Solution robustness

Workshop on robust and stochastic optimization method
2021/11/19

ALES Zacharie[†,‡] ELLOUMI Sourour[†,‡]

[†] UMA, ENSTA Paris, Institut Polytechnique de Paris, 91120 Palaiseau, France

[‡] CEDRIC, CNAM, Paris, France

Sommaire

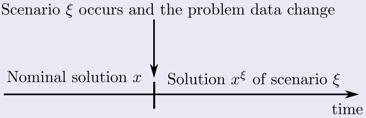
1. Solution robustness approach
2. Complexity results
3. Numerical results

Nominal optimization problem (without uncertainty)

Nominal cost

$$\overline{P}\begin{cases} \min & c(x) \\ \text{s.t.} & x \in X \end{cases}$$

Adaptation to uncertainty



Objective of most robust optimization approaches

Minimize the nominal cost $c(x)$ or $c(x^{\xi})$

Quality robustness

What should also be optimized

Minimize a distance $d(x, x^{\xi})$

Solution robustness

Solution robustness is important when...

... the nominal solution x is implemented on a regular basis

Risk of human errors

A traveler may miss a train which leaves earlier than usual
An operator may activate a railroad switch no longer required

... the uncertainty is revealed late

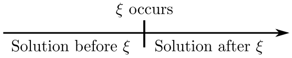
Not enough time to implement many changes

Transfers of trains between stations take time
Changes may require synchronization between different services

Definition - **Solution cost**

Quantify the cost of adapting x into x^{ξ}

May disrupt the realization of x^{ξ} or even make it infeasible
$\rightarrow$ can be more important than the nominal cost



Approach	Solution Before ξ	Solution After ξ	Nominal cost Before ξ	Nominal cost After ξ	Solution cost
Classical robustness [Soyster, 1973, Bertsimas and Sim, 2004]	x	x No modification	Minimized	-	$= 0$
Recoverable robustness [Liebchen et al., 2009]	x, A Algorithm	$A(x, \xi)$	Minimized	-	A can limit $d(x, A(x, \xi))$

ξ occurs

Solution before ξ | Solution after ξ

Approach	Solution Before ξ	Solution After ξ	Nominal cost Before ξ	Nominal cost After ξ	Solution cost
Classical robustness	x	x	Minimized	-	$= 0$
[Soyster, 1973, Bertsimas and Sim, 2004]		No modification			
Recoverable robustness	x, A	$A(x, \xi)$	Minimized	-	A can limit $d(x, A(x, \xi))$
[Liebchen et al., 2009]	Algorithm				
k-distance	x	x^{ξ}	Minimized	-	$d(x, x^{\xi}) \leq k$
[Büsing, 2012]					
Anchored	$(x_a, x_{\overline{a}})$	$(x_a, x_{\overline{a}}^{\xi})$	Constrained	-	Maximize anchored variables
[Bendotti et al., 2019]	Anchored variables				
Recovery to optimality	x	x^{ξ}	-	Minimized	Minimized (secondary objective)
[Goerigk and Schöbel, 2014]	Not necessarily feasible				

ξ occurs

Solution before ξ | Solution after ξ

Approach	Solution: Before ξ	Solution: After ξ	Nominal cost: Before ξ	Nominal cost: After ξ	Solution cost
Classical robustness [Soyster, 1973, Bertsimas and Sim, 2004]	x	x No modification	Minimized	-	$= 0$
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Anchored [Bendotti et al., 2019]	$(x_a, x_{\bar{a}})$ Anchored variables	$(x_a, x_{\bar{a}}^{\xi})$	Constrained	-	Maximize anchored variables
Recovery to optimality [Goerigk and Schöbel, 2014]	x Not necessarily feasible	x^{ξ}	-	Minimized	Minimized (secondary objective)
Reactive [Ales and Elloumi, 2021]	x	x^{ξ}	-	-	Minimize $d(x, x^{\xi})$ **a posteriori**
Proactive [Ales and Elloumi, 2021]	x	x^{ξ}	Constrained	-	Minimize $d(x, x^{\xi})$ **a priori**

