\psi\left[f_\varepsilon\right](t,x,v) = \int_{\mathbb{B}^{d-1}} f_\varepsilon(t,x,w)\psi(v,w)dw,$$

is relatively compact in $L^1_{loc}(\mathbb{R}_t^+\times\mathbb{R}_x^d\times\mathbb{B}_v^{d-1})$.

A straighforward consequence of Proposition 3.3 is the compactness in L^1_{loc} of $\left(T_k\left(\{f_\varepsilon\}\right)\right)_{\varepsilon>0}$ which a key argument in the proof of Theorem 2.2.

Lemma 3.4. *Let $f_\varepsilon \equiv f_\varepsilon(t,x,v)$ be the family of solutions of the initial boundary value problem. Then the families*

$$T_k\left(\{f_\varepsilon\}\right) = \{T_k\left(f_\varepsilon\right)\}$$

and

$$\int_{\mathbb{B}^2}\{f_\varepsilon\}\,dv$$

are relatively compact in $L^1_{loc}(\mathbb{R}_t^+\times\mathbb{R}_x^d\times\mathbb{B}_v^{d-1})$ strong.

Proof. By the Maximum Principle,

$$|f_\varepsilon(t,x,v)| \le \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{B}^2)}$$

a.e. in $t \ge 0$, $x \in Z_\varepsilon$ and $v \in \mathbb{B}^2$, so that

$$\sup_\varepsilon\|\{f_\varepsilon\}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2)}.$$

We recall that $\{f_\varepsilon\}$ satisfies

$$\left(\partial_t + v.\nabla_x\right)\{f_\varepsilon\} + \sigma(\{f_\varepsilon\} - T_k\left(\{f_\varepsilon\}\right)) = (v.n_x)f_\varepsilon\mid_{\partial Z_\varepsilon\times\mathbb{B}^2}\delta_{\partial Z_\varepsilon}$$

in $\mathcal{D}'(\mathbb{R}_+^* \times \mathbb{R}^2 \times \mathbb{B}^2)$. By the inequality above and given that k is nonnegative, one has

$$\|\sigma(T_k(\{f_\varepsilon\}) - \{f_\varepsilon\})\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2)} \leq 2\sigma\|\{f_\varepsilon\}\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2)}$$

since $T_k(1) = 1$. Besides we know that the family of Radon measures

$$\mu_\varepsilon := f_\varepsilon \mid_{\partial Z_\varepsilon \times \mathbb{B}^2} (v.n_x)\delta_{\partial Z_\varepsilon}$$

satisfies

$$\sup_{\varepsilon>0} \int_{[0,T]\times[-R,R]^2\times\mathbb{B}^2} |\mu_\varepsilon| < +\infty$$

for each $T > 0$ and $R > 0$. Consequently, Proposition 3.3 implies that the family

$$T_k(\{f_\varepsilon\})$$

is relatively compact in $L^1_{loc}(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{B}^2)$. □

3.5. **Uniqueness for the homogenized equation.** Consider the Cauchy problem with unknown $G \equiv G(t, s, x, v)$

$$\begin{cases} (\partial_t + v\cdot\nabla_x + \partial_s)G = -\sigma G + |v|\dfrac{\dot p(|v|(t\wedge s))}{p(|v|(t\wedge s))}G, & t, s > 0, x \in \mathbb{R}^2, v \in \mathbb{B}^2, \\ G(t, 0, x, v) = S(t, x, v), & t > 0, (x, v) \in \mathbb{R}^2 \times \mathbb{B}^2, \\ G(0, s, x, v) = G^{in}(s, x, v), & s > 0, (x, v) \in \mathbb{R}^2 \times \mathbb{B}^2. \end{cases}$$

If, for a.e. $(t, s, x, v) \in \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2$, the function $\tau \mapsto G(t+\tau, s+\tau, x+\tau v, v)$ is C^1 in $\tau > 0$, then, since the function $p \in C^1(\mathbb{R}_+)$ and $p > 0$ on $\mathbb{R}_+$, one has

$$\begin{aligned}\left(\frac{d}{d\tau} + \sigma - |v|\frac{\dot p(|v|(t\wedge s+\tau))}{p(|v|(t\wedge s+\tau))}\right) G(t+\tau, s+\tau, x+\tau v, v) \\ = e^{-\sigma\tau}p(|v|(t\wedge s+\tau))\frac{d}{d\tau}\left(\frac{e^{\sigma\tau}G(t+\tau, s+\tau, x+\tau v, v)}{p(|v|(t\wedge s+\tau))}\right) = 0\,.\end{aligned}$$

Hence

$$\Gamma : \tau \mapsto \frac{e^{\sigma\tau}G(t+\tau, s+\tau, x+\tau v, v)}{p(|v|(t\wedge s+\tau))}$$

is a constant. Therefore

$$\Gamma(0) = \begin{cases} \Gamma(-t) & \text{if } t < s, \\ \Gamma(-s) & \text{if } s < t, \end{cases}$$

so that

(3.4)
$$G(t, s, x, v) = \mathbb{1}_{t<s}e^{-\sigma t}p(|v|t)G^{in}(s-t, x-tv, v) + \mathbb{1}_{s<t}e^{-\sigma s}p(|v|s)S(t-s, x-sv, v)\,.$$

Proposition 3.5. *Under the assumptions and with the notations above, the problem* (Σ) *has a unique mild solution* F *such that*

$$(t, x, v) \mapsto \int_0^\infty |F(t, s, x, v)|ds \in L^\infty\left(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{B}^2\right).$$

This solution satisfies

$$\begin{aligned} F(t, s, x, v) &= \mathbb{1}_{t<s}\sigma e^{-\sigma t}p\left(|v|t\right) f^{in}(x - tv, v) \\ &\quad + \mathbb{1}_{s<t}\sigma e^{-\sigma s}\int_0^\infty (T_kF)(t-s, \tau, x-sv, v)d\tau \end{aligned}$$

for a.e. $(t, s, x, v) \in \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2$. *Besides,* $F \geq 0$ *a.e. on* $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2$ *if* $f^{in} \geq 0$ *a.e. on* $\mathbb{R}^2 \times \mathbb{B}^2$.

Proof. If a mild solution exists, it must satisfies the integral equation worked out before the proposition. For each $T>0$ define $\mathcal{Y}_T$ the set of measurable functions G defined a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2$ such that

$$(t,x,v)\mapsto\int_0^\infty |G(t,s,x,v)|ds\in L^\infty\left(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2\right).$$

It is obviously a Banach space for the norm

$$\|G\|_{\mathcal{Y}_T}:=\left\|\int_0^\infty |G|ds\right\|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2)}.$$

Let the operator $\mathcal{Q}\in\mathcal{L}\left(\mathcal{Y}_T\right)$ defined by

$$\mathcal{Q}G=\mathbb{1}_{s<t}\sigma e^{-\sigma s}\int_0^\infty\left(T_kG\right)(t-s,\tau,x-sv,v)d\tau$$

for each $G\in\mathcal{Y}_T$. A computation similar as in the proof of Proposition 3.1 shows that for each $n\geq 1$

$$\|\mathcal{Q}^nG\|_{\mathcal{Y}_T}\leq\frac{\left(\sigma\|k\|_{L^\infty(\mathbb{B}^2\times\mathbb{B}^2)}\right)^n}{n!}\|G\|_{\mathcal{Y}_T}.$$

Besides integral equation (3.4) can be recasted in the form

$$F=F_1+\mathcal{Q}F$$

where

$$F_1(t,s,x,v):=\mathbb{1}_{t<s}\sigma e^{-\sigma t}p\left(|v|t\right)f^{in}(x-tv,v).$$

Consequently, as established in the proof of Proposition 3.1, the integral equation has a unique mild solution F in $\mathcal{Y}_T$ given by

$$F=\sum_{n\geq 0}\mathcal{Q}^nF_1.$$

Finally, we observe that if $G\geq 0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2$ then for each $n\geq 0$ $\mathcal{Q}^n\geq 0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2$. Therefore, if $f^{in}\geq 0$ a.e. on $\mathbb{R}^2\times\mathbb{B}^2$, we have $F_1\geq 0$ a.e. on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2$ and thus $F\geq 0$ on $\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2$. □

3.6. **Proof of Theorem** 2.2.

Proof. We recall that the solution F_ε of (Σ_ε) admits a decomposition (see equality (3.1))

$$F_\varepsilon=F_{1,\varepsilon}+F_{2,\varepsilon}$$

with

$$F_{1,\varepsilon}(t,s,x,v)=\mathbb{1}_{s<t}\mathbb{1}_{s<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}\sigma e^{-\sigma s}\int_0^\infty T_k\left(F_\varepsilon\right)(t-s,\tau,x-sv,v)d\tau,$$

and

$$F_{2,\varepsilon}(t,s,x,v):=\mathbb{1}_{t<s}\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}\sigma e^{-\sigma s}f^{in}(x-tv,v).$$

Passing to the limit as $\varepsilon\to 0^+$ in the term $F_{2,\varepsilon}$ is easy. By Lemma 3.2 we have

$$\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}\rightharpoonup p\left(|v|t\right)\tag{3.5}$$

in $L^\infty\left(\mathbb{R}^2\times\mathbb{B}^2\right)$ weak-* as $\varepsilon\to 0^+$ for each $t\geq 0$. Hence

$$\begin{aligned}F_{2,\varepsilon}(t,s,x,v)&=\mathbb{1}_{t<s}\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)}\sigma e^{-\sigma s}f^{in}(x-tv,v)\\&\rightharpoonup\mathbb{1}_{t<s}p\left(|v|t\right)\sigma e^{-\sigma s}f^{in}(x-tv,v)\end{aligned}$$

in $L^\infty\left(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2\right)$. We denote in the sequel

$$F_2=\mathbb{1}_{t<s}p\left(|v|t\right)\sigma e^{-\sigma s}f^{in}(x-tv,v)$$

Next, consider the term $F_{1,\varepsilon}$ We have the uniform bound

$$\sup_{\varepsilon>0} \| \{f_\varepsilon\} \|_{L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2)} \leq \|f^{in}\|_{L^\infty(\mathbb{R}^2\times\mathbb{B}^2)}$$

thus by the Banach-Alaoglu theorem

$$\{f_\varepsilon\} \to f \text{ in } L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2) \text{ weak-*} \tag{3.6}$$

for some $f \in L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2)$ after a possible extraction of subsequence $\varepsilon \to 0^+$. Therefore, by the strong compactness lemma implies then

$$T_k(\{f_\varepsilon\}) \to T_k(f) \text{ in } L^1_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2),$$

as $\varepsilon \to 0^+$. Hence we have

$$\begin{aligned}\{F_\varepsilon\} &\rightharpoonup F_2 + \mathbb{1}_{s<t} p(|v|s)\,\sigma e^{-\sigma s}(T_k f)(t-s, x-sv, v),\\ &=: \tilde{F}\end{aligned}$$

in $L^1_{loc}\left(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2\right)$. Fix $T>0$, we remark that for each $t\in[0,T]$, on has

$$\int_0^T F_\varepsilon(t,s,x,v)ds = \int_0^T F_{1,\varepsilon}(t,s,x,v)ds + e^{-\sigma t}\mathbb{1}_{t<\varepsilon\tau_\varepsilon\left(\frac{x}{\varepsilon},v\right)} f^{in}(x-tv,v)$$

since $F_{1,\varepsilon}$ is supported in $s\leq t\leq T$ so that

$$\int_0^T F_\varepsilon(t,s,x,v)ds \rightharpoonup \int_0^T \tilde{F}(t,s,x,v)ds$$

in $L^1_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2)$-weak as $\varepsilon\to 0^+$. And on the other hand we have

$$\int_0^\infty \{F_\varepsilon\}\,ds \rightharpoonup f$$

in $L^\infty(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2)$ weak-* as $\varepsilon\to0^+$ and therefore in $L^1_{loc}(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2)$ weak as $\varepsilon\to0^+$. By uniqueness of the limit, we have

$$\int_0^\infty \tilde{F} ds = f \tag{3.7}$$

and so that $\tilde{F}$ satisfies

$$\tilde{F}(t,s,x,v) = F_2 + \mathbb{1}_{s<t} p(|v|s)\,\sigma e^{-\sigma s}\int_0^\infty \left(T_k\tilde{F}\right)(t-s,\tau,x-sv,v)d\tau.$$

By Proposition 3.5 the integral equation above has a unique mild solution that is F and thus

$$\tilde{F} = F$$

and

$$F_\varepsilon \rightharpoonup F \text{ in } L^1_{loc}\left(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2\right) - \text{weak}$$

as $\varepsilon\to0^+$. Finally, (3.6) and (3.7) imply

$$\{f_\varepsilon\} \rightharpoonup f = \int_0^\infty F ds$$

in $L^1_{loc}\left(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2\right)$-weak as $\varepsilon\to0^+$ which concludes the proof of Theorem 2.2. □

4. The mass equations

4.1. **The Renewal PDE governing the total mass of the homogenized system.** We begin with a proof of Corollary 2.3.

Proof. Should a mild solution M of the Renewal PDE above exist, it must satisfy
(4.1)
$$\mu(t,s,v) = \mathbb{1}_{t<s}\sigma e^{-\sigma s}p(|v|t)\mu^{in}(v) + \mathbb{1}_{s<t}e^{-\sigma s}p(|v|s)\int_0^\infty T_k(\mu)(t-s,\tau,v)d\tau.$$

We define for each $T>0$, $\mathcal{V}_T$ the set of measurable function G defined on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{B}^2$ such that
$$(t,v) \mapsto \int_0^\infty |M(t,s,v)|\,ds \in L^\infty\left([0,T];L^1\left(\mathbb{B}^2\right)\right).$$
It is a Banach space for the norm
$$\|M\|_{\mathcal{V}_T} := \left\|\int_{\mathbb{B}^2}\int_0^\infty |M(t,s,v)|\,dsdv\right\|_{L^\infty([0,T])}.$$
Introduce here $\mathcal{K}:\mathcal{V}_T \mapsto \mathcal{V}_T$ defined by
$$\mathcal{K}(M) := \mathbb{1}_{s<t}e^{-\sigma s}p(|v|s)\int_0^\infty T_k(M)(t-s,\tau,v)d\tau$$
for each $M \in \mathcal{V}_T$. As for each $v \in \mathbb{B}^2$ and $t \geq 0$, $p(|v|) \leq 1$, we show, with a argument similar as in Proposition 3.1, that for each $n \geq 0$
$$\|\mathcal{K}^n M\|_{\mathcal{V}_T} \leq \frac{\left(\sigma\|k\|_{L^\infty(\mathbb{B}^2\times\mathbb{B}^2)}\right)^n}{n!}\|M\|_{\mathcal{V}_T}.$$
Equation (4.1) is equivalent to
$$\mu(t,s,v) = \mathbb{1}_{t<s}\sigma e^{-\sigma s}p(|v|t)\mu^{in}(v) + (\mathcal{K}\mu)(t,s,v).$$
It has a solution $\mu \in \mathcal{V}_T$ for each $T>0$ given by the serie
$$\mu = \sum_{n\geq 0}\mathcal{K}^n\nu^{in} \tag{4.2}$$
with
$$\nu^{in}(t,s,v) := \mathbb{1}_{t<s}\sigma e^{-\sigma s}p(|v|t)\mu^{in}(v) \in \mathcal{V}_T.$$
This serie converges normally in the Banach $\mathcal{V}_T$ since
$$\sum_{n\geq 0}\left\|\mathcal{K}^n\nu^{in}\right\|_{\mathcal{V}_T} \leq \sum_{n\geq 0}\frac{\left(\sigma\|k\|_{L^\infty(\mathbb{B}^2\times\mathbb{B}^2)}\right)^n}{n!}\left\|\mu^{in}\right\|_{\mathcal{V}_T} < +\infty.$$
Moreover, this solution is unique in $\mathcal{V}_T$ since if μ' est another solution in $\mathcal{V}_T$, we would have
$$\begin{aligned}\|\mu-\mu'\|_{\mathcal{V}_T} &= \|\mathcal{K}^n(\mu-\mu')\|_{\mathcal{V}_T}\\ &\leq \frac{\left(\sigma\|k\|_{L^\infty(\mathbb{B}^2\times\mathbb{B}^2)}\right)^n}{n!}\|\mu-\mu'\|_{\mathcal{V}_T}\\ &\to 0 \text{ as } n\to+\infty.\end{aligned}$$
Observe that since $k \geq 0$, $\mathcal{K}\left(\mathcal{V}_T^+\right) \subseteq \mathcal{V}_T^+$ where $\mathcal{V}_T^+$ is the cone of nonnegative functions of $\mathcal{V}_T$. Therefore (4.2) implies, as μ^{in} is nonnegative, that
$$\mu \text{ is nonnegative on } \mathbb{R}_+\times\mathbb{R}_+\times\mathbb{B}^2.$$

Finally, let F be the mild solution of the problem (Σ) obtained in Proposition 3.1. Since $F \geq 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{B}^2$ is measurable, one can apply the Fubini theorem to show that

$$\begin{aligned}
m(t,s,v) :&= \int_{\mathbb{R}^2} F(t,s,x,v)dx \\
&= \mathbb{1}_{t<s}\sigma e^{-\sigma t} p(|v|\, t) \int_{\mathbb{R}^2} f^{in}(x-tv,v)dx \\
&+ \mathbb{1}_{s<t}\sigma e^{-\sigma s} p(|v|\, s) \int_0^\infty \int_{\mathbb{R}^2} T_k\left(F\right)(t-s,\tau,x-sv,v)dxd\tau \\
&= \mathbb{1}_{t<s}\sigma e^{-\sigma t} p(|v|\, t) \int_{\mathbb{R}^2} f^{in}(y,v)dy \\
&+ \mathbb{1}_{s<t}\sigma e^{-\sigma s} p(|v|\, s) \int_0^\infty \int_{\mathbb{R}^2} T_k\left(F\right)(t-s,\tau,y,v)dyd\tau \\
&= \mathbb{1}_{t<s}\sigma e^{-\sigma t} p(|v|\, t) \int_{\mathbb{R}^2} f^{in}(y,v)dy \\
&+ \mathbb{1}_{s<t}\sigma e^{-\sigma s} p(|v|\, s) \int_0^\infty \int_{\mathbb{B}^2} k(v,w) \int_{\mathbb{R}^2} F(t-s,\tau,y,w)dydwd\tau \\
&= \mathbb{1}_{t<s}\sigma e^{-\sigma t} p(|v|\, t)\mu^{in}(v) \\
&+ \mathbb{1}_{s<t}\sigma e^{-\sigma s} p(|v|\, s) \int_0^\infty T_k(m)(t-s,\tau)d\tau\,,
\end{aligned}$$

where the second equality follows from the substitution $y = x - tv$ that leaves the Lebesgue measure invariant. In other words,

$$m \text{ satisfies the same integral equation as } \mu.$$

Now the solution f_ε of (Ξ_ε) satisfies

$$f_\varepsilon \geq 0 \text{ a.e. on } \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{B}^2 \text{ and } \iint_{\mathbb{R}^2\times\mathbb{B}^2} f_\varepsilon(t,y,v)dydv \leq \iint_{\mathbb{R}^2\times\mathbb{B}^2} f^{in}(y,v)dydv\,,$$

which implies by Theorem 2.2 that

$$\int_{|y|\leq R}\int_{\mathbb{B}^2} f_\varepsilon(t,y,v)dvdy \rightharpoonup \int_0^{+\infty}\int_{|y|\leq R}\int_{\mathbb{B}^2} F(t,s,y,v)dvdyds\,.$$

Hence, by Fatou's lemma

$$\begin{aligned}
\int_0^{+\infty}\int_{|y|\leq R}\int_{\mathbb{B}^2} F(t,s,y,v)dvdyds &\leq \varliminf_{\varepsilon\to 0^+} \iint_{\mathbb{R}^2\times\mathbb{B}^2} f_\varepsilon(t,x,v)dxdv \\
&\leq \iint_{\mathbb{R}^2\times\mathbb{B}^2} f^{in}(y,v)dydv\,,
\end{aligned}$$

a.e. in $t \geq 0$.

Letting $R \to +\infty$ in the inequality above, we see that $m \in L^\infty(\mathbb{R}_t^+; L^1(\mathbb{R}_s^+ \times \mathbb{B}^2))$ and so that for each $T > 0$

$$m \in \mathcal{V}_T.$$

Therefore, since the mild solution of the Renewal PDE is unique and nonegative a.e. on $\mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{B}^2$, one has

$$\iiint_{\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2} F(t,s,x,v)dsdxdv = \iint_{\mathbb{R}_+\times\mathbb{B}^2} \mu(t,s,v) \quad \text{a.e. in } t \in \mathbb{R}_+.$$

$\square$

4.2. **The total mass in the vanishing ε limit.** By Theorem 2.2, the solution f_ε of (Ξ_ε) satisfies

$$\{f_\varepsilon\} \rightharpoonup \int_0^{+\infty} F ds \text{ in } L^\infty(\mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{B}^2) \text{ weak-}*;$$

therefore, checking that

$$\iint_{\mathbb{R}^2\times\mathbb{B}^2} \{f_\varepsilon\} dxdv \rightharpoonup \int_0^{+\infty} \iint_{\mathbb{R}^2\times\mathbb{B}^2} F dxdvds =: M(t)$$

reduces to proving that there is no mass loss at infinity in the x variable.

Lemma 4.1. *Under the same assumptions as in Theorem 2.2*

$$\iint_{Z_\varepsilon\times\mathbb{B}^2} f_\varepsilon(t,x,v)dxdv = \iint_{\mathbb{R}^2\times\mathbb{B}^2} \{f_\varepsilon\}(t,x,v)dxdv \to M(t)$$

strongly in $L^1_{loc}(\mathbb{R}_+)$ *as* $\varepsilon \to 0^+$.

Proof. Going back to the proof of Proposition 3.1 (whose notations are kept in the present discussion), we have seen that

$$F_\varepsilon = \sum_{n\geq 0} \mathcal{T}^n F_{2,\varepsilon} \quad \text{on } \mathbb{R}_+ \times \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{B}^2\,,$$

with the notation

$$F_{2,\varepsilon}(t,s,x,v) = \mathbb{1}_{t<\varepsilon\tau_\varepsilon(\frac{x}{\varepsilon},v)} \mathbb{1}_{t<s} \sigma e^{-\sigma s} f^{in}(x-tv,v)\,.$$

Since $\mathcal{T}\Phi \geq 0$ a.e. whenever $\Phi \geq 0$ a.e., the formula above implies that

$$F_\varepsilon \leq G := \sum_{n\geq 0} \mathcal{T}^n G_2 \text{ a.e. in } (t,s,x,v) \in \mathbb{R}_+ \times \mathbb{R}_+ \times Z_\varepsilon \times \mathbb{B}^2\,,$$

where

$$G_2(t,s,x,v) := \mathbb{1}_{t<s} \sigma e^{-\sigma s} f^{in}(x-tv,v)\,.$$

Thus, G satisfies the integral equation

$$G = G_2 + \mathcal{T}G$$

meaning that G is the mild solution of

$$\begin{cases} (\partial_t + v\cdot\nabla_x + \partial_s)G = -\sigma G\,, & t,s>0\,,\ (x,v)\in\mathbb{R}^2\times\mathbb{B}^2\,, \\ G(t,0,x,v) = \sigma \displaystyle\int_0^{+\infty} KG(t,s,x,v)ds\,, & t>0\,,\ (x,v)\in\mathbb{R}^2\times\mathbb{B}^2\,, \\ G(0,s,x,v) = f^{in}(x,v)\sigma e^{-\sigma s}\,, & s>0\,,\ (x,v)\in\mathbb{R}^2\times\mathbb{B}^2\,, \end{cases}$$

Reasoning as in Proposition 3.1 shows that

$$g(t,x,v) := \int_0^{+\infty} G(t,s,x,v)ds$$

is the solution of the linear Boltzmann equation

$$\begin{cases} (\partial_t + v\cdot\nabla_x)g + \sigma(g - Kg) = 0\,, & t>0\,,\ x\in\mathbb{R}^2\,,\ |v|=1\,, \\ g(0,x,v) = f^{in}(x,v)\,, & x\in\mathbb{R}^2\,,\ |v|=1\,. \end{cases}$$

In view of the assumption (1.3) bearing on f^{in}, we know that

$$G \geq 0 \text{ a.e. on } \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^2 \times \mathbb{B}^2$$

and

$$\int_0^{+\infty}\iint_{\mathbb{R}^2\times\mathbb{B}^2} G(t,s,x,v)dxdvds = \iint_{\mathbb{R}^2\times\mathbb{B}^2} g(t,x,v)dxdv$$
$$= \iint_{\mathbb{R}^2\times\mathbb{B}^2} f^{in}(x,v)dxdv$$

for each $t\geq 0$.

Summarizing, we have

$$0\leq\{F_\varepsilon\}\leq G$$

and

$$\iiint_{\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2} G(t,s,x,v)dsdxdv = \iint_{\mathbb{R}^2\times\mathbb{B}^2} f^{in}(x,v)dxdv < +\infty\,.$$

Then we conclude as follows: for each $R>0$, one has

$$\begin{aligned}
&\iint_{Z_\varepsilon\times\mathbb{B}^2} f_\varepsilon(t,x,v)dxdv - \int_0^{+\infty}\iint_{\mathbb{R}^2\times\mathbb{B}^2} F(t,s,x,v)dxdvds\\
&\qquad = \int_0^{+\infty}\int_{|x|>R}\int_{\mathbb{B}^2}\{F_\varepsilon\}(t,s,x,v)dvdxds\\
&\qquad + \int_0^{+\infty}\int_{|x|\leq R}\int_{\mathbb{B}^2}\left(\{F_\varepsilon\}-F\right)(t,s,x,v)dvdxds\\
&\qquad - \int_0^{+\infty}\int_{|x|>R}\int_{\mathbb{B}^2}\{F\}(t,s,x,v)dvdxds = I_{R,\varepsilon}(t)+II_{R,\varepsilon}(t)+III_R(t)\,.
\end{aligned}$$

First, for a.e. $t>0$, the term $I_{R,\varepsilon}(t)\to 0$ as $R\to+\infty$ uniformly in $\varepsilon>0$ since $0\leq\{F_\varepsilon\}\leq G$ and $G\in L^\infty(\mathbb{R}_+;L^1(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2))$.

Next, the term $II_{R,\varepsilon}(t)\to 0$ strongly in $L^1_{loc}(\mathbb{R}_+)$ as $\varepsilon\to 0^+$ for each $R>0$ by Lemma 3.4.

Finally, since $\{F_\varepsilon\}\rightharpoonup F$ in $L^1_{loc}(\mathbb{R}_+\times\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2)$ weak as $\varepsilon\to 0^+$, one has $0\leq\{F\}\leq G$, so that $F\in L^\infty(\mathbb{R}_+;L^1(\mathbb{R}_+\times\mathbb{R}^2\times\mathbb{B}^2))$. Hence the term $III_R(t)\to 0$ as $R\to+\infty$ for a.e. $t\geq 0$.

Thus we have proved that

$$\iint_{Z_\varepsilon\times\mathbb{B}^2} f_\varepsilon(t,x,v)dxdv \to \int_0^{+\infty}\iint_{\mathbb{R}^2\times\mathbb{B}^2} F(t,s,x,v)dxdvds$$

in $L^1_{loc}(\mathbb{R}_+)$ and therefore for a.e. $t\geq 0$, possibly after extraction of a subsequence of $\varepsilon\to 0^+$. □

4.3. **An integral equation for** $t,v\mapsto m(t,v)$**.** We recall the notation

$$m(t,v) := \int_0^\infty \mu(t,s,v)ds.$$

with μ is the unique mild solution of the renewal PDE.

Lemma 4.2. *The function m satisfies the integral equation*

$$m(t,v) = m(0,v)p\left(|v|t\right)e^{-\sigma t} + \sigma\int_0^t p\left(|v|s\right)e^{-\sigma s}T_k(m)(t-s,v)ds.$$

Proof. We apply the same method as in Proposition 3.5 for deriving the explicit representation formula for F in order to find an exact formula for m. Indeed, by the method of characteristics,

$$\begin{aligned}\mu(t,s,v) &= \mathbb{1}_{s<t}p(|v|\,s)e^{-\sigma s}\mu(t-s,0,v) + \mathbb{1}_{t<s}p(|v|\,t)e^{-\sigma t}\mu(0,s-t,v)\\ &= \mathbb{1}_{s<t}p(|v|\,s)\sigma e^{-\sigma s}\int_0^\infty T_k\left(\mu\right)(t-s,u,v)du \\ &\quad + \mathbb{1}_{t<s}p(|v|\,t)\sigma e^{-\sigma s}\int_{\mathbb{R}^2} f^{in}(x,v)dv\,.\end{aligned} \tag{4.3}$$

We notice that

$$\begin{aligned}\int_0^\infty T_k\left(\mu\right)(t-s,u,v)du &= \int_0^\infty \int_{\mathbb{B}^2} k(v,w)\mu(t-s,u,w)dwdu\\ &= \int_{\mathbb{B}^2} k(v,w)m(t-s,w)dw = T_k(m)(t-s,v).\end{aligned}$$

Therefore, integrating both sides of (4.3) in $s \in \mathbb{R}_+$ gives

$$m(t,v) = e^{-\sigma t}p(|v|\,t)m(0,v) + \int_0^t \sigma e^{-\sigma s}p\left(|v|\,s\right)T_k(m)(t-s,v)$$

a.e. in $t \geq 0$ and $v \in \mathbb{B}^2$ with

$$m(0,v) = \int_{\mathbb{R}^2} f^{in}(x,v)dx,$$

which is precisely the desired integral equation for m. $\square$

5. Asymptotic behavior of the mass in the long time limit in the irreducible case

We recall that we henceforth assume for simplicity that the operator T_k has finite rank. That means there exists $n \in \mathbf{N}$ and two finite sequences $(\phi_i)_{1\leq i\leq n} \in L^\infty(\mathbb{R}^2)$ and $(\psi_i)_{1\leq i\leq n} \in L^1(\mathbb{R}^2)$ such that for each $f \in L^1(\mathbb{R}^2)$

$$T_k(f)(v) = \sum_{i=1}^n \left(\int_{\mathbb{R}^2} f(w)\phi_i(w)dw\right)\psi_i(v). \tag{5.1}$$

In other words $T_k \in L^\infty(\mathbb{R}^2)\otimes L^1(\mathbb{R}^2) \subset \mathcal{L}\left(L^1\left(\mathbb{B}^2\right)\right)$, and

$$T_k = \sum_{i=1}^n \phi_i \otimes \psi_i \text{ in } \mathcal{L}(L^1(\mathbb{R}^2)).$$

5.1. **A system of integral equations.** Let us return to the integral equation

$$m(t,v) = m(0,v)p(|v|t)e^{-\sigma t} + \sigma\int_0^t p(|v|s)e^{-\sigma s}T_k(m)(t-s,v)$$

Define

$$\mu_i(t) := \int_{\mathbb{R}^2} m(t,v)\phi_i(v)dv,\ 1\leq i\leq n, \tag{5.2}$$

so that

$$\begin{aligned}m(t,v) &= m(0,v)p(|v|t)e^{-\sigma t} + \sigma\int_0^t p(|v|s)e^{-\sigma s}T_k(m)(t-s,v)ds\\ &= m(0,v)p(|v|t)e^{-\sigma t}\\ &+\sigma\int_0^t p(|v|s)e^{-\sigma s}\sum_{i=1}^n\left(\int_{\mathbb{R}^2}\phi_i(w)m(t-s,w)dw\right)\psi_i(v)ds\\ &= m(0,v)p(|v|t)e^{-\sigma t} + \sigma\sum_{i=1}^n\int_0^t \mu_i(t-s)p(|v|s)e^{-\sigma s}\psi_i(v)ds.\end{aligned}$$

If the asymptotic behavior of μ_i in the long time limit are known for each $i \in [|1,n|]$, then one deduces from the equality above the asymptotic behavior of $m(.,v)$. We begin with the integral equation governing the evolution of $(\mu_i)_{1\le i\le n}$.

Introduce here the matrix convolution product:

Definition 5.1. *Let $n, m, p \in \mathbf{N}$ and let F be a $n \times p$ matrix valued function and G be a $p \times m$ matrix-valued function whose entries belong to $L^1(\mathbb{R}_+)$. We define $H := F * G$ the matrix convolution product by the formula*

$$h_{ij}(t) := \sum_{k=1}^{p} \int_0^t f_{ik}(t-s) g_{kj}(s) ds, \quad \begin{matrix} 1 \le i \le n, \\ 1 \le j \le m, \end{matrix}$$

where $f_{ik}(t), g_{kj}(t)$ and $h_{ij}(t)$ designate the entries of $F(t), G(t)$ and $H(t)$ respectively.

In the case $n = p = m = 1$, we recover the classical convolution product of two functions defined on the half-line. We have thus defined the convolution exactly as we define matrix multiplication except that we convolve entries instead of multiplying them. Like matrix multiplication, matrix convolution product is associative but is noncommutative except in the scalar case $n = m = p = 1$.

Let us return to equation above:

$$m(t,v) = m(0,v)p(|v|t)e^{-\sigma t} + \sigma \sum_{i=1}^{n} \int_0^t \mu_i(t-s)p(|v|s)e^{-\sigma s}\psi_i(v)ds.$$

Multiplying both sides of the equality above by ϕ_j, integrating in $v \in \mathbb{R}^2$ and applying Fubini's Theorem leads

$$\begin{aligned} \mu_j(t) = &\int_{\mathbb{R}^2} m(0,v)p(|v|t)e^{-\sigma t}\phi_j(v)dv \\ &+ \sigma \sum_{i=1}^{n} \int_0^t \mu_i(t-s) \left(\int_{\mathbb{R}^2} p(|v|s)\psi_i(v)\phi_j(v)dv \right) e^{-\sigma s} ds. \end{aligned} \tag{5.3}$$

We define the matrix-valued function

$$P(t) := (p_{ij}(t))_{1\le i,j\le n} \qquad \text{for each } t \ge 0, \tag{5.4}$$

where

$$p_{ij}(t) := \int_{\mathbb{R}^2} \phi_i(v)\psi_j(v)p(|v|t)dv \text{ for each } t \ge 0,\ 1 \le i,j \le n$$

together with

$$K(t) := \sigma e^{-\sigma t} P(t) \qquad \text{for each } t \ge 0. \tag{5.5}$$

We also define

$$g(t) := \begin{pmatrix} g_1(t) \\ \vdots \\ g_n(t) \end{pmatrix} \tag{5.6}$$

with

$$g_i(t) := \int_{\mathbb{R}^2} m(0,v)p(|v|t)e^{-\sigma t}\phi_i(v)dv \qquad \text{for each } t \ge 0 \text{ and } 1 \le i \le n.$$

With these notations, (5.3) is transformed into the system of integral equations

$$\mu_j(t) = g_j(t) + \sum_{i=1}^{n} (k_{ji} * \mu_i)(t) \text{ for each } 1 \le j \le n. \tag{5.7}$$

In other words, the vector-valued function μ defined by

$$\mu(t) := \begin{pmatrix} \mu_1(t) \\ \vdots \\ \mu_n(t) \end{pmatrix}$$

is a solution of the following vector-valued integral equation

$$\mu(t) = g(t) + (K * \mu)(t). \tag{5.8}$$

The equation above is of renewal type. At variance with the scalar Renewal equation (see [11], pages?) the asymptotic behavior of the vector-valued case is much more involved, as it relies on the algebraic structure of the matrix kernel K. More precisely, if the matrix

$$\int_0^\infty K(s)ds$$

is irreducible then the asymptotic behavior of μ is homogeneous (see [8],[10] or [13]). But dropping the irreducibility assumption may lead to a much intricate behavior, as we shall see below (see also [9] for an other example).