Our problems

Approach	Solution		Nominal cost		Solution cost
	Before ξ	After ξ	Before ξ	After ξ	
Reactive	x	x^ξ	-	-	Minimize $d(x, x^\xi)$ **a posteriori**
Proactive	x	x^ξ	Constrained	-	Minimized $d(x, x^\xi)$ **a priori**

Reactive problem

$$P^r(\underbrace{\xi_i}_{\text{a scenario}}, x) \begin{cases} \min & d(x^i, x) \\ \text{s.t.} & x^i \in \underbrace{X_i}_{\text{Feasible set of } \xi_i} \end{cases}$$

Returns

- one reactive solution x^i

Proactive problem

$$P^p(\underbrace{\mathcal{S}}_{\text{Set of scenarios}}, c^*) \begin{cases} \min & \sum_{\xi_i \in \mathcal{S}} \underbrace{w_i}_{\text{Scenarios weight}} d(x, x^i) \\ \text{s.t.} & x \in X \\ & x^i \in X_i \qquad \xi_i \in \mathcal{S} \\ & c(x) = c^* \end{cases}$$

Returns

- one proactive solution x and
- $|\mathcal{S}|$ scenario solutions $\{x^i\}_{i \in \mathcal{S}}$

Two solution cost distances

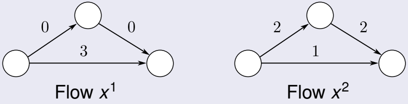
Distance in values (ℓ_1 norm)

$$d_{val}(x^1, x^2) \triangleq \sum_{j=1}^{n} |x_j^1 - x_j^2|$$

Distance in structure

$$d_{struct}(x^1, x^2) \triangleq \sum_{j=1}^{n} |\mathbb{1}_{x_j^1 > 0} - \mathbb{1}_{x_j^2 > 0}|$$

Example



- $d_{val}(x^1, x^2) = 6$
- $d_{struct}(x^1, x^2) = 2$

Sommaire

1. Solution robustness approach
2. Complexity results
3. Numerical results

Polynomial problems considered

- Integer min-cost flow
  Uncertain arc demands
- Integer max-flow
  Uncertain capacities

Complexity results for both problems

Problem	Distance	
	d_{val}	d_{struct}
Proactive	?	?
Reactive	?	?

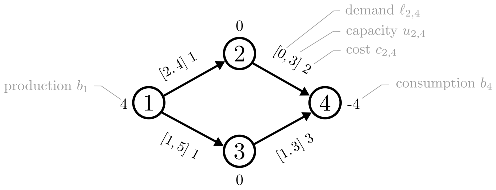
Integer min-cost flow

Data

- $G = (V, A)$
- b_v: production/consumption of $v \in V$ $\sum_v b_v = 0$
- c_a: unitary flow cost of $a \in A$
- ℓ_a / u_a: minimal demand / maximal capacity of $a \in A$

Definition (polynomial)

Find an integer flow of minimal cost which satisfies the productions, consumptions, demands and capacities



Uncertainty

Arc demands ℓ are uncertain

Integer min-cost flow

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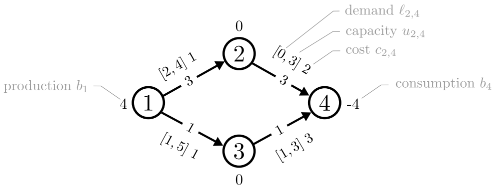


Uncertainty

Arc demands ℓ are uncertain

Theorem

$MCF^r_{d_{val}}$ is polynomial

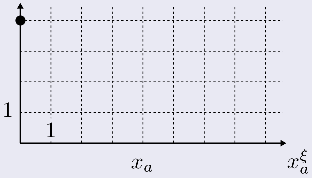
Reactive problem with distance d_{val}

Problem	Distance	
	d_{val}	d_{struct}
Proactive	?	?
Reactive	Polynomial	?