So that, the present section is devoted to the study of the asymptotic behavior of (μ) in the long time limit with the further assumption if irreducibility of the operator T_k. The evolution of (μ) when T_k is reducible is discussed in the next section.

For each measurable $A \subseteq \mathbb{B}^2$, we denote

$$I_A := \left\{ f \in L^1\left(\mathbb{B}^2\right) | \text{supp } f \subseteq A \right\},$$

and denote by λ the uniform probability measure on $\mathbb{B}^2$. In other words, for each measurable $A \subseteq \mathbb{B}^2$,

$$\lambda(A) := \int_{v \in A} dv.$$

We next give the definition of irreducibility for an operator of $L^1\left(\mathbb{B}^2\right)$.

Definition 5.2. *Let $T \in \mathcal{L}\left(L^1\left(\mathbb{B}^2\right)\right)$. The operator T is irreducible if and only if*

$$T\left(I_A\right) \subseteq I_A \Leftrightarrow \lambda(A) \in \{0, 1\}.$$

We call reducible an operator that is not irreducible.

Notice that T_k being irreducible is equivalent to

$$\int_{\mathbb{B}^2 \setminus \Omega} \left(\int_\Omega k(w,v)dw \right) dv > 0$$

holding for every subset $\Omega \subset \mathbb{B}^2$ satisfying

$$0 < \lambda\left(\Omega\right) < 1.$$

We establish in the present section

Proposition 5.3. *Assume that T_k is irreducible. Then there exists $\xi_\sigma \in (-\sigma, 0)$ and $c_i > 0$ for each $i \in [|1, n|]$ such that*

$$\mu_i(t) \sim c_i e^{\xi_\sigma t} \text{ as } t \to +\infty.$$

Before going further, we establish a lemma about T_k. We will establish that assumption (0.3) excludes precisely the possibility of T_k nilpotent.

Lemma 5.4. *Let $T_k \in \mathcal{L}\left(L^1\left(\mathbb{B}^2\right)\right)$ be as in (0.3). Then for each nonnegative $f \in L^1\left(\mathbb{B}^2\right)$*

$$\int_{\mathbb{B}^2} T_k(f)(v)dv = \int_{\mathbb{B}^2} f(v)dv.$$

Proof. Denote $L^1_+(\mathbb{R}^2)$ the cone of nonnegative integrable function. Since the kernel is nonnegative, $T_k\left(L^1_+(\mathbb{R}^2)\right) \subset L^1_+(\mathbb{R}^2)$ and for each $f \in L^1_+(\mathbb{R}^2)$, by the Fubini-Tonelli Theorem

$$\begin{aligned}\int_{\mathbb{R}^2} T_k\,(f)\,(v)dv &= \int_{\mathbb{R}^2}\left(\int_{\mathbb{R}^2} k(v,w)f(w)dw\right)dv,\\ &= \int_{\mathbb{R}^2} f(w)dw.\end{aligned}$$

□

5.2. **A renewal theorem for vector-valued equations.**

5.2.1. *The Banach space $\mathcal{M}_n\left(L^1\left(\mathbb{R}\right)\right)$ and the renewal equation.* We denote $\mathcal{M}_n(\mathbb{R})$ the real $n\times n$ square matrix algebra and $\mathcal{M}_n(\mathbb{R}_+)$ the subset of matrices whoses entries are nonnegative. Notice that it is a cone, i.e. a closed convex set K such that $\lambda K \subset K$ for all $\lambda \geq 0$. Moreover it is proper: $K \cap (-K) = \{0\}$. As proper cone, $\mathcal{M}_n(\mathbb{R}_+)$ induces a partial order $\leq$ on $\mathcal{M}_n(\mathbb{R})$ by the following rule:

$$\text{For each } A, B \in \mathcal{M}_n(\mathbb{R}),\ A \leq B \text{ if and only if } B - A \in \mathcal{M}_n(\mathbb{R}_+).$$

In the same way, we denote $A < B$ if and only if $B - A$ is a matrix whose each entry is positive.
As matrix norm, we take

$$\|A\|_{\infty,\infty} := \sup_{1\leq i\leq n}\sum_{j=1}^{n}|a_{ij}|.$$

Define now $\mathcal{M}_n\left(L^1\left(\mathbb{R}\right)\right)$ the vector space of matrices whoses entries are integrable functions defined on $\mathbb{R}_+$ with values in $\mathbb{R}$. It is a Banach space for the norm

$$|||F|||_1 := \sup_{1\leq i\leq n}\sum_{j=1}^{n}\int_0^\infty |f_{ij}(s)|\,ds.$$

Moreover, we notice that for each $F, G \in \mathcal{M}_n\left(L^1\left(\mathbb{R}\right)\right)$ we have

$$\begin{aligned}|||F*G|||_1 &= \sup_{1\leq i\leq n}\sum_{j=1}^{n}\int_0^\infty\left|\sum_{k=1}^{n}\int_0^t f_{ik}(t-s)g_{kj}(s)ds\right|dt,\\ &\leq \sup_{1\leq i\leq n}\sum_{j=1}^{n}\int_0^\infty\sum_{k=1}^{n}\int_0^t |f_{ik}(t-s)|\,|g_{kj}(s)|\,dsdt,\\ &\leq \sup_{1\leq i\leq n}\sum_{j=1}^{n}\int_0^\infty |f_{ik}(s)|\,ds\int_0^\infty |g_{kj}(s)|\,ds.\end{aligned}$$

In other words

$$\begin{aligned}|||F*G|||_1 &\leq \left\|\int_0^\infty F(s)ds\int_0^\infty G(s)ds\right\|_{\infty,\infty},\\ &\leq \left\|\int_0^\infty F(s)ds\right\|_{\infty,\infty}\left\|\int_0^\infty G(s)ds\right\|_{\infty,\infty} = |||F|||_1\,|||G|||_1\,,\end{aligned}\tag{5.9}$$

since $\|\cdot\|_{\infty,\infty}$ is a norm for the Banach algebra $\mathcal{M}_n\left(\mathbb{R}\right)$.

Hence $\left(\mathcal{M}_n\left(L^1\left(\mathbb{R}\right)\right), *, |||.|||_1\right)$ is a Banach algebra. Eventually, introduce $\mathcal{M}_n\left(L^\infty\left(\mathbb{R}\right)\right)$ the Banach space of matrices whose entries are essentially bounded functions with the norm

$$|||F|||_\infty = \sup_{1\leq i,j\leq n}\sup_{t\in\mathbb{R}_+}|f_{ij}(t)|\,.$$

We check easily that for each $F \in \mathcal{M}_n\left(L^1(\mathbb{R})\right)$ and $G \in L^\infty(\mathbb{R}_+; \mathcal{M}_n(\mathbb{R}))$ we have $F * G, G * F \in \mathcal{M}_n\left(L^1(\mathbb{R})\right)$ with

$$|||F * G|||_1 \leq |||F|||_1 \, |||G|||_\infty ,$$

and

$$|||G * F|||_1 \leq |||G|||_\infty \, |||F|||_1 .$$

We henceforth denote for each $F \in \mathcal{M}_n\left(L^1(\mathbb{R})\right)$

$$\int_0^\infty F(s)ds := \left(\int_0^\infty f_{ij}(s)ds\right)_{1 \leq i,j \leq n} .$$

Now $(L^p(\mathbb{R}_+))$ with $p \in [1, +\infty]$ denotes the vector spaces of vectors whose entries are L^p-integrable functions defined on $\mathbb{R}_+$ with values in $\mathbb{R}$. It is a Banach space for the norm

$$\|f\|_{\infty,p} := \sup_{1 \leq j \leq n} \|f_j\|_{L^p(\mathbb{R}_+)} .$$

With the Young inequality and a similar argument as in 5.9, we have for each $F \in \mathcal{M}_n\left(L^1(\mathbb{R})\right)$ and for each $f \in (L^p(\mathbb{R}_+))$ with $p \in [1, +\infty]$, $F * f \in (L^p(\mathbb{R}_+))$ with

$$\|F * f\|_{\infty,p} \leq |||F|||_1 \, \|f\|_{\infty,p} . \tag{5.10}$$

Let $A \in \mathcal{M}_n(\mathbb{R})$, we designate its spectral radius with $\rho(A)$. We now give

Proposition 5.5. *Let F be in $\mathcal{M}_n\left(L^1(\mathbb{R})\right)$ with $F \geq 0$ such that*

$$\rho\left(\int_0^\infty F(s)ds\right) < 1$$

and let g be in $(L^p(\mathbb{R}_+))^n$ with $p \in [1, +\infty]$ then the renewal equation

$$f = g + F * f$$

has a unique solution in $(L^p(\mathbb{R}_+))^n$ that is

$$\sum_{n \geq 0}^{\infty} F^{*n} * g,$$

with $F^{*0} := \delta I_n$ where δ is the identity element for the scalar convolution product and I_n the matrix identity and $F^{*n} := \underbrace{F * F * \cdots * F}_{n\ factors}$.

Proof. For each $N \geq 0$ one has

$$\begin{aligned}
\left\| \sum_{n \geq 0}^{N} F^{*n} * g \right\|_{\infty,p} &\leq \sum_{n \geq 0}^{N} \|F^{*n} * g\|_{\infty,p} \\
&\leq \sum_{n \geq 0}^{N} |||F^{*n}|||_1 \, \|g\|_{\infty,p} \text{ by (5.10)} \\
&\leq \|g\|_{\infty,p} \sum_{n \geq 0}^{N} \left\| \left(\int_0^\infty F(s)ds\right)^n \right\|_{\infty,\infty} \text{ by (5.9).}
\end{aligned}$$

Noticing that

$$\left(\left\| \left(\int_0^\infty F(s)ds\right)^n \right\|_{\infty,\infty}\right)^{\frac{1}{n}} \to \rho\left(\int_0^\infty F(s)ds\right) < 1 \text{ as } n \to +\infty,$$

one concludes by the root test that the series

$$\sum_{n\geq 0}^{N} F^{*n} * g$$

converges absolutely in $\left(L^p\left(\mathbb{R}_+\right)\right)^n$ and so that

$$\sum_{n\geq 0}^{\infty} F^{*n} * g \in \left(L^p\left(\mathbb{R}_+\right)\right)^n .$$

Now, one observe that

$$\begin{aligned}\sum_{n\geq 0}^{\infty} F^{*n} * g \in \left(L^p\left(\mathbb{R}_+\right)\right)^n &= (\delta I_n) * g + \sum_{n\geq 1}^{\infty} F^{*n} * g \\ &= g + f * \left(\sum_{n\geq 0}^{\infty} F^{*n} * g\right)\end{aligned}$$

hence, it a solution of the Renewal equation. Finally, for the uniqueness. Assume that f_1 and f_2 are solution, so that $f := f_1 - f_2$ is a solution of

$$f = F * f$$

and verifies for each $n \geq 1$

$$f = F^{*n} * f.$$

That implies that for each $n \geq 1$

$$\|f\|_{\infty,p} \leq \left\| \left(\int_0^\infty F(s)ds\right)^n \right\|_{\infty,\infty} \|f\|_{\infty,p} .$$

Since

$$\rho\left(\int_0^\infty F(s)ds\right) < 1$$

one has

$$\left\| \left(\int_0^\infty F(s)ds\right)^n \right\|_{\infty,\infty} \to 0 \text{ as } n \to +\infty,$$

so that

$$\|f\|_{\infty,p} = 0,$$

meaning that $f_1 = f_2$ in $\left(L^p\left(\mathbb{R}_+\right)\right)^n$. □

With a similar argument, one has

Proposition 5.6. *Let F be in $\mathcal{M}_n\left(L^1\left(\mathbb{R}\right)\right)$ with $F \geq 0$ such that*

$$\rho\left(\int_0^\infty F(s)ds\right) < 1$$

and let g be in $\left(L^p_{loc}\left(\mathbb{R}_+\right)\right)^n$ with $p \in [1, +\infty]$ then the renewal equation

$$f = g + F * f$$

has a unique solution in $\left(L^p_{loc}\left(\mathbb{R}_+\right)\right)^n$ that is

$$\sum_{n\geq 0}^{\infty} F^{*n} * g.$$

5.2.2. *Irreducibility.* Recall now the notion of irreducibility for matrix.

Definition 5.7. *A square matrix M is said reducible if and only if there exists a permutation matrix P such that*

$$PMP^T = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$

with A,B and C are matrices. If M is not reducible, it is said irreducible.

We give here two criteria characterizing the reducibility of a matrix.

Criterion 1. *A matrix A is reducible if and only if there exists a non trivial partition I, J of $[|1, n|]$ such that for each $(i, j) \in I \times J$ one has $a_{ij} = 0$.*

For the other criterion, we first give a definition.

Definition 5.8. *Let A a matrix and $i, j \in 1, \cdots, n$. One said that there is a chain joining i and j if and only if there exists a finite sequence $(k_l)_{l=1}^{p} \in [|1, n|]$ with $p \leq n$,$k_0 = i$ and $k_p = j$ such that $a_{k_l k_{l+1}} \neq 0$ for every $0 \leq l < p$. A such sequence (k_l) is denoted C_{ij} and is called a chain joining i and j.*

The second criterion is then

Criterion 2. *A matrix A is irreducible if and only if for each $i, j \in [|1, n|]$, there exists a chain joining i and j.*

We recall then the Perron-Frobenius Theorem

Theorem 5.9. *Let A be an irreducible matrix with nonnegative entries then*

(1) *$\rho(A)$ is a simple eigenvalue of A and there exists two vectors x, y with positive entries*
$$Ax = \rho(A)x$$
and
$$(y)^T A = \rho(A)(y)^T.$$
(2) *Let $B \in \mathcal{M}_n(\mathbb{R}_+)$ such that $0 \leq B \leq A$ then $\rho(B) \leq \rho(A)$. Moreover $B \neq A$ implies $\rho(B) < \rho(A)$.*

5.2.3. *The Renewal Theorem.* Let $M \in \mathcal{M}_n\left(L^1(\mathbb{R}_+)\right)$, we denote

$$M_\infty := \left(\int_0^\infty m_{ij}(s)ds\right)_{1 \leq i,j \leq n}.$$

One assumes M_∞ is irreducible with $\rho(M_\infty) = 1$. By The Perron-Frobenius Theorem, there exists a positive right eigenvector m_r and a positive left eigenvector m_l such that

$$M_\infty m_r = m_r$$

and

$$m_l M_\infty = m_l.$$

We construct a matrix-valued renewal measure H by the formula

$$H(dt) := \sum_{n=0} M^{*n}(t)dt,$$

and each solution of the renewal equation

$$x(t) = y(t) + (M * x)(t) \tag{5.11}$$

with $y \in L^1(\mathbb{R}_+; \mathbb{R}^n)$, has the form

$$x(t) := \int_0^t H(ds)y(t-s), \forall t \geq 0.$$

Theorem 5.10. *Under the assumptions and with the notations above, the matrix-valued renewal measure H is decomposed as*

$$H = H_1 + H_2,$$

where H_2 is a finite matrix-valued measure on $\mathbb{R}_+$ and $H_1(dt) := (h_{ij}(t))_{i,j=1}^n dt$ where for each $1 \le i, j \le n$, h_{ij} is bounded and continuous with

$$\lim_{t\to+\infty} (h_{ij}(t))_{i,j=1}^n = \frac{m_r m_l}{m_l \left(\int_0^\infty sM(s)ds\right) m_r}. \tag{5.12}$$

In other words, for each $f \in L^1(\mathbb{R}_+;\mathbb{R}^n)$

$$\int_0^t H(ds)f(t-s) = (H_1 * f)(t) + \int_0^t H_2(ds)f(t-s) \ \forall t \ge 0$$

with $\int_0^\infty H_2(ds) < +\infty$.

For the proof of this theorem, see for instance [10, 13]. Any decomposition above is known as a Stone decomposition. It is not unique but the limit (5.12) is unique. It enables to remove the directly Riemann integrable assumption (see [11], pp. 348-349) and gives us an short proof of the following renewal theorem

Theorem 5.11. *Assume moreover that $y \in L^\infty(\mathbb{R}_+;\mathbb{R}^n)$ and $\lim_{t\to+\infty} y(t) = 0$. Then*

$$\lim_{t\to+\infty} x(t) = D_\infty \int_0^\infty y(s)ds$$

with

$$D_\infty := \frac{m_r m_l}{m_l \left(\int_0^\infty sM(s)ds\right) m_r}.$$

Proof. We have for each $t \ge 0$

$$x(t) = \int_0^t H(ds)y(t-s).$$

And let (H_1, H_2), a Stone decomposition, then

$$x(t) = (H_1 * y)(t) + \int_0^t H_2(ds)y(t-s).$$

First, since $y \in L^\infty(\mathbb{R}_+;\mathbb{R}^n)$, there exists $0 < C < \infty$ such that $|y(t)| \le C$ a.e. in $t \in \mathbb{R}_+$. Since H_2 is finite measure, $t \mapsto C \in L^1_{H_2}(\mathbb{R}_+;\mathbb{R}^n)$ so that $(y(t-s))_{t\ge s}$ is dominated by a H_2-integrable function. Besides $y(t) \to 0$ as $t \to +\infty$. Consequently, one has by dominated convergence that

$$\int_0^t H_2(ds)y(t-s) \to 0 \text{ as } t \to +\infty.$$

We establish with a similar argument that

$$\lim_{t\to+\infty} (H_1 * y)(t) = \frac{m_r m_l}{m_l \left(\int_0^\infty sM(s)ds\right) m_r} \int_0^\infty y(s)ds.$$

□

5.3. **The vector-valued intergral equation** (5.8) **and the matrix kernel** K. We have noticed that the essential assumption for the Renewal Theorem is the irreducibility of the matrix M_∞. Thus, we have to discuss the long-time limit behavior of the mass according the irreducibility of the matrix kernel K.

5.3.1. *The irreducibility of* $P(0)$. We recall that we have assumed in the present section that the operator $T_k \in \mathcal{L}\left(L^1\left(\mathbb{B}^2\right)\right)$ is irreducible (see Definition 5.2). We now give the relation between the two notions of irreducibility presented above.

Proposition 5.12. *Let* T_k *and* P *be as in* (5.1) *and* (5.4).*The following statements are equivalent.*

(1) *the matrix* $P(0)$ *is irreducible,*
(2) *the operator* T_k *is irreducible.*

Proof. We first show that (2) implies (1). Indeed, assume that $P(0)$ is reducible. By criterion 1, there exists a non trivial partition I, J of $[|1,n|]$ such that

$$p_{ij}(0) = \int_{\mathbb{B}^2} \phi_i \psi_j = 0 \text{ for } (i,j) \in I \times J.$$

We define

$$A := \bigcup_{j \in J} \text{supp } \psi_j$$

and

$$B := \bigcup_{i \in I} \text{supp } \phi_i.$$

As $(\psi_j, \phi_i)_{1 \leq i,j \leq n}$ are nonnegative,

$$\int_{\mathbb{B}^2} \phi_i(v)\psi_j(v)dv = 0$$

for $(i,j) \in I \times J$ implies that $\lambda\left(A \cap B\right) = 0$. Besides, $0 < \lambda\left(A\right) < 1$. That being said, we have for each $f \in I_A$

$$\begin{aligned}
T_k\left(f\right) &= \left(\sum_{i=1}^{n} \phi_i \otimes \psi_i\right)\left(f\right) \\
&= \sum_{i=1}^{n} \left(\int_{\mathbb{B}^2} \phi_i(v)f(v)dv\right)\psi_i \\
&= \sum_{i\in I} \left(\int_{\mathbb{B}^2} \phi_i(v)f(v)dv\right)\psi_i + \sum_{i\in J} \left(\int_{\mathbb{B}^2} \phi_i(v)f(v)dv\right)\psi_i \\
&= \sum_{i\in J} \left(\int_{\mathbb{B}^2} \phi_i(v)f(v)dv\right)\psi_i \in I_A.
\end{aligned}$$

Therefore, T_k is reducible, and by contraposition, if T_k is irreducible, the matrix $P(0)$ is irreducible.

Now show that the irreducibility of $P(0)$ implies the one of T_k. Assume that T_k is reducible. There exists $A \subset \mathbb{B}^2$ with $0 < \lambda(A) < 1$ such that

$$T_k\left(I_A\right) \subseteq I_A.$$

In particular, we have

$$\begin{aligned}
T_k\left(\mathbb{1}_A\right) &= \left(\sum_{i=1}^{n} \phi_i \otimes \psi_i\right)\left(\mathbb{1}_A\right) = \sum_{i=1}^{n} \left(\int_A \phi_i(v)dv\right)\psi_i \\
&= \sum_{i\in I} \left(\int_A \phi_i(v)dv\right)\psi_i,
\end{aligned}$$

with

$$I := \left\{i \in [|1,n|] | \lambda\left(A \cap supp\ \phi_i\right) > 0\right\}.$$

Define

$$J := [|1,n|] \setminus I.$$

These set I, J form a partition of $[|1,n|]$. Since $T_k(\mathbb{1}_A) \in I_A$ and $\left(\int_A \phi_i(v)dv\right) > 0$ for each $i \in I$, we have *supp* $\psi_i \subseteq A$ for each $i \in I$ and $\int_{\mathbb{B}^2} \phi_j(v)\psi_i(v)dv = 0$ for each $j \in J$. In other words,

$$\text{for each } i,j \in I \times J \ p_{ij}(0) = 0.$$

Consequently, by Criterion 1, if I, J is a nontrivial partition of $[|1,n|]$, the matrix $P(0)$ is reducible. Therefore it remains to show that I, J is nontrivial, meaning that $I \neq \varnothing$ and $I \neq [|1,n|]$.
If I was empty, it would imply that

$$T_k(\mathbb{1}_A) = 0$$

but we have

$$\int_{\mathbb{B}^2} T_k(\mathbb{1}_A) = \lambda(A) > 0.$$

If $I = [|1,n|]$, it would imply that $\int_A \phi_i(v)dv > 0$ for each $i \in [|1,n|]$ and, since $T_k(\mathbb{1}_A) \in I_A$, for each $i \in [|1,n|]$, *supp* $\psi_i \subseteq A$. Yet, the assumptions

$$\int_{\mathbb{B}^2} k(v,w)dv = 1$$

and

$$k(v,w) = k(w,v), \quad \text{for each } (v,w) \in \mathbb{B}^2$$

imply

$$\int_{\mathbb{B}^2} k(v,w)dw = 1.$$

In other words

$$\int_{\mathbb{B}^2} k(v,w)dw = \sum_{i=1}^{n} \left(\int_{\mathbb{B}^2} \phi_i(w)dw\right)\psi_i(v) = \sum_{i=1}^{n} \alpha_i\psi_i(v) = 1 \text{ for each } w \in \mathbb{B}^2.$$

This is obviously incompatible with the condition *supp* $\psi_i \subseteq A$ for each $i \in [|1,n|]$. Therefore, $I \neq \varnothing$ and $I \neq [|1,n|]$, so that I, J is a nontrivial partition of $[|1,n|]$. □

Lemma 5.13. *Assume that $P(0)$ is irreducible, then $P(t), K(t)$ are irreducible for each $t > 0$, and $\int_0^\infty e^{-\xi t}K(t)dt$ is irreducible for each $\xi > -\sigma$.*

Proof. We recall that the function $(t,v) \mapsto p(|v|t)$ is positive and that $(\psi_i, \phi_i)_{1\leq i\leq n}$ are nonnegative. As

$$p_{ij}(t) = \int_{\mathbb{B}^2} \phi_i(v)\psi_j(v)p(|v|t)dv.$$

for each $t \geq 0$, $i,j \in [|1,n|]$, one has $p_{ij}(0) \neq 0$ if and only if $p_{ij}(t) \neq 0$ for each $t \geq 0$, $i,j \in [|1,n|]$.

Assume now that $P(0)$ is irreducible, there exists for each i,j a chain C_{ij} joining i and j, hence by the remark above this very chain C_{ij} also joins i and j in $P(t)$ for each $t \geq 0$. Therefore by Criterion 2, $P(t)$ is irreducible for each $t > 0$.

Finally, we establish that $K(t)$ and $\int_0^\infty e^{-\xi t}K(t)dt$ are irreducible for each $t > 0, \xi > -\sigma$ with a similar argument. □

Proposition 5.14. *Let T_k and $P(0)$ be as in* (5.1) *and* (5.4). *Assume moreover that T_k is irreducible, then*

$$\rho\left(\int_0^\infty K(t)dt\right) < 1.$$

Proof. By Proposition 5.12 the irreducibility of T_k entails the irreducibility of $P(0)$. We shall show that

$$\rho\left(\int_0^\infty K(t)dt\right) < \rho(P(0)).$$

Notice that for each $v \in \mathbb{B}^2 \setminus \{0\}$, and we recall that $\dot{p}(t) < 0$ for each $t \geq 0$. Therefore for each $i, j \in [|1, n|]$ such that $p_{ij}(0) \neq 0$ the function p_{ij} is decreasing with $\dot{p}_{ij}(t) < 0$ for each $t \geq 0$. We have

$$\begin{aligned} \int_0^\infty k_{ij}(t)dt &= \int_0^\infty \sigma e^{-\sigma t} p_{ij}(t)dt \\ &= \int_0^\infty e^{-\sigma t}\dot{p}_{ij}(t)dt + p_{ij}(0) \\ &\leq p_{ij}(0). \end{aligned}$$

The inequality above is strict if $p_{ij}(0) \neq 0$ since the term $\int_0^\infty e^{-\sigma t}\dot{p}_{ij}(t)dt$ is negative. So that

$$0 \leq \int_0^\infty K(t)dt \leq P(0),$$

with

$$\int_0^\infty K(t)dt \neq P(0).$$

That implies by the Perron-Frobenius theorem that

$$\rho\left(\int_0^\infty K(t)dt\right) < \rho(P(0)).$$

We now establish that

$$\rho(P(0)) = 1.$$

As $P(0)$ is irreducible, by the Perron-Frobenius Theorem, there exists a vector x whose entries are positive such that

$$P(0)x = \rho(P(0))x.$$

We have for each $j \in [|1, n|]$

$$\begin{aligned} T_k\left(\psi_j\right) &= \sum_{i=1}^n \left(\int_0^\infty \phi_i(v)\psi_j(v)dv\right)\psi_i \\ &= \sum_{i=1}^n p_{ij}(0)\psi_i. \end{aligned}$$

As $\int_{\mathbb{B}^2} T_k\left(f\right) = \int_{\mathbb{B}^2} f$ for each nonnegative function, the equality above entails that

$$\sum_{i=1}^n \left(\int_{\mathbb{B}^2} \psi_i(v)dv\right) p_{ij}(0) = \int_{\mathbb{B}^2} \psi_j(v)dv.$$

In other words,

$$y^T P(0) = y^T$$

with

$$y := \left(\int_{\mathbb{B}^2} \psi_i(v)dv\right)_{1 \leq i \leq n}.$$

That implies that

$$\rho\left(P(0)\right) y^T x = y^T x > 0$$

since $y^T x$ is a matrix whose entries are positive. Thus

$$\rho\left(P(0)\right) = 1.$$

$\square$

Henceforth we denote

$$H(t) := \sum_{n=0}^{\infty} K^{*n}(t) \text{ for each } t \geq 0. \tag{5.13}$$

5.3.2. *An explicit representation formula for μ.*

Proposition 5.15. *Under the assumptions and with the notations above, Equation* (5.8) *has a unique solution in $L^1(\mathbb{R}_+;\mathbb{R}^n)$ that is $H * g$.*

Proof. It is an immediate consequence of Proposition 5.14 and Proposition 5.5. □

5.4. **The characteristic exponent ξ_σ.**

Proposition 5.16. *Let T_k and P be as in* (5.1) *and* (5.4) *and let*

$$K(t) := \sigma e^{-\sigma t} P(t)$$

for each $t \geq 0$. Assume moreover that T_k is irreducibleo then for each $\sigma > 0$, the equation

$$\rho\left(\int_0^\infty e^{-\xi s} K(s) ds\right) = 1$$

with unknown ξ has a unique real solution ξ_σ. Moreover, one has

$$-\sigma < \xi_\sigma < 0.$$

Proof. Consider the Laplace transform of the matrix-valued function

$$\begin{aligned}\mathcal{L}\left[K\right](\xi) &:= \left(\mathcal{L}\left[k_{ij}\right](\xi)\right)_{1\leq i,j\leq n} \\ &= \left(\int_0^\infty \sigma e^{-(\sigma+\xi)t} p_{ij}(t) dt\right)_{1\leq i,j\leq n}.\end{aligned}$$

We define

$$\lambda : (-\sigma,+\infty) \ni \xi \mapsto \rho(\mathcal{L}\left[K\right](\xi)) \in \mathbb{R}_+. \tag{5.14}$$

To prove the proposition, we must show that there exists a unique $\xi_\sigma \in (-\sigma, 0)$ such that

$$\lambda\left(\xi_\sigma\right) = 1.$$

As $0 \leq p \leq 1$ and the functions $(\phi_i,\psi_i)_{1\leq i\leq n} \in L^\infty\left(\mathbb{B}^2\right) \times L^1\left(\mathbb{B}^2\right)$ and are non-negative, the function

$$\mathcal{L}\left[k_{ij}\right](\xi)$$

is of class C^1 on $(-\sigma,+\infty)$ for each $i,j \in [|1,n|]$, and

$$\frac{d}{d\xi}\mathcal{L}\left[k_{ij}\right] = -\int_0^\infty \sigma e^{-(\sigma+\xi)t} t \int_{\mathbb{R}^2} p(|v|t)\phi_i(v)\psi_j(v)dv \leq 0,$$

with strict inequality whenever $p_{ij}(0) \neq 0$. Therefore one has, for each $\xi_1 < \xi_2$,

$$\mathcal{L}\left[K\right](\xi_2) \leq \mathcal{L}\left[K\right](\xi_1),$$

with

$$\mathcal{L}\left[K\right](\xi_2) \neq \mathcal{L}\left[K\right](\xi_1),$$

and $\mathcal{L}\left[K\right](\xi_1)$ being irreducible by Lemma 5.13. So that by the Perron-Frobenius Theorem

$$\rho(\mathcal{L}\left[K\right](\xi_2)) < \rho(\mathcal{L}\left[K\right](\xi_1)).$$

In other words, the function λ is decreasing on $]-\sigma,+\infty[$.

Moreover the spectral radius is continuous on $\mathcal{M}_n(\mathbb{R})$ and thus λ is also continuous on $]-\sigma,+\infty[$.

Notice that whenever $p_{ij}(0) \neq 0$,

$$k_{ij}(t)e^{-\xi t} \to 0^+ \text{ as } \xi \to +\infty,$$

for each $t > 0$ and $i, j \in [|1, n|]$ while

$$k_{ij}(t)e^{-\xi t} \leq C\sigma e^{-\sigma t}$$

where

$$C := \sup_{1 \leq i,j \leq n} p_{ij}(0)$$

Therefore one obtains by dominated convergence,

$$\mathcal{L}[K](\xi) \to 0^+ \text{ as } \xi \to +\infty,$$

and since ρ is continuous, one concludes that

$$\lambda(\xi) \to 0^+ \text{ as } \xi \to +\infty.$$

We denote

$$\tilde{K} \equiv \left(\tilde{k}_{ij}\right)_{1 \leq i,j \leq n} := \begin{cases} 1 \text{ if } p_{ij}(0) > 0, \\ 0 \text{ otherwise.} \end{cases}$$

As $P(0)$ is irreducible, $\tilde{K}$ is irreducible by a similar argument as in the proof of Lemma 5.13 and thus by the Perron-Frobenius Theorem, we have

$$\rho\left(\tilde{K}\right) > 0.$$

That being said for each $t > 0, i, j \in [|1, n|]$, whenever $p_{ij}(0) \neq 0$

$$k_{ij}(t) \uparrow \sigma p_{ij}(t) \text{ as } \xi \downarrow -\sigma^+.$$

As $(t, v) \mapsto p(|v|t)\phi_i(v)\psi_j(v)$ is nonnegative, applying the Tonelli Theorem for each (i, j) such that $p_{ij}(0) \neq 0$ shows that

$$\begin{aligned} \int_0^\infty \sigma p_{ij}(t)dt &= \sigma \int_{\mathbb{R}^2} \phi_i(v)\psi_j(v) \left(\int_0^\infty p(|v|t)dt\right) dv \\ &= +\infty \end{aligned}$$

since $p \notin L^1(\mathbb{R}_+)$. So that by monotone convergence, for each $(i, j) \in [|1, n|]$

$$\mathcal{L}[k_{ij}](\xi) \to +\infty \text{ as } \xi \downarrow -\sigma^+.$$

whenever $p_{ij}(0) \neq 0$. Hence, for each $M > 0$, there exists, ξ_0 such that $\xi < \xi_0$ implies $\mathcal{L}[k_{ij}](\xi) \geq M$ for each i, j such that $p_{ij}(0) \neq 0$. In other words, for each $M > 0$, there exists, ξ_0 such that for each $\xi < \xi_0$ one have

$$\mathcal{L}[K](\xi) \geq M\tilde{K}$$

and thus

$$\rho(\mathcal{L}[K](\xi)) \geq M\rho(\tilde{K}).$$

Therefore

$$\lambda(\xi) \to +\infty \text{ as } \xi \downarrow -\sigma^+.$$

The function λ is a continuous decreasing function on $(-\sigma, +\infty)$ with

$$\lambda(\xi) \to +\infty \text{ as } \xi \downarrow -\sigma^+,$$

and

$$\lambda(\xi) \to 0 \text{ as } \xi \uparrow +\infty,$$

so that, by the intermediate value theorem, there exists a unique $\xi_\sigma > -\sigma$ such that

$$\lambda(\xi_\sigma) = 1.$$

Finally, we show that ξ_σ is negative. By corollary 5.14, we have

$$\rho\left(\int_0^\infty K(t)dt\right) = \rho(\mathcal{L}[K](0)) = \lambda(0) < 1 = \lambda(\xi_\sigma);$$

and since λ is decreasing,

$$\xi_\sigma < 0.$$

□

5.5. The asymptotic behavior of μ in the irreducible case.

Proof. First, for each $\eta \in \mathbb{R}$ and each locally bounded measurable matrix-valued function $M : \mathbb{R} \mapsto \mathcal{M}_{n,p}(\mathbb{R})$ supported in $\mathbb{R}_+$, denote

$$ {}_\eta M := \left(e^{\eta t} m_{ij}(t)\right)_{1\le i\le n, 1\le j\le p} \text{ for each } t \in \mathbb{R}. \tag{5.15}$$

Notice that for each such M, N, we have

$$e^{\eta t}(M * N)(t) = ({}_\eta M *_\eta N)(t) \text{ for each } t \in \mathbb{R}.$$

Hence, if μ is a solution of the equation (5.8), the function $\mu_{-\xi_\sigma}$ satisfies

$$ {}_{-\xi_\sigma}\mu(t) = {}_{-\xi_\sigma} g(t) + ({}_{-\xi_\sigma}K *_{-\xi_\sigma} \mu)(t) \tag{5.16}$$

which is a system of renewal equations in the sense of Theorem 5.11. Thus, applying this theorem to the equation above shows that

$$\mu(t)e^{-\xi_\sigma t} \to m_r m_l \frac{\int_0^\infty e^{-\xi_\sigma s} g(s)ds}{m_l \left(\int_0^\infty s e^{-(\sigma+\xi_\sigma)t} P(s)ds\right) m_r}.$$