Property

The cost of an arc is a piecewise linear convex function of its flow

$|x_a - x_a^\xi|$



$\Rightarrow$ $MCF^r_{d_{val}}$ is a convex cost flow

Theorem [Ahuja et al., 1988]

A convex cost flow problem with piecewise linear convex cost functions can be transformed into a min-cost flow problem

Theorem

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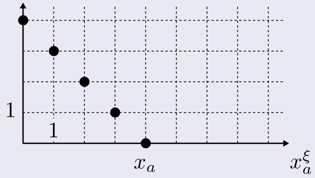

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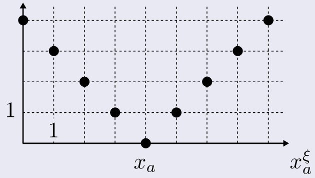

$\Rightarrow MCF^r_{d_{val}}$ is a convex cost flow

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Theorem

Proactive problem with distance d_{val}

$MCF^p_{d_{val}}$ is $\mathcal{NP}$-hard

Reduction from 3-SAT

Problem	Distance	
	d_{val}	d_{struct}
Proactive	$\mathcal{NP}$-hard	?
Reactive	Polynomial	?

3-SAT

x_i or $\overline{x}_i$

- n boolean variables $X = \{x_1, ..., x_n\}$
- m clauses $\{C_1, ..., C_m\}$ each containing a conjunction of three literals

Is there an affectation of X which satisfies all clauses?

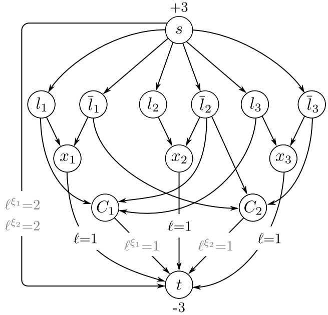
e.g.: $C_1 = (x_1 \vee \overline{x}_2 \vee x_3)$ and $C_2 = (\overline{x}_1 \vee \overline{x}_2 \vee \overline{x}_3)$

Sketch of the proof

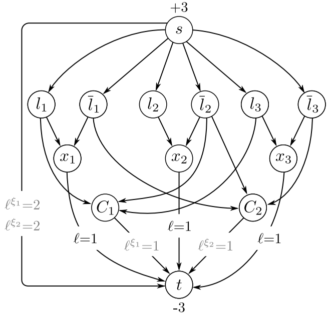
$$\forall I_{SAT} \quad (\exists I_{MCF} \text{ of } d_{val} \leq 4mn \text{ iff } I_{SAT} \text{ is true})$$

Instance of 3-SAT

Instance of the proactive problem with d_{val}



Arc demands which are $\neq 0$



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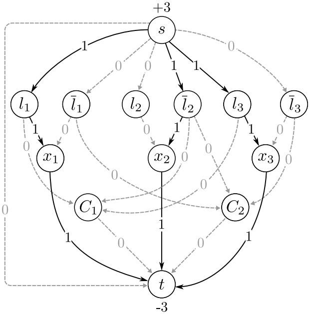
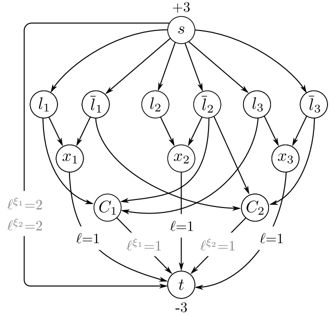


Nominal flow



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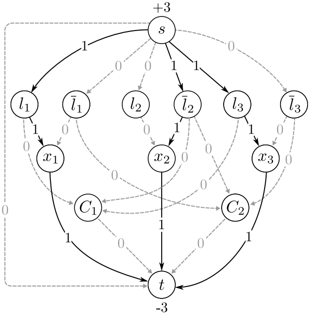
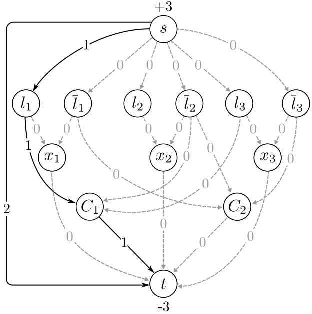
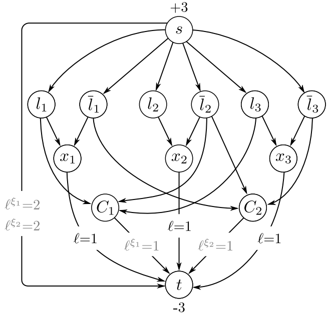