□

6. The asymptotic behavior of μ in the reducible case

6.1. Summary of the previous section and main result. We have seen in the previous section that the function m verifies

$$m(t, v) = m(0, v)p(|v|t)e^{-\sigma t} + \sigma \sum_{i=1}^{n} \int_0^t \mu_i(t - s)p(|v|s)e^{-\sigma s}\psi_i(v)ds.$$

with μ defined in (5.2). Moreover, μ is solution of the vector-valued equation

$$\mu = g + K * \mu \tag{6.1}$$

where g is defined in (5.6) and

$$K(t) := \sigma e^{-\sigma t} P(t)$$

for each $t \ge 0$ with P defined in (5.4). Besides, if T_k is irreducible, then there exists $\xi_\sigma \in (-\sigma, 0)$ and a vector c with positive entries such that

$$\mu(t) \sim e^{\xi_\sigma t} c \text{ as } t \to +\infty.$$

In other words, the irreducibility assumption entails both an exponential-type decay and a uniform behavior for $(\mu_i)_{1\le i\le n}$. In the present section, we discuss the asymptotic behavior of μ in the long time limit without the irreducibility assumption. The main result is

Proposition 6.1. *Let T_k be defined as in* (5.1) *and μ be the solution of equation* (6.1). *Then for each $i \in [|1, n|]$ there exists $c_i > 0$, $\xi_i \in (-\sigma, 0)$ and $n_i \in [|0, n|]$ such that*

$$\mu_i(t) \sim c_i t^{n_i} e^{\xi_i t} \text{ as } t \to +\infty.$$

6.2. **A new formulation for the Renewal Equation in the reducible case.** Return to the equation (6.1)

$$\mu(t) = g(t) + (K * \mu)(t),$$

for each $t \geq 0$ with P defined in (5.4). By Proposition 5.12, the reducibility of T_k entails that $P(0)$ is reducible, meaning that there exists a permutation matrix Π such that

$$\Pi P(0)\Pi^T = \begin{bmatrix} P_{11}(0) & \cdots & \cdots & P_{1k}(0) \\ 0 & P_{22}(0) & \cdots & P_{2k}(0) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{kk}(0) \end{bmatrix} \tag{6.2}$$

with $k \leq n$ and, for each $i \in [|1,k|]$, either $P_{ii}(0) = 0$ or is an irreducible matrix. Since Π is a permutation matrix $\Pi^T = \Pi^{-1}$, so that

$$Sp(P(0)) = \cup_{1\leq i\leq k} Sp(P_{ii}(0)).$$

Likewise as $p_{ij}(t) \neq 0$ if and only if $p_{ij}(0) \neq 0$, the same permutation induces

$$\Pi P(t)\Pi^T = \begin{bmatrix} P_{11}(t) & \cdots & \cdots & P_{1k}(t) \\ 0 & P_{22}(t) & \cdots & P_{2k}(t) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{kk}(t) \end{bmatrix}$$

with $P_{ii}(t) = 0$ or irreducible for each $t \geq 0$ (see the argument for Lemma 5.13). Returning to equation (6.1) and multiplying both sides by Π leads to

$$\begin{aligned} \Pi\mu(t) &= \Pi g(t) + (\Pi K * \mu)(t) \\ &= \Pi g(t) + ((\Pi K \Pi^T) * \Pi\mu)(t) \text{ for each } t \geq 0, \end{aligned} \tag{6.3}$$

with for each $t \geq 0$

$$\begin{aligned} \Pi K(t)\Pi^T &= \Pi\sigma e^{-\sigma t} P(t)\Pi^T \\ &= \sigma e^{-\sigma t}\Pi P(t)\Pi^T \\ &= \sigma e^{-\sigma t} \begin{bmatrix} P_{11}(t) & \cdots & \cdots & P_{1k}(t) \\ 0 & P_{22}(t) & \cdots & P_{2k}(t) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & P_{kk}(t) \end{bmatrix} \\ &= \begin{bmatrix} K_{11}(t) & \cdots & \cdots & K_{1k}(t) \\ 0 & K_{22}(t) & \cdots & K_{2k}(t) \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & K_{kk}(t) \end{bmatrix} \end{aligned} \tag{6.4}$$

where $K_{ab}(t) := \sigma e^{-\sigma t} P_{ab}(t)$ for each $1 \leq a, b \leq k$. Each permutation in $\mathfrak{S}_n$ defines a partition of $[|1,n|]$ (the subsets of $[|1,n|]$ invariant under that permutation). We therefore deduce from ((6.4) a partition of $[|1,n|]$ into k disjoint classes $C_1, C_2, \cdots, C_k$ such that

$$K_{a,b}(t) = (k_{ij}(t))_{i\in C_a, j\in C_b} \text{ for each } t \geq 0. \tag{6.5}$$

For $a, b = 1, 2, \cdots, k$ define

$$\nu_a(t) := (\mu_i(t))_{i\in C_a} \text{ for each } t \geq 0,$$

and

$$h_a(t) := (g_i(t))_{i\in C_a} \text{ for each } t \geq 0.$$

Therefore the system of equations (6.3) can be written in the form

$$\nu_a(t) = h_a(t) + \sum_{b>a} (K_{ab} * \nu_b)(t) + (K_{aa} * \nu_a)(t), \quad a = 1, \cdots, k. \tag{6.6}$$

We shall prove the following proposition that entails obviously Proposition 6.1.

Proposition 6.2. *Under the assumptions and with the notations above, for each* $a \in [|1, k|]$

(1) *either* $K_{aa}(t)$ *is a zero matrix for all* $t \geq 0$*, then for each* $i \in C_a$*, there exists* $c_i > 0$*,* $n_i \in [|0, k-a|]$ *and* $\xi_i \in (-\sigma, 0)$ *such that*

$$\mu_i(t) \sim c_i t^{n_i} e^{\xi_i t} \text{ as } t \to +\infty;$$

(2) *or* $K_{aa}(t)$ *is an irreducible matrix for all* $t \geq 0$*, then there exists a vector* c_a *with positive entries,* $\xi_a \in (-\sigma, 0)$ *and an integer* $n_a \in [|0, k-a|]$ *such that*

$$\nu_a(t) \sim c_a t^{n_a} e^{\xi_a t} \text{ as } t \to +\infty.$$

6.2.1. *Summary of the argument.* Equation (6.6) shows that the behavior of ν_a is determined by that of $(\nu_b)_{b>a}$. Hence, we proceed as follows.

First, we prove that Proposition 6.2 holds for ν_k that is the solution of the equation

$$\nu_k(t) = h_k(t) + (K_{kk} * \nu_k)(t). \tag{6.7}$$

Then, we eprove the desired estimate for each ν_a with $a \in [|1, k-1|]$ assuming that it holds for ν_b with $b > a$.

6.3. **A few technical lemmas.** Before going further, we give two technical lemmas

Lemma 6.3. *Let* f, g *be two nonnegative functions on* $\mathbb{R}_+$*. We assume that there exists* $c > 0$ *and* $n \in \mathbf{N}$ *and* $\alpha \leq 0$ *such that*

$$f(t) \sim ct^n e^{\alpha t} \text{ as } t \to +\infty.$$

(1) *If* $t \mapsto e^{-\alpha t} g(t) \in L^1(\mathbb{R}_+)$ *then*

$$(f * g)(t) \sim c \left(\int_0^\infty e^{-\alpha s} g(s) ds \right) t^n e^{\alpha t} \text{ as } t \to +\infty;$$

(2) *if there exists* $d > 0$ *such that*

$$\lim_{t \to +\infty} e^{-\alpha t} g(t) = d$$

then

$$(f * g)(t) \sim \frac{cd}{n+1} t^{n+1} e^{\alpha t} \text{ as } t \to +\infty.$$

The (elementary) proof is given in Appendix.

Lemma 6.4. *Let* $P(0)$ *be defined as in* (5.4) *then*

$$\rho(P(0)) \leq 1.$$

Proof. First recall given a nonnegative matrix A, then for each r and each x with positive entries such that

$$Ax \leq rx,$$

then

$$\rho(A) \leq r.$$

That being said, since $\int_{\mathbb{B}^2} T_k(f)(v) dv = \int_{\mathbb{B}^2} f(v) dv$, for each $j \in [|1, n|]$

$$\int_{\mathbb{B}^2} T_k(\psi_j)(v) dv = \sum_{i=1}^n \left(\int_{\mathbb{B}^2} \phi_i(v) \psi_j(v) dv \right) \int_{\mathbb{B}^2} \psi_i(v) dv.$$

In other words, on has

$$P(0)x \leq x$$

with

$$x = \begin{pmatrix} \int_{\mathbb{B}^2} \psi_1(v)dv \\ \vdots \\ \int_{\mathbb{B}^2} \psi_n(v)dv \end{pmatrix}.$$

Thus, by the argument above,

$$\rho\,(P(0)) < 1.$$

□

6.4. **The asymptotic behavior of ν_k in the long time limit.** Consider the case $a = k$ for which equation (6.6) reads

$$\nu_k(t) = h_k(t) + (K_{kk} * \nu_k)(t).$$

First, we show that $P_{kk}(0)$ is irreducible. Assume that $P_{kk}(0)$ is a zero matrix, then for each $i, j \in C_k$, with C_k defined in (6.5), one has

$$p_{ij}(0) = \int_{\mathbb{B}^2} \phi_i(v)\psi_j(v)dv = 0$$

And by (6.2),

$$\int_{\mathbb{B}^2} \phi_i(v)\psi_j(v)dv = 0$$

for each $i \in C_k,\ j \in [|1, n|]$. That implies, as $(\phi_i, \psi_j)_{1 \leq i \leq n}$ are nonnegative,

$$\lambda \left(Supp\ \phi_i \bigcap \left(\bigcup_{j=1}^{n} Supp\ \psi_j \right) \right) = 0,$$

for each $i \in C_k,\ j \in [|1, n|]$, with λ the uniform probability measure on $\mathbb{B}^2$. The assumption

$$\int_{\mathbb{B}^2} k(v, w)dw = 1$$

implies

$$\int_{\mathbb{B}^2} k(v, w)dw = \sum_{i=1}^{n} \left(\int_{\mathbb{B}^2} \phi_i(w)dw \right) \psi_i(v) = \sum_{i=1}^{n} \alpha_i \psi_i(v) = 1 \text{ for each } v \in \mathbb{B}^2$$

so that

$$\bigcup_{j=1}^{n} Supp\ \psi_j = \mathbb{B}^2$$

which means that

$$\phi_i \equiv 0 \text{ for each } i \in C_k.$$

Hence $P_{kk}(0)$ is irreducible and therefore K_{kk} is irreducible (see Lemma 5.13). Besides, since $\rho\,(P_{kk}(0)) \in Sp\,(P(0))$, one has by Lemma 6.4

$$\rho\,(P_{kk}(0)) \leq 1.$$

Therefore, with a similar argument as in the section devoted to the irreducible case, we show that there exists a unique $\xi_k \in (-\sigma, 0)$ such that

$$\rho \left(\int_0^{\infty} e^{-\xi_k s} K_{kk}(s)ds \right) = 1,$$

and conclude that there exists a vector c_k with positive entries such that

$$\nu_k(t) \sim c_k e^{\xi_k t} \text{ as } t \to +\infty.$$

Therefore Proposition 6.2 holds for ν_k.

6.5. **The asymptotic behavior of $(\nu_a)_{1\leq a<k}$ in the long time limit with $K_{aa}(t) = 0$ for each $t \geq 0$.** Let $a \in [|1, k-1|]$. We assume that Proposition 6.2 holds for each $(\nu_b)_{b>a}$, and consider the case where $K_{aa}(t) = 0$ for each $t \geq 0$. By Equation (6.6)

$$\nu_a(t) = h_a(t) + \sum_{b>a} (K_{ab} * \nu_b)(t) \text{ for each } t \geq 0.$$

which means that for each $i \in C_a$

$$\begin{aligned} \mu_i(t) &= g_i(t) + \sum_{b>a} (K_{ab} * \nu_b)^{(i)}(t) \\ &= g_i(t) + \sum_{b>a} \sum_{j \in C_b} (k_{ij} * \mu_j)(t). \end{aligned} \tag{6.8}$$

We define

$$I_a^{(i)} := \{ j \in C_b,\ b > a \,|p_{ij}(0) \neq 0 \}.$$

We first show that $I_a^{(i)}$ is not empty whatever $i \in C_a$. Assume that there exists $i \in C_a$ such that

$$I_a^{(i)} = \varnothing.$$

That means that for each $j \in C_b$ with $b > a$, one has $p_{ij}(0) = 0$. But one has already $p_{ij}(0) = 0$ for each $j \in C_b, b \leq a$. As $(C_b)_{b\in[|1,k|]}$ is a partition of $[|1, n|]$, that implies that for each $j \in [|1, n|]$

$$p_{ij}(0) = \int_{\mathbb{B}^2} \phi_i(v)\psi_j(v)dv = 0$$

which contradicts

$$\bigcup_{j=1}^{n} Supp\ \psi_j = \mathbb{B}^2.$$

Hence $I_a^{(i)}$ is not empty for each $i \in C_a$. Let $i \in C_a$, one deduces from equation (6.8)

$$\mu_i(t) = g_i(t) + \sum_{j \in I_a^{(i)}} (k_{ij} * \mu_j)(t). \tag{6.9}$$

By definition, for each $j \in I_a^{(i)}$, there exists $c_j > 0$, $\xi_j \in (-\sigma, 0)$ and an integer $n_j \in [|0, k-b|]$ (b denoting the class C_b containing j) such that

$$\mu_j(t) \sim c_j t^{n_j} e^{\xi_j t} \text{ as } t \to +\infty. \tag{6.10}$$

That being said, we define for each $i \in C_a$,

$$\xi_i := \max_{j \in I_a^{(i)}} \xi_j,$$

and

$$J_a^{(i)} := \left\{ j \in I_a^{(i)} \,|\xi_j = \xi_i \right\},$$

together with

$$n_i := \max_{j \in J_a^{(i)}} n_j,$$

and

$$N_a^{(i)} := \left\{ j \in J_a^{(i)} \,|n_j = n_i \right\}.$$

Obviously, $\xi_i \in (-\sigma, 0)$ and $n_i \in [|0, k-a|]$. Since $k_{ij}(t) = \sigma e^{-\sigma t} p_{ij}(t)$, for each $j \in I_a^{(i)}$, the function $t \mapsto e^{-\xi_i t} k_{ij}(t) \in L^1(\mathbb{R}_+)$. With (6.10) and Lemma 6.3, this implies that

$$(k_{ij} * \mu_j)(t) \sim c_j \left(\int_0^\infty e^{-\xi_i s} k_{ij}(s) ds \right) t^{n_j} e^{\xi_j t} \text{ as } t \to +\infty.$$

Hence for each $j \in I_a^{(i)} \setminus N_a^{(i)}$

$$(k_{ij} * \mu_j)(t) = o\left(t^{n_i} e^{\xi_i t}\right) \text{ as } t \to +\infty.$$

Besides, one has

$$g_i(t) = o\left(t^{n_i} e^{\xi_i t}\right) \text{ as } t \to +\infty.$$

Therefore in view of (6.9), one deduces that for each $i \in C_a$

$$\mu_i(t) \sim c_i t^{n_i} e^{\xi_i t} \text{ as } t \to +\infty,$$

with

$$c_i := \sum_{j \in N_a^{(i)}} c_j \left(\int_0^\infty e^{-\xi_i s} k_{ij}(s) ds \right) > 0.$$

Thus Proposition 6.2 holds for ν_a.

6.6. **The asymptotic behavior of ν_a with $1 \le a < k$ in the long time limit with $K_{aa}(t)$ irreducible for each $t \ge 0$.** We consider here the other case, where K_{aa} is an irreducible matrix. By Equation (6.6), one has

$$\nu_a(t) = h_a(t) + \left(\sum_{b>a} K_{ab} * \nu_b \right)(t) + (K_{aa} * \nu_a)(t).$$

With a similar argument as for the equation governing the evolution of ν_k, we show that there exists a unique $\alpha_a \in (-\sigma, 0)$ such that

$$\rho \left(\int_0^\infty e^{-\alpha_a s} K_{aa}(s) ds \right) = 1.$$

As in the case $K_{aa}(t) = 0$ for each $t \ge 0$, we define

$$I_a^{(i)} := \{ j \in C_b,\ b > a \,|\, p_{ij}(0) \neq 0 \},$$

and

$$I_a := \bigcup_{i \in C_a} I_a^{(i)}.$$

Finally, we set

$$\beta_\alpha := \max_{i \in I_a} \xi_i.$$

Notice that

$$I_a = \{ j \in C_b,\ b > a \,|\, \exists\, i \in C_a \text{ s.t. } p_{ij}(0) \neq 0 \}.$$

We distinguish three cases, namely $\beta_a < \alpha_a$, $\beta_a > \alpha_a$ and $\beta_a = \alpha_a$.

6.6.1. *The case $\beta_a < \alpha_a$.* Returning to the equation above

$$\nu_a(t) = h_a(t) + \left(\sum_{b>a} K_{ab} * \nu_b\right)(t) + (K_{aa} * \nu_a)(t),$$

and multiplying both sides by $e^{-\alpha_a t}$ leads to

$$_{-\alpha_a}\nu_a(t) =_{-\alpha_a} h_a(t) + \sum_{b>a} e^{-\alpha_a t}(K_{ab} * \nu_b)(t) + (_{-\alpha_a}K_{aa} *_{-\alpha_a} \nu_a)(t)$$

with the notation used in (5.15).

Proposition 6.2 holds for $(\nu_b)_{b>a}$, meaning that for each $j \in C_b$ with $b > a$, there exists $c_j > 0$, $n_j \in [|0, k-b|]$ and $\xi_i \in (-\sigma, 0)$ such that

$$\mu_j(t) \sim c_j t^{n_j} e^{\xi_j t} \text{ as } t \to +\infty.$$

Since $k_{ij}(t) = \sigma e^{-\sigma t} p_{ij}(t)$ for each $i, j \in [|1, n|]$ and $\xi_j \in (-\sigma, 0)$ for each $j \in C_b$ with $b > a$, one has by Lemma 6.3

$$(k_{ij} * \mu_j)(t) \sim c_j \left(\int_0^\infty e^{-\xi_j s} k_{ij}(s) ds\right) t^{n_j} e^{\xi_j t} \text{ as } t \to +\infty.$$

Since $\xi_j \leq \beta_a < \alpha_a$ for each $j \in I_a$, the long time limit asymptotic behavior established above implies that $t \mapsto \sum_{b>a} e^{-\alpha_a t}(K_{ab} * \nu_b)(t)$ is a vector whose all of entries are integrable. Besides $_{-\alpha_a}h_a$ is also a vector with integrable entries.

Hence, by a similar argument as in the section devoted to the irreducible case, one has

$$\nu_a(t) \sim c e^{\alpha_a t} \text{ as } t \to +\infty,$$

where c is a vector with positive entries. Therefore Proposition 6.2 holds for ν_a.

6.6.2. *The case $\beta_a > \alpha_a$.* Returning to the equation

$$\nu_a(t) = h_a(t) + \left(\sum_{b>a} K_{ab} * \nu_b\right)(t) + (K_{aa} * \nu_a)(t).$$

and multiplying both sides by $e^{-\beta_a t}$ leads to

$$_{-\beta_a}\nu_a(t) =_{-\beta_a} h_a(t) + \left(\sum_{b>a} {}_{-\beta_a}K_{ab} *_{-\beta_a} \nu_b\right)(t) + (_{-\beta_a}K_{aa} *_{-\beta_a} \nu_a)(t).$$

Obviously $t \mapsto_{-\beta_a} h_a(t) + \left(\sum_{b>a} {}_{-\beta_a}K_{ab} *_{-\beta_a} \nu_b\right)(t)$ is locally bounded and by a similar argument as in the proof of Proposition 5.16, one has

$$\rho\left(\int_0^\infty e^{-\beta_a s} K_{aa}(s) ds\right) < 1.$$

Therefore by Proposition 5.6,

$$\begin{aligned} _{-\beta_a}\nu_a(t) = {} & {}_{-\beta_a}h_a(t) + \left(\sum_{b>a} {}_{-\beta_a}K_{ab} *_{-\beta_a} \nu_b\right)(t) \\ & + (F_a * {}_{-\beta_a}h_a)(t) + \left(F_a * \left(\sum_{b>a} {}_{-\beta_a}K_{ab} *_{-\beta_a} \nu_b\right)\right)(t) \end{aligned}$$

with

$$F_a := \sum_{n \geq 1} ({}_{-\beta_a}K_{aa})^{*n}.$$

Define

$$\zeta_a(t) := e^{-\beta_a t} h_a(t) + e^{-\beta_a t}\left(\sum_{b>a} K_{ab} * \nu_b\right)(t),$$

and for each $b > 0$

$$F_{ab}(t) := \left(F_a *_{-\beta_a} K_{ab}\right)(t).$$

The equation above is then recast as

$$_{-\beta_a}\nu_a(t) = \zeta_a(t) + \sum_{b>a} \left(F_{ab} *_{-\beta_a} \nu_b\right)(t).$$

or equivalently, with the notation

$$F_{ab}(t) = \left(f_{ij}^{(ab)}\right)_{i\in C_a, j\in C_b},$$

as

$$\mu_i(t) = e^{\beta_a t}\zeta_i(t) + e^{\beta_a t}\sum_{b>a}\sum_{j\in C_b}\int_0^t f_{ij}^{(ab)}(t-s)e^{-\beta_a s}\mu_j(s)ds, \tag{6.11}$$

for each $i \in C_a$. We denote

$$\beta_i := \max_{j\in I_a^{(i)}} \beta_j,$$

and

$$J_a^{(i)} := \left\{ j \in I_a^{(i)} \,|\beta_i = \xi_j \right\}$$

while

$$\gamma_i := \max_{j\in J_a^{(i)}} n_j$$

and

$$N_a^{(i)} := \left\{ j \in J_a^{(i)} \,|\gamma_i = n_j \right\}.$$

By definition, one has for each $i \in C_a$

$$\begin{aligned} e^{\beta_a t}\zeta_i &= g_i(t) + \sum_{b>a}\sum_{j\in C_b}\int_0^t k_{ij}(t-s)\mu_j(s)ds \\ &= g_i(t) + \sum_{j\in I_a^{(i)}}\int_0^t k_{ij}(t-s)\mu_j(s)ds. \end{aligned} \tag{6.12}$$

Since Proposition 6.2 holds for $(\nu_b)_{b>a}$, and $t \mapsto e^{-\xi_j s}k_{ij}(s)ds \in L^1(\mathbb{R}_+)$, one deduces from by Lemma 6.3 that

$$\int_0^t k_{ij}(t-s)\mu_j(s)ds \sim c_j\left(\int_0^\infty k_{ij}(s)ds\right)t^{n_j}e^{\xi_j t} \text{ as } t\to+\infty.$$

Therefore, in view of the equality (6.12), one has

$$e^{\beta_a t}\zeta_i(t) \sim Ct^{\gamma_i}e^{\beta_i t} \text{ as } t\to+\infty \tag{6.13}$$

with

$$C := \sum_{j\in N_a^{(i)}} c_j\left(\int_0^\infty k_{ij}(s)ds\right).$$

We now discuss the asymptotic beahvior of

$$t \mapsto e^{\beta_a t}\sum_{b>a}\sum_{j\in C_b}\int_0^t f_{ij}^{(ab)}(t-s)e^{-\beta_s}\mu_j(s)ds$$

in the long time limit. Before going further in the computation of the long time asymptotic behavior of ν_a, we show that $\int_0^\infty F_a(s)ds$ is a matrix with positive entries.We recall that

$$\int_0^\infty F_a(s)ds = \int_0^\infty \sum_{n\geq 1} \left({}_{-\beta_a} K_{aa} \right)^{*n}(s)ds$$
$$= \sum_{n\geq 1} \left(\int_0^\infty e^{-\beta_a s} K_{aa}(s)ds \right)^n .$$

With a similar argument as in Lemma 5.13, we show that the matrix

$$\int_0^\infty e^{-\beta_a s} K_{aa}(s)ds$$

is irreducible meaning that for each i, j, there exists $n \geq 1$ such that

$$\left(\left(\int_0^\infty e^{-\beta_a s} K_{aa}(s)ds \right)^n \right)_{i,j} > 0.$$

Hence, $\int_0^\infty F_a(s)ds$ is a matrix with positive entries. Besides for each $b > a$,

$$\int_0^\infty F_a *_{\beta_a} K_{ab}(s)ds = \int_0^\infty F_a(s)ds \int_0^\infty e^{-\beta_a s} K_{ab}(s)ds.$$

As $\int_0^\infty F_a(s)ds$ is a matrix with positive entries, a j-th column of the matrix $\int_0^\infty F_a *_{\beta_a} K_{ab}(s)ds$ is a zero vector if and only if the j-th column of $\int_0^\infty e^{-\beta_a s} K_{ab}(s)ds$ is a zero vector. And in the case where the j-th column of $\int_0^\infty e^{-\beta_a s} K_{ab}(s)ds$ is not a zero vector, each entry of the j-th column of the matrix $\int_0^\infty F_a *_{\beta_a} K_{ab}(s)ds$ is positive. Indeed $f_{ij}^{(ab)} \neq 0$ if and only if $j \in I_a$. Hence

$$e^{\beta_a t} \sum_{b>a} \sum_{j\in C_b} \int_0^t f_{ij}^{(ab)}(t-s) e^{-\beta_a s} \mu_j(s)ds = e^{\beta_a t} \sum_{j\in I_a} \int_0^t f_{ij}^{(ab)}(t-s) e^{-\beta_a s} \mu_j(s)ds.$$

Now, we show that for each $i \in C_a,\ j \in I_a$ the function $t \mapsto e^{(\beta_a - \xi_j)t} f_{ij}^{(ab)}(t) \in L^1(\mathbb{R}_+)$. Observe indeed that

$$e^{(\beta_a-\xi_j)t} F_{ab}(t) = e^{(\beta_a-\xi_j)t} \left(F_a *_{-\beta_a} K_{ab} \right)(t)$$
$$= e^{(\beta_a-\xi_j)t} \left(\left(\sum_{n\geq 1} \left({}_{-\beta_a} K_{aa} \right)^{*n} \right) *_{-\beta_a} K_{ab} \right)(t)$$
$$= e^{-\xi_j t} \left(\left(\sum_{n\geq 1} \left(K_{aa} \right)^{*n} \right) * K_{ab} \right)(t).$$

Thus, since $K(t) = \sigma e^{-\sigma t} P(t)$, the function $t \mapsto e^{(\beta_a-\xi_j)t} F_{ab}(t) \in \mathcal{M}_{\mathrm{Card}\ C_a, \mathrm{Card}\ C_b}\left(L^1(\mathbb{R}_+) \right)$. Therefore, by Lemma 6.3, for each $(i, j) \in C_a \times I_a$

(6.14)
$$e^{\beta_a t} \int_0^t f_{ij}^{(ab)}(t-s) e^{-\beta_a s} \mu_j(s)ds \sim c_j \left(\int_0^\infty e^{(\beta_a-\xi_j)s} f_{ij}^{(ab)}(s)ds \right) t^{n_j} e^{\xi_j t} \text{ as } t \to +\infty.$$

Let

$$J_a := \{ j \in I_a \,|\xi_j = \beta_a \}$$

with

$$n_a := \max_{j\in I_a} n_j$$

and

$$N_a := \{ j \in J_a \,|n_j = n_a \}.$$

In view of equality (6.11), one easily deduce from (6.13) and (6.14) that

$$\mu_i(t) \sim ct^{n_a} e^{\beta_a t} \text{ as } t \to +\infty$$

with

$$c := \begin{pmatrix} c_1 \\ \vdots \\ c_m \end{pmatrix},$$

where $m = \text{Card } C_a$ and

$$c_i = \begin{cases} \sum_{j \in N_a^{(i)}} c_j \left(\int_0^\infty k_{ij}(s)ds\right) + \sum_{j \in N_a} c_j \left(\int_0^\infty f_{ij}^{(ab)}(s)ds\right) & \text{if } (\beta_i, \gamma_i) = (\beta_a, n_a), \\ \sum_{j \in N_a} c_j \left(\int_0^\infty f_{ij}^{(ab)}(s)ds\right) & \text{otherwise.} \end{cases}$$

Thus Proposition 6.2 holds for ν_a.

6.6.3. *The case $\beta_a = \alpha_a$.* Returning to the equation

$$\nu_a(t) = h_a(t) + \left(\sum_{ba} K_{ab} * \nu_b\right)(t) + (K_{aa} * \nu_a)(t).$$

and multiplying both sides by $e^{-\alpha_a}$ leads to

$${}_{-\alpha_a}\nu_a(t) = {}_{-\alpha_a} h_a(t) + \sum_{b>a} e^{-\alpha_a t} (K_{ab} * \nu_b)(t) + ({}_{-\alpha_a}K_{aa} *_{-\alpha_a} \nu_a)(t)$$

with the notation in (5.15). We denote the renewal measure by

$$H_a(t) := \sum_{n \geq 0} ({}_{-\alpha_a}K_{aa})^{*n}(t),$$

where ${}_{-\alpha_a}K_{aa} : t \mapsto e^{-\alpha_a t} K_{aa}(t)$. And let $\left(H_a^{(1)} H_a^{(2)}\right)$ be a Stone decomposition of the renewal measure:

$$H_a = H_a^{(1)} + H_a^{(2)} \text{ in } \mathcal{M}(\mathbb{R}_+). \tag{6.15}$$

where $H_a^{(2)}$ is a matrix whose entries are finite measures on $\mathbb{R}_+$ and $H_a^{(1)}$ is a matrix of absolutely continuous measures with bounded continuous densities $H_a^{(1)} := (h_a)_{ij}$ such that there exists a positive matrix H_∞ (see Theorem 5.10) verifying

$$\lim_{t \to +\infty} H_a^{(1)}(t) = H_\infty \equiv \left(h_{ij}^{(\infty)}\right)_{1 \leq i,j \leq \text{Card } C_a}. \tag{6.16}$$

We have

$$\nu_a(t) = e^{\alpha_a t} (H_a *_{-\alpha_a} h_a)(t) + e^{\alpha_a t} \sum_{b>a} (H_a *_{-\alpha_a} K_{ab} *_{-\alpha_a} \nu_b)(t). \tag{6.17}$$

We recall — see the cases discussed above — that, since Proposition holds 6.2 holds for $(\nu_b)_{b>a}$, for each $(i,j) \in C_a \times \left(\bigcup_{b>a} C_b\right)$,

$$\int_0^t k_{ij}(t-s)\mu_j(s)ds \sim c_j \left(\int_0^\infty k_{ij}(s)ds\right) t^{n_j} e^{\xi_j t} \text{ as } t \to +\infty,$$

or equivalently

$$e^{-\alpha_a t} \int_0^t k_{ij}(t-s)\mu_j(s)ds \sim c_j \left(\int_0^\infty k_{ij}(s)ds\right) t^{n_j} e^{(\xi_j - \alpha_a)t} \text{ as } t \to +\infty.$$

We denote

$$J_a := \{j \in I_a \,|\xi_j = \alpha_a\}.$$

If $j \notin J_a$, the function $t \mapsto e^{-\alpha_a t} \int_0^t k_{ij}(t-s)\mu_j(s)ds \in L^1(\mathbb{R}_+)$, so that by dominated convergence, for each $l \in C_a$,

$$\int_0^t h_{lj}^{(1)}(t-\tau)e^{-\alpha_a \tau} \int_0^t k_{ij}(\tau - s)\mu_j(s)dsd\tau$$
$$\to h_{lj}^{(\infty)} \int_0^\infty e^{-\alpha_a s} k_{ij}(s)ds \int_0^\infty e^{-\alpha_a s}\mu_j(s)ds$$

as $t \to +\infty$, and

$$\int_0^{t-d\tau} e^{-\alpha_a(t-\tau)} \int_0^t k_{ij}(t-\tau-s)\mu_j(s)h_{lj}^{(2)}(ds) \to 0$$

as $t \to +\infty$. If $j \in J_a$,

$$e^{-\alpha_a t}\int_0^t k_{ij}(t-s)\mu_j(s)ds \sim c_j \left(\int_0^\infty k_{ij}(s)ds\right) t^{n_j} \text{ as } t \to +\infty$$

and we recall that

$$h_{lj}(t) \to h_{lj}^{(\infty)} > 0 \text{ as } t \to +\infty,$$

so that, by Lemma 6.3,

$$\int_0^t h_{lj}^{(1)}(t-\tau)e^{-\alpha_a \tau}\int_0^t k_{ij}(\tau-s)\mu_j(s)dsd\tau \sim \frac{h_{lj}^{(\infty)}c_j}{n_j+1}\left(\int_0^\infty e^{-\alpha_a s}k_{ij}(s)ds\right)t^{n_j+1}$$

while

$$\int_0^t h_{lj}^{(2)}(d\tau)e^{-\alpha_a(\tau-t)}\int_0^{\tau-t} k_{ij}(\tau-t-s)\mu_j(s)ds = O\left(t^{n_j}\right)$$
$$= o\left(t^{n_j+1}\right) \text{ as } t \to +\infty,$$

since $h_{ij}^{(2)}$ is a finite measure on $\mathbb{R}_+$. Consequently, denoting

$$n_a := \max_{j \in J_a} n_j$$

there exists a vector c with positive entries such that

$$\sum_{b>a}\left(H_a *_{-\alpha_a} K_{ab} *_{-\alpha_a} \nu_b\right) \sim ct^{n_a+1} \text{ as } t \to +\infty.$$

With a similar argument, we show that

$$\left(H_a *_{-\alpha_a} h_a\right)(t) = o\left(t^{n_a+1}\right) \text{ as } t \to +\infty.$$

In view of (6.17) that implies

$$\nu_a(t) \sim ct^{n_a+1}e^{\alpha_a t} \text{ as } t \to +\infty.$$

And we conclude by observing that since $(n_j)_{j\in N_a} \in [|0, k-(a+1)|]$, one has

$$n_a + 1 \in [|0, k-a|].$$

Thus Proposition 6.2 holds for ν_a.

7. Proof of statement (2) and (3) of Theorem 2.4

We conclude the proof of Theorem 2.4.