Nominal flow



Flow of the first scenario



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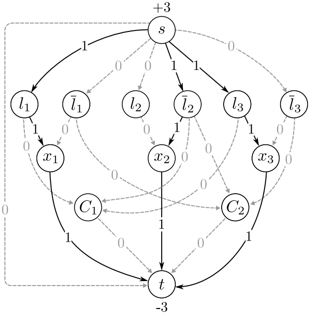

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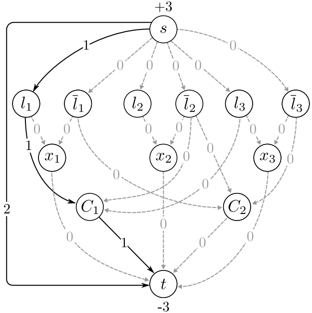

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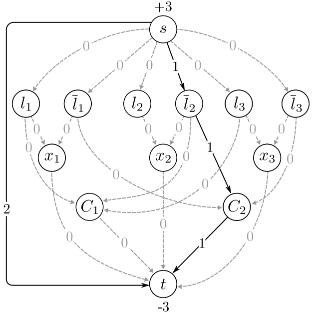

Flow of the second scenario

Complexity results summary for both flow problems

Problem	Distance	
	d_{val}	d_{struct}
Proactive	$\mathcal{NP}$-hard	$\mathcal{NP}$-hard even with 1 scenario
Reactive	Polynomial	$\mathcal{NP}$-hard

Sommaire

1 Solution robustness approach

2 Complexity results

3 Numerical results

Approaches comparison

Approach	Solution		Nominal cost		Solution cost
	Before ξ	After ξ	Before ξ	After ξ	
Anchored [Bendotti et al., 2019]	$(x_a, x_{\bar{a}})$ Anchored variables	$(x_a, x_{\bar{a}}^{\xi})$	Constrained	-	Maximize anchored variables
k-distance [Büsing, 2012]	x	x^{ξ}	Minimized	-	$d(x, x^{\xi}) \leq k$
Proactive [Ales and Elloumi, 2021]	x	x^{ξ}	Constrained	-	Minimize $d(x, x^{\xi})$

Problem considered

A railway scheduling problem in which lines frequency must be fixed

Uncertainty

Number of passengers for each possible (origin, destination)
20% of the values are unchanged
80% of the values are in $\pm$ 10% of their nominal value

Modelisation

MILP compact formulations solved with CPLEX

Approaches comparison

$\|\mathcal{S}\|$	Method	Nominal cost $c(x)$	Solution cost d_{val}	Number of anchored lines
2	Proactive	**81742**	**50**	196 (-2%)
	Anchored	81742	76 (+52%)	**200**
	0-distance	89028 (+9%)	0 (-100%)	210 (+5%)
	1-distance	86663 (+6%)	12 (-76%)	208 (+4%)
	2-distance	84740 (+4%)	19 (-62%)	206 (+3%)
	4-distance	82156 (+1%)	37 (-26%)	202 (+1%)
	10-distance	81742	81 (+62%)	196 (-2%)
6	Proactive	**81742**	**120**	186 (-7%)
	Anchored	81742	246 (+105%)	**199**
	0-distance	91998 (+13%)	0 (-100%)	210 (+6%)
	1-distance	87274 (+7%)	28 (-77%)	204 (+3%)
	2-distance	85178 (+4%)	53 (-56%)	200 (+1%)
	4-distance	82158 (+1%)	105 (-12%)	193 (-3%)
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Conclusion

- New robust approach
- Optimize the solution robustness
- Complexity results for two network problems and two distances
- Numerical results which highlight advantages over other approaches

Ales, A., and Elloumi, S. (2021).
A solution robustness approach applied to network optimization problems.
Available on arXiv: https://arxiv.org/abs/2110.11647

Perspectives

- Identification of polynomial proactive integer problems
- Uncertainty sets rather than set of scenarios
  Box, budgeted, polytope, ...
- Bi-objective resolution
  Solution cost and nominal cost

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Theorem

Proactive problem with distance d_{struct}

$MCF^p_{d_{struct}}$ is $\mathcal{NP}$-hard

Reduction from 3-Partition

Problem	Distance d_{val}	Distance d_{struct}
Proactive	$\mathcal{NP}$-hard	$\mathcal{NP}$-hard even with 1 scenario
Reactive	Polynomial	?

3-Partition

- $E = \{1, 2, ..., 3m\}$: elements
- $s(i)$: size of $i \in E$ in $]\frac{B}{4}, \frac{B}{2}[$
  $B \in \mathbb{N}$

Is there a partition $\{E_1, ..., E_m\}$ of E such that $\sum_{i \in E_j} s(i) = B$?

e.g.: $B = 100$, $m = 2$, elements size $\{30, 30, 30, 35, 35, 40\}$

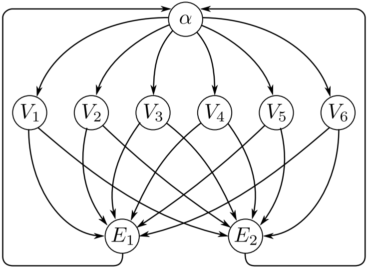
Sketch of the proof

$$\forall I_P \quad (\exists I_{MCF} \text{ of } d_{struct} \leq 3m^2 + 4m \text{ iff } I_P \text{ is true})$$

Instance of 3-partition (I_P) — Instance of the proactive problem with d_{struct} (I_{MCF})

Example of instance I_P

- $B = 100$
- Partition with 2 subsets
- Elements size: $\{30, 30, 30, 35, 35, 40\}$



I_{MCF} graph ($b = \ell = 0$, $c = 1$, $c^* = 0$, $u = 100$)

Nominal flow is empty

$\Rightarrow d_{struct} =$ # or arcs in the scenario flow

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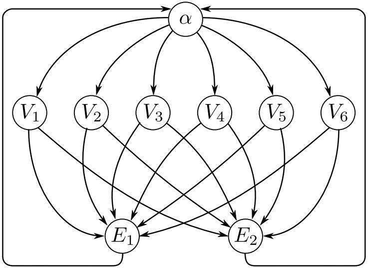

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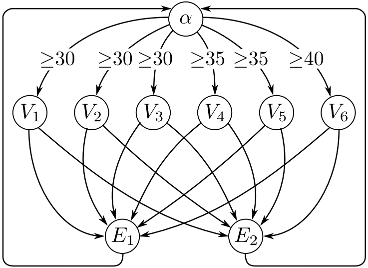

Scenario demands $\ell^{\xi} \neq 0$

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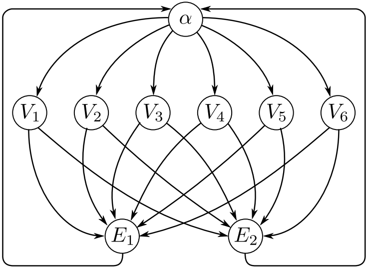

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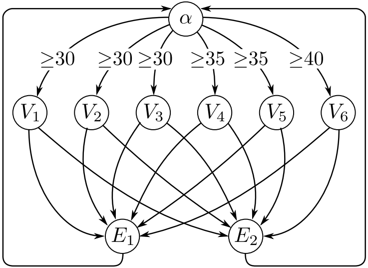

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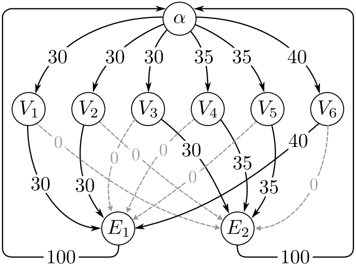

Proactive flow

Theorem

$MCF^r_{d_{struct}}$ is $\mathcal{NP}$-hard.

Reactive problem with distance d_{struct}

Problem	Distance	
	d_{val}	d_{struct}
Proactive	$\mathcal{NP}$-hard	$\mathcal{NP}$-hard even with 1 scenario
Reactive	Polynomial	$\mathcal{NP}$-hard

Proof

Same reduction but with the unique optimal flow as an input rather than an output

Integer max-flow problem