Proof. We recall that the function $t, v \mapsto m(t,v)$ verifies the equality

$$m(t,v) = m(0,v)p(|v|t)e^{-\sigma t} + \sigma \sum_{i=1}^n \int_0^t \mu_i(t-s)p(|v|s)e^{-\sigma s}\psi_i(v)ds. \tag{7.1}$$

We denote

$$h(t) := e^{-\sigma t}\int_{\mathbb{R}^2} m(0,v)p(|v|t)dv, \ t \geq 0,$$

and, for each $1 \leq i \leq n$,

$$p_i(t) := \sigma e^{-\sigma s} \int_{\mathbb{R}^2} p(|v|s)\psi_i(v)dv.$$

Notice that for each $\alpha \in (-\sigma, 0)$ and for each $i \in [|1, n|]$ the function $t \mapsto e^{\alpha t} p_i(t) \in L^1(\mathbb{R}_+)$. That being said, integrating equality (7.1) in $v \in \mathbb{B}^2$ gives

$$m(t) = h(t) + \sum_{i=1}^{n} (\mu_i * p_i)(t), \ t \geq 0. \tag{7.2}$$

We have seen that for each $i \in [|1, n|]$, there exists $c_i > 0$, $n_i \in \mathbf{N}$ and $\xi_i \in (-\sigma, 0)$ such that

$$\mu_i(t) \sim c_i t^{n_i} e^{\xi_i t} \text{ as } t \to +\infty.$$

Denote

$$\beta := \max_{i \in [|1,n|]} \xi_i,$$

and

$$I := \{i \in [|1, n|] \, | \xi_i = \beta\},$$

while

$$\gamma := \max_{i \in I} n_i,$$

and

$$J := \{i \in I \, | n_i = \gamma\}.$$

By Lemma 6.3 for each $i \in [|1, n|]$

$$(\mu_i * p_i)(t) \sim c_i \left(\int_0^\infty e^{-\xi_i s} p_i(s) ds \right) t^{n_i} e^{\xi_i t} \text{ as } t \to +\infty.$$

Obviously

$$h(t) = o\left(t^\gamma e^{\beta t}\right) \text{ as } t \to +\infty,$$

Therefore, (7.2) implies that

$$m(t) \sim c t^\gamma e^{\beta t} \text{ as } t \to +\infty$$

with

$$c := \sum_{i \in J} c_i \int_0^\infty e^{-\beta s} p_i(s) ds.$$

We finish with a discussion of the asymptotic behavior of ξ_σ in the collisionless regime $\sigma \to 0^+$. Denote $\lambda_\sigma := \xi_\sigma + \sigma$. Establishing that $\xi_\sigma \sim -\sigma$ as $\sigma \to 0^+$ amounts to proving that $\lambda_\sigma = o(\sigma)$. Observe that, since $\xi_\sigma \in (-\sigma, 0)$,

$$0 < \lambda_\sigma < \sigma$$

so $\lambda_\sigma \to 0^+$ as $\sigma \to 0^+$. Keeping this in mind, we have by definition of ξ_σ

$$\rho\left(\int_0^\infty \sigma e^{-\lambda_\sigma s} P(s) ds\right) = 1,$$

where P is defined in (5.4). Substituting $z = \lambda_\sigma s$ in the integral above, we deduce from the equality above

$$\frac{\lambda_\sigma}{\sigma} = \rho\left(\int_0^\infty e^{-z} P\left(\frac{z}{\lambda_\sigma}\right) dz\right). \tag{7.3}$$

Since $\lambda_\sigma \to 0^+$ as $\sigma \to 0^+$ and $p_{ij}(t) \to 0^+$ as $t \to +\infty$ for each $i, j \in [|1, n|]$, one has $p_{ij}(z/\lambda_\sigma) \to 0^+$ as $\sigma \to 0^+$ for each $i, j \in [|1, n|]$. Besides, $0 \leq e^{-z} p_{ij}(z/\lambda_\sigma) \leq e^{-z} \int_{\mathbb{B}^2} \phi_i(v)\psi_j(v)dv$ so that by dominated convergence

$$\int_0^\infty e^{-z} P\left(\frac{z}{\lambda_\sigma}\right) dz \to 0 \text{ as } \sigma \to 0^+.$$

Therefore (7.3) implies by continuity of the spectral radius that

$$\frac{\lambda_\sigma}{\sigma} \to 0^+ \text{ as } \sigma \to 0^+.$$

□

8. Annexe

We here establish Lemma 6.3.

Lemma 8.1. *Let f, g be two nonnegative functions on $\mathbb{R}_+$. We assume that there exits $c > 0$ and $n \in \mathbf{N}$ such that*

$$f(t) \sim ct^n \text{ as } t \to +\infty.$$

(1) *If $g \in L^1(\mathbb{R}_+)$ then*

$$(f * g)(t) \sim c \left(\int_0^\infty g(s) ds \right) t^n \text{ as } t \to +\infty;$$

(2) *If $f \in L^\infty_{loc}(\mathbb{R}_+)$ and there exists $d > 0$ such that*

$$\lim_{t \to +\infty} g(t) = d$$

then

$$(f * g)(t) \sim \frac{cd}{n+1} t^{n+1} \text{ as } t \to +\infty.$$

Proof. (1) We recall that

$$(f * g)(t) = \int_0^t f(t-s) g(s) ds, \ \forall t \geq 0.$$

By assumption, one has for each $s \geq 0$

$$\frac{1}{t^n} f(t-s) \to c \text{ as } t \to +\infty$$

and

$$\left(\frac{1}{t^n} f(t-s) g(s) \right) \leq C g(s) \in L^1(\mathbb{R}_+)$$

so that one obtains by dominated convergence

$$\lim_{t \to +\infty} \frac{1}{t^n} \int_0^t f(t-s) g(s) ds = \left(\int_0^\infty g(s) ds \right) c.$$

(2) One first recalls that if f, g are two nonnegative functions on $\mathbb{R}_+$ such that

$$f(t) \sim g(t) \text{ as } t \to +\infty$$

and their integrals diverge at infinity then

$$\int_0^t f(s) ds \sim \int_0^t g(s) ds \text{ as } t \to +\infty.$$

Therefore one has for f verifying the hypothesis

$$\int_0^t f(s) ds \sim \frac{c}{n+1} t^{n+1} \text{ as } t \to +\infty.$$

To obtain the desired conclusion, it remains to show that for each nonnegative function g such that $g(t) \to a \neq 0$ as $t \to +\infty$,

$$(f * g)(t) \sim a \int_0^t f(s) ds \text{ as } t \to +\infty.$$

By assumption, for each $\varepsilon > 0$, there exists $A > 0$ such that $t \geq A$ implies $|g(t) - a| \leq \varepsilon$. Thus

$$\begin{aligned}
\left|\int_0^t g(t-s)f(s)ds - a\int_0^t f(s)ds\right| &\leq \int_0^t |g(s)-a|f(t-s)ds \\
&\leq \int_0^A |g(s)-a|f(t-s)ds \\
&\quad + \int_A^t |g(s)-a|f(t-s)ds \\
&\leq \int_0^A |g(s)-a|f(t-s)ds + \varepsilon\int_A^t f(t-s)ds \\
&\leq M\int_0^A |g(s)-a|ds + \frac{\varepsilon}{a}a\int_0^t f(s)ds,
\end{aligned}$$

where M verifies

$$f(t) \leq M \text{ a.e. in } [0, A].$$

So that

$$\frac{|\int_0^t g(t-s)f(s)ds - a\int_0^t f(s)ds|}{a\int_0^t f(s)ds} \leq \frac{\varepsilon}{a} + \frac{M\int_0^A |g(s)-a|ds}{a\int_0^t f(s)ds}. \tag{8.1}$$

As $\int_0^t f(s)ds$ diverges, for each $\varepsilon > 0$ there exists $B > 0$ such that $t \geq B$ implies

$$\int_0^t f(s)ds \geq \frac{M\int_0^A |g(s)-a|ds}{a\varepsilon}$$

or

$$\frac{M\int_0^A |g(s)-a|ds}{a\int_0^t f(s)ds} \leq \varepsilon.$$

So that $t \geq \max(A, B)$ implies that

$$\frac{|\int_0^t g(t-s)f(s)ds - a\int_0^t f(s)ds|}{a\int_0^t f(s)ds} \leq \frac{a+1}{a}\varepsilon.$$

That is the wanted result.

□

And we conclude by noticing that Lemma 8.1 obviously entails Lemma 6.3.

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Chapitre 5

Homogenization of transport problems and semigroups

CHAPTER V
HOMOGENIZATION OF TRANSPORT PROBLEMS AND SEMIGROUPS

1. Introduction

The mathematical modeling of the response of composite materials to external fields usually involves partial differential equations with oscillating coefficients. Specifically, the wavelengths of these oscillations correspond with the spatial scales defined by the microscopic structure of the composite, i.e. the scale at which the elementary constituents of the composite are assembled. When investigating the macroscopic properties of such a composite material, a first step is to average out the oscillations of the coefficients at microscopic scale, and to filter the high frequency oscillations they induce in the response fields one is interested in. This mathematical process is called homogenization, since it may be viewed as the replacement of a composite material by an equivalent homogeneous material. In the most favorable cases, this would be done by simply replacing the response coefficients oscillating at microscopic scale in the field equation with coefficients for the equivalent homogeneous material where the oscillations at microscopic scale have been eliminated.

Unfortunately, this picture is outrageously optimistic. In many cases, a single response coefficient oscillating at microscopic scale will be replaced with several homogenized equivalent coefficients, for instance due to the persistence of anisotropy effects in the microscopic structure of the composite material. Worse, the structure of the partial differential equation itself can be modified after taking the homogenization limit, and this is precisely our concern in the present work. A striking example of such a change in the structure of the homogenized equation was given by Tartar [10], who observed that the homogenization limit of the simplest imaginable ordinary differential equation would lead to an integro-differential equation (i.e. involving memory terms). In other words, the group property of the original evolution equation can be destroyed by the homogenization limit.

Some time later, Vanderhaegen [11, 12], Levermore-Pomraning-Sanzo-Wong [5], and Sentis [8] studied in detail the homogenization problem for the absorption coefficient in transport theory (for either neutrons or photons). Their work also leads to integro-differential equations as in the simple example considered by Tartar — and for the same basic reason.

The phenomena observed by Tartar in his simple example — i.e. the fact that the group property satisfied by the solutions of an evolution equation can be destroyed by the homogenization limit — also occurs in very different contexts. It has been very recently identified in a classical problem in nonequilibrium statistical mechanics, namely the Boltzmann-Grad limit of the periodic Lorentz gas, by E. Caglioti and the second author, and by J. Marklof and Strömbergsson in [3, 2, 6], as well as in a homogenization problem for the linear Boltzmann equation in a periodically perforated domain, by the two first authors and E. Caglioti [1]. In all these works, the solution of the equation at microscopic scale is given by a semigroup, and, in order to keep the semigroup property after passing to the macroscopic limit, it is necessary to consider an enlarged phase space involving additional variables. The present paper explains how the ideas in [2, 6, 1] can be applied in the context of the homogenization of opacities considered in [11, 12, 5, 7, 8].

Semigroups and kinetic models have been among Aldo Belleni-Morante's favorite scientific subjects. His own ideas have had a great influence on the development of this field of mathematical analysis. In view of his own particular interest in photonics and, more generally, transport problems in astrophysics, we dedicate this modest contribution to his memory.

2. Homogenization of an ODE

Our starting point is the following elementary, yet fairly instructive example, due to L. Tartar [10].

Let $a \in L^\infty(\mathbf{T}^N)$, assume without loss of generality that $a \geq 0$ a.e. on $\mathbf{T}^N$, and consider, for each $\epsilon > 0$, the ODE with unknown $u_\epsilon \equiv u_\epsilon(t,z) \in \mathbf{R}$:

$$\left\{\begin{array}{l}\dfrac{du_\epsilon}{dt} + a\left(\dfrac{z}{\epsilon}\right)u_\epsilon = 0\,, \quad t > 0\,,\ z \in \mathbf{R}^N\,, \\ u_\epsilon(0,z) = u^{in}(z)\,,\end{array}\right. \tag{2.1}$$

where $u^{in} \in L^2(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$. Obviously, for each $\epsilon > 0$, one has

$$u_\epsilon(t,z) = u^{in}(z)e^{-ta(z/\epsilon)}\,, \quad t > 0\,,\ z \in \mathbf{R}^N\,,$$

so that, in the limit as $\epsilon \to 0^+$, one has

$$u_\epsilon \rightharpoonup u \text{ in } L^\infty(\mathbf{R}_+ \times \mathbf{R}^N) \text{ weak-*}$$

where the limit u is explicitly given by the formula

$$u(t,z) = u^{in}(z)\Phi(t)\,, \quad t \geq 0\,,\ z \in \mathbf{R}^N\,, \tag{2.2}$$

with

$$\Phi(t) = \int_{\mathbf{T}^N} e^{-ta(y)}dy\,, \quad t \geq 0\,. \tag{2.3}$$

This example shows that the homogenized solution u does not satisfy the equation

$$\frac{du}{dt} + \overline{a}u = 0$$

where $\overline{a}$ is the average of a on $\mathbf{T}^N$, i.e.

$$\overline{a} = \int_{\mathbf{T}^N} a(y)dy\,,$$

as someone unfamiliar with the intricacies of homogenization might (naively) expect. Worse, unless a is a.e. constant on $\mathbf{T}^N$, there does not exist any $A \in \mathbf{R}$ such that

$$\frac{du}{dt} + Au = 0\,.$$

(Should such an A exist, it would be referred to as the "homogenized coefficient" equivalent to the oscillating coefficient $a(z/\epsilon)$.) Equivalently, although for each $\epsilon > 0$ the solution u_ϵ is defined in terms of u^{in} by the semigroup $S_\epsilon(t)$ defined on $L^2(\mathbf{R}^N)$ by the formula

$$S_\epsilon(t)\phi = \phi(z)e^{-ta(z/\epsilon)}\,, \tag{2.4}$$

the homogenized solution u is not given in terms of u^{in} by a semigroup acting on $L^2(\mathbf{R}^N)$, since (by convexity of the exponential)

$$\Phi(t_1 + t_2) \neq \Phi(t_1)\Phi(t_2)\,, \quad t_1, t_2 > 0\,,$$

unless a is a.e. constant on $\mathbf{T}^N$ — meaning that there are no fast oscillations in the original problem (2.1), so that there is no need for homogenization in this case.

In fact, L. Tartar (see lecture 35 in [10]) proved that the homogenized solution satisfies the following integro-differential equation

$$\tag{2.5} \begin{cases} \dfrac{du}{dt}(t,z) + \overline{a}u(t,z) = \displaystyle\int_0^t K(t-s)u(s,z)ds\,, \quad t>0\,,\ z\in\mathbf{R}^N\,, \\ u(0,z) = u^{in}(z)\,, \end{cases}$$

where the Laplace transform of K is given by the expression

$$\tilde{K}(p) := \int_0^\infty e^{-pt}K(t)dt = \int_{\mathbf{T}^N}(p+a(y))dy - \left(\int_{\mathbf{T}^N}\frac{dy}{p+a(y)}dy\right)^{-1}, \quad p>0\,.$$

Concerning the appearance of an integro-differential equation such as (2.5) as the homogenization limit of an ODE, it is instructive to compare the situation above with the problem

$$\tag{2.6} \begin{cases} \dfrac{dv_\epsilon}{dt} + b\left(\dfrac{t}{\epsilon}\right)v_\epsilon = 0\,, \quad t>0\,, \\ v_\epsilon(0) = v^{in}\,, \end{cases}$$

with unknown $v_\epsilon \equiv v_\epsilon(t) \in \mathbf{R}$, where $b\in L^\infty(\mathbf{T}^1)$. In this case

$$v_\epsilon(t) = v^{in}\exp\left(-\int_0^t b(s/\epsilon)ds\right) \to v^{in}e^{-Bt} = v(t)$$

for each $t\geq 0$ as $\epsilon\to 0^+$, where

$$B := \int_0^1 b(\sigma)d\sigma\,.$$

Indeed,

$$\frac{1}{t}\int_0^t b(s/\epsilon)ds = \frac{\epsilon}{t}\int_0^{t/\epsilon} b(\sigma)d\sigma \to \lim_{t\to+\infty}\frac{1}{t}\int_0^t b(\sigma)d\sigma = B$$

as $\epsilon\to 0^+$. Hence, the homogenized equation obtained from (2.6) is

$$\begin{cases} \dfrac{dv}{dt} + Bv = 0\,, \quad t>0\,, \\ v(0) = v^{in}\,, \end{cases}$$

and in this case, B is the equivalent absorption coefficient obtained from the oscillating absorption coefficient $b(t/\epsilon)$ by homogenization.

The difference between the homogenization of problems (2.1) and (2.6) is that in the latter case, the oscillating variable in the coefficient b is the time variable, and the equation (2.6) provides a bound on the time derivative of the solution v_ϵ, thereby excluding the possibility of fast oscillations in t in the solution v_ϵ.

At variance with this case, in Tartar's example (2.1), the oscillating variable is z, and the equation (2.1) does not involve derivatives in z to prevent the buildup of fast oscillations in z in the solution u_ϵ. In that example, the fast oscillations in z in both $a(z/\epsilon)$ and u_ϵ combine to produce the integral term on the right-hand side of (2.5).

Obviously, the example (2.1) can be generalized to the case where the quasi-periodic oscillating coefficient $a(z/\epsilon)$ is replaced with a bounded family $a_\epsilon\equiv a_\epsilon(z)$ of functions in $L^\infty(\mathbf{R}^N)$ converging in the sense of Young measures as $\epsilon\to 0^+$.

3. A semigroup in extended phase space

As a warm-up, we shall in this section consider again Tartar's example above, and express the homogenization limit of (2.1) in terms of a semigroup defined on an extended phase space — i.e. acting on functions with additional variables.

Let $a_\epsilon \equiv a_\epsilon(z)$ be a bounded family of functions in $L^\infty(\mathbf{R}^N)$ converging in the sense of Young measures to $(\mu_z)_{z\in\mathbf{R}^N}$ (see [9] for a lucid presentation of the notion of Young measures.) In other words, $(\mu_z)_{z\in\mathbf{R}^N}$ is a family of probability measures on $\mathbf{R}$ that measurably depends on z, and satisfies, for each $f \in C_b(\mathbf{R})$

$$f(a_\epsilon) \rightharpoonup F_a \text{ in } L^\infty(\mathbf{R}^N) \text{ weak-}*\,, \text{ with } F_a(z) = \int_{\mathbf{R}} f(\lambda) d\mu_z(\lambda) =: \langle \mu_z, f\rangle$$

in the limit as $\epsilon \to 0^+$. Without loss of generality, we henceforth assume that $a_\epsilon \geq \alpha > 0$ a.e. on $\mathbf{R}^N$.

For each $\epsilon > 0$, let $u_\epsilon \equiv u_\epsilon(t,z)$ be the solution of

$$\left\{\begin{aligned} &\frac{du_\epsilon}{dt} + a_\epsilon(z)u_\epsilon = 0\,, \quad t>0\,,\ z\in\mathbf{R}^N\,,\\ &u_\epsilon(0,z) = u^{in}(z)\,, \end{aligned}\right. \tag{3.1}$$

where $u^{in} \in L^1 \cap L^\infty(\mathbf{R}^N)$.

Proposition 3.1. *In the limit as $\epsilon \to 0^+$, one has*

$$u_\epsilon \rightharpoonup u = \int_0^{+\infty} U ds \text{ in } L^\infty(\mathbf{R}_+ \times \mathbf{R}^N) \text{ weak-}*,$$

where $U \equiv U(t,s,z)$ is the solution of

$$\left\{\begin{aligned} &\partial_t U - \partial_s U = 0\,, \quad t,s>0\,,\ z\in\mathbf{R}^N\,,\\ &U(0,s,z) = -u^{in}(z)\frac{d\tilde{\mu}_z}{ds}(s)\,. \end{aligned}\right. \tag{3.2}$$

(We recall that the notation $\tilde{\mu}_z$ designates the Laplace transform of μ_z.)

Before giving the (elementary) proof of this result, a few remarks are in order.

First, the equation satisfied by U is a free transport equation, where $s \in \mathbf{R}_+$ is the space variable. Since the vector field $-\partial_s$ is outgoing on the boundary of the half-line $\mathbf{R}_+$, there is no need of a boundary condition for $s = 0$, so that the problem (3.2) is well-posed — in $L^2(\mathbf{R}_+ \times \mathbf{R}^N; e^{-s}dsdz)$, for instance.

Next, although the homogenization limit u of u_ϵ as $\epsilon \to 0^+$ is not of the form $u(t,\cdot) = S(t)u^{in}$ with $S(t)$ a semigroup on $L^2(\mathbf{R}^N)$, the function U is defined by a semigroup in terms of its initial data (since the equation satisfied by U is a free transport equation.) Specifically

$$U(t,s,z) = \Sigma(t)U(0,s,z) \text{ with } \Sigma(t)\psi(s,z) = \psi(t+s,z)\,, \quad t,s>0,\ z\in\mathbf{R}^N.$$

In other words, while there does not exist any semigroup $S(t)$ acting on $L^2(\mathbf{R}^N)$ such that $S_\epsilon(t) \to S(t)$ in the weak operator topology for each $t > 0$ as $\epsilon \to 0^+$, one has

$$S_\epsilon(t) \to \int_0^{+\infty} \Sigma(t) ds$$

in that same topology.

Proof. For each $\epsilon > 0$, define

$$U_\epsilon(t,s,z) := u_\epsilon(t,z)a_\epsilon(z)e^{-sa_\epsilon(z)}\,, \quad t,s\geq 0\,,\ z\in\mathbf{R}^N\,.$$

Obviously

$$(\partial_t - \partial_s)U_\epsilon(t,s,z) = a_\epsilon(z)e^{-sa_\epsilon(z)}\left(\frac{du_\epsilon}{dt}(t,z) + a_\epsilon(z)u_\epsilon(t,z)\right)$$

so that U_ϵ satisfies

$$(3.3)\qquad \begin{cases} \partial_t U_\epsilon - \partial_s U_\epsilon = 0\,, \quad t,s>0\,,\ z\in\mathbf{R}^N\,,\\ U_\epsilon(0,s,z) = u^{in}(z)a_\epsilon(z)e^{-sa_\epsilon(z)}\,. \end{cases}$$

Since $a_\epsilon > 0$ a.e. on $\mathbf{R}^N$, one has $\|U_\epsilon(t,\cdot,\cdot)\|_{L^\infty(\mathbf{R}_+\times\mathbf{R}^N)} \le \|u^{in}\|_{L^\infty(\mathbf{R}^N)}$. Hence U_ϵ is bounded and therefore (by the Banach-Alaoglu theorem) relatively weak-* compact in $L^\infty(\mathbf{R}_+\times\mathbf{R}_+\times\mathbf{R}^N))$. If U is a weak-* limit point of U_ϵ as $\epsilon\to 0^+$, by passing to the limit in the sense of distributions in the free transport equation satisfied by U_ϵ, we conclude that $(\partial_t - \partial_s)U = 0$.

Since U_ϵ is bounded in $L^\infty(\mathbf{R}_+\times\mathbf{R}_+\times\mathbf{R}^N)$, the free transport equation satisfied by U_ϵ implies that $\partial_t U_\epsilon$ is bounded in $L^\infty(\mathbf{R}_+\times\mathbf{R}^N;W^{-1,\infty}(\mathbf{R}_+))$. Therefore, $U_{\epsilon_n}\big|_{t=0}\rightharpoonup U\big|_{t=0}$ in $L^\infty(\mathbf{R}^N;W^{-1,\infty}(\mathbf{R}_+)))$ weak-* for each subsequence $\epsilon_n\downarrow 0$ such that $U_{\epsilon_n}\rightharpoonup U$ in $L^\infty(\mathbf{R}_+\times\mathbf{R}_+\times\mathbf{R}^N)$ weak-*. Since

$$U_\epsilon\big|_{t=0}\rightharpoonup u^{in}(z)\int_0^{+\infty} ae^{-sa}d\mu_z(a) = -u^{in}(z)\frac{d\tilde\mu_z}{ds}(s)$$
$$\text{in } L^\infty(\mathbf{R}_+\times\mathbf{R}^N) \text{ weak-*}$$

and the problem (3.2) has a unique solution, $U_\epsilon\rightharpoonup U$ in $L^\infty(\mathbf{R}_+\times\mathbf{R}_+\times\mathbf{R}^N)$ weak-* as $\epsilon\to 0^+$.

Since $a_\epsilon(z)\ge\alpha>0$ a.e. in $z\in\mathbf{R}^N$, one has

$$\int_T^{+\infty}|U_\epsilon(t,s,z)|ds = e^{-Ta_\epsilon(z)}|u_\epsilon(t,z)| \le e^{-T\alpha}\|u^{in}\|_{L^\infty(\mathbf{R}^N)}$$

so that, for each test function $\phi\in L^1(\mathbf{R}_+\times\mathbf{R}^N)$,

$$\int_0^{+\infty}\int_{\mathbf{R}^N}\left(\int_T^{+\infty}|U_\epsilon(t,s,z)|ds\right)|\phi(t,z)|dtdz \to 0 \text{ uniformly in } \epsilon>0$$

as $T\to+\infty$, by dominated convergence. Since on the other hand

$$u_\epsilon(t,z) = \int_0^{+\infty}U_\epsilon(t,s,z)ds$$

we conclude that

$$u_\epsilon = \int_0^{+\infty}U_\epsilon ds \rightharpoonup \int_0^{+\infty}U ds$$

in $L^\infty(\mathbf{R}_+\times\mathbf{R}^N)$ weak-*. □

4. Homogenization of opacities in radiative transfer

In this section, we shall apply the method described above to the equation of radiative transfer.

Radiative transfer is a kinetic theory for a gas of photons exchanging energy with a background material (such as a plasma, a stellar or a planetary atmosphere). This energy exchange is the result of absorption, emission or scattering of photons by the atoms in the background matter. The state at time t of the population of photons is given by the specific radiative intensity denoted $I(t,x,\omega,\nu)$ that is $ch\nu$ times the number density of photons with frequency ν located at the position x with direction ω. Here, c is the speed of light while h is Planck's constant.

Neglecting scattering processes, the radiative intensity satisfies the radiative transfer equation

$$(4.1)\qquad \frac{1}{c}\partial_t I + \omega\cdot\nabla_x I = \sigma(\nu,T)B_\nu(T) - \sigma(\nu,T)I\,.$$

Here $B_\nu(T)$ is the specific radiative intensity at frequency ν of a black body at temperature T, while $\sigma(\nu,T)>0$ is the opacity, or absorption cross-section per

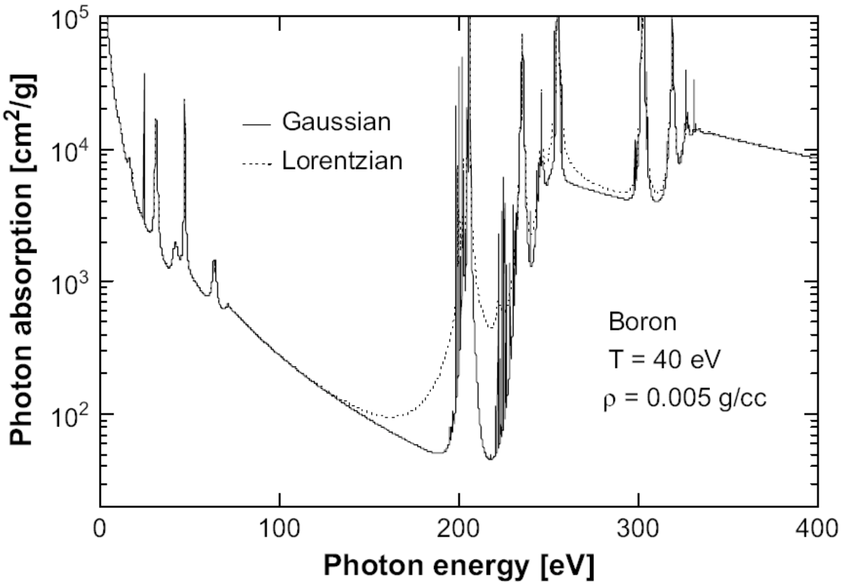

FIGURE 1. Opacity of a boron plasma (see [4] on p. 98)

unit volume, of the background material at temperature T for an incident radiation with frequency ν. While $B_\nu(T)$ has the explicit expression

$$B_\nu(T) = \tfrac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}\,,$$

the opacity $\sigma(\nu,T)$ is in general not known explicitly but tabulated. What is worse, the dependence of $\sigma(\nu,T)$ in either ν or T is quite involved, and the function $\nu \mapsto \sigma(\nu,T)$ can be wildly oscillating, even for T fixed, as can be seen on the graph given above.

We recognize in (4.1) the same type of difficulty that was handled in the two previous sections, since oscillations in the opacity $\sigma(\nu,T)$ are due to the dependence of that coefficient in the frequency ν, while the streaming (or free transport) operator $\frac{1}{c}\partial_t + \omega\cdot\nabla_x$ acts on the variables t and x only.

Henceforth, we assume for simplicity that the temperature $T \equiv T(t,x)$ is given in the background medium which occupies the Euclidian space $\mathbf{R}^3$. We consider the following model problem:

$$\left\{\begin{aligned}&\frac{1}{c}\partial_t I_\epsilon + \omega\cdot\nabla_x I_\epsilon = \sigma_\epsilon(\nu,T)B_\nu(T) - \sigma_\epsilon(\nu,T)I_\epsilon\,,\\&I_\epsilon\big|_{t=0} = I^{in}(x,\omega,\nu)\,,\end{aligned}\right. \tag{4.2}$$

posed for $(t,x,\omega,\nu)\in\mathbf{R}_+^*\times\mathbf{R}^3\times\mathbf{S}^2\times\mathbf{R}_+^*$. Here the oscillations of the opacity are recorded by the small parameter ϵ that is the typical "oscillation wavelength" in the variable ν.

We henceforth assume that the given temperature profile T is bounded away from 0 and $+\infty$, i.e. that $T\in[\theta,\Theta]$ for some constants $0<\theta<\Theta$, and that the family $(\sigma_\epsilon(\nu,T))_{\epsilon>0}$ satisfies the uniform bound

$$0 < m \le \sigma_\epsilon(\nu,T) \le M\,,\qquad \text{for each } \epsilon,\nu>0 \text{ and } T\in[\theta,\Theta]\,.$$

Furthermore, we assume that, for each $T>0$, the family $\sigma_\epsilon(\cdot,T)$ converges in the sense of Young measures to $(\mu_\nu^T)_{\nu>0}$ as $\epsilon\to 0^+$. By the method introduced in the previous section, we can formulate a theorem on the homogenized limit of the model problem (4.2) in the following manner.

Theorem 4.1. *In the limit as $\epsilon \to 0^+$, one has*

$$I_\epsilon \rightharpoonup I = \int_0^{+\infty} J ds \text{ in } L^\infty(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{S}^2 \times \mathbf{R}_+) \text{ weak-}*,$$

where $J \equiv J(t, s, x, \omega, \nu)$ is the solution of

$$\begin{cases} \frac{1}{c}\partial_t J + \omega \cdot \nabla_x J - \partial_s J = \frac{d^2 \tilde{\mu}_\nu^T}{ds^2} B_\nu(T)\,, \\ J\big|_{t=0} = -I^{in}(x, \omega, \nu) \frac{d\tilde{\mu}_\nu^T}{ds}(s)\,, \end{cases} \tag{4.3}$$

posed for $(t, s, x, \omega, \nu) \in \mathbf{R}_+^* \times \mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{S}^2 \times \mathbf{R}_+^*$, where the notation $\tilde{\mu}_\nu^T$ denotes the Laplace transform of μ_ν^T.

The proof of this theorem is essentially the same as that of Proposition 3.1 — except for the source term in (4.2) — and we do not repeat it.

Observe that the homogenized problem (4.3) is a transport equation where the space variables are x and s, and therefore defines a semigroup acting on the extended phase space $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{S}^2 \times \mathbf{R}_+ = \{(s, x, \omega, \nu)\}$, instead of the usual phase space $\mathbf{R}^3 \times \mathbf{S}^2 \times \mathbf{R}_+ = \{(x, \omega, \nu)\}$ familiar in radiative transfer problems. More precisely, the solution of (4.3) is given in terms of the Duhamel formula for the transport semigroup in extended phase space defined by the left-hand side of that equation. This is at variance with the homogenized radiative transfer equations obtained in [5, 8] which are written in the usual phase space, but involve memory terms as in Tartar's example — and precisely for the same reason.

Notice that we have assumed that the initial data I^{in} does not have fast oscillations in the ν variable — as is the case of B_ν. In general, treating the case of an oscillating initial data I_ϵ^{in} (in the ν variable, say) would require considering the joint Young measure of I_ϵ^{in} and σ_ϵ — i.e. the Young measure of the couple $(I_\epsilon^{in}, \sigma_\epsilon)$ viewed as a function of ν with values in $\mathbf{R}^2$. The complexity of the resulting model could be reduced in the case where the oscillations of I_ϵ^{in} and σ_ϵ are independent so that the joint Young measure is the tensor product of the Young measure of I_ϵ^{in} by that of σ_ϵ.

A few words about the meaning of the additional variable s appearing in the homogenized equation (4.3) are in order. The original equation (4.2) can be viewed as a balance equation for the number density of photons with frequency ν located at the position x with direction ω at time t, that is $\frac{1}{ch\nu} I_\epsilon(t, x, \omega, \nu)$. The loss term $-\sigma_\epsilon(\nu, T) I_\epsilon(t, x, \omega, \nu)$ on the right-hand side of (4.2) models the absorption of photons by the matter as follows. Assuming for simplicity that $\sigma_\epsilon \equiv \sigma_\epsilon(\nu)$ is independent of temperature, the probability that a photon with frequency ν is not absorbed in the time interval $[0, t]$ is $e^{-t\sigma_\epsilon(\nu)}$. In the homogenized equation (4.3), the loss of photons due to absorption by the atoms of the surrounding matter is described by the term $-\partial_s J$ on the left-hand side. Any characteristic line of the streaming operator $\frac{1}{c}\partial_t + \omega \cdot \nabla_x - \partial_s$ being of the form $t \mapsto (x + ct\omega, s - t)$, the unknown quantity $\frac{1}{ch\nu} J(t, s, x, \omega, \nu)$ in (4.3) should be viewed as the number density at time t of photons with frequency ν at the position x in the direction ω which will be absorbed precisely at time $s + t$. In other words, in the homogenized model (4.3), the additional variable s should be viewed as the "life expectancy" of photons, and their number density is disintegrated with respect to — in probabilistic terms, conditioned relatively to — this new variable. The absorption of photons is described by characteristic lines of the streaming operator on the left-hand side of equation (4.3) leaving the phase space $s > 0$, and not by prescribing the probability that a photon of frequency ν is absorbed in the infinitesimal interval of time $[t, t+dt]$.

5. Conclusion

We have explained how the notion of a "kinetic theory in extended phase space" introduced in [2] can be used in the homogenization problem for opacities in radiative transfer (Theorem 4.1), and how it avoids considering integro-differential equations whose solutions do not have the semigroup property, as in Tartar's elementary example.

The formalism presented here could be applied to various problems of the same nature. For instance, as mentioned above, opacities are strongly oscillating functions of the frequency variable, which seriously complicates the discretization of the radiative transfer equation. Usually, this is done by replacing the radiative intensity $I(\nu)$ with the vector $(I_j)_{1\le j\le n}$, where

$$I_j \simeq \int_{\nu_j}^{\nu_{j+1}} I(\nu)d\nu\,,$$

and where the frequency groups — i.e. the intervals (ν_j, ν_{j+1}) — are chosen appropriately. Of course the main difficulty is to understand what to do with the absorption term

$$\int_{\nu_j}^{\nu_{j+1}} \sigma(\nu)I(\nu)d\nu\,.$$

The projection of the radiative intensity on frequency groups as above is an instance of homogenization process, and one could hope that the considerations outlined in Theorem 4.1 might be helpful in this context.

Similar difficulties exist in the theory of neutron transport — with the neutron kinetic energy being the analogue of the frequency in radiative transfer. One could hope to apply the same method as above to this type of problem also; however, scattering processes are more important and should be taken into consideration, at variance with the discussion in the present paper. We hope to return to these questions in a forthcoming publication.

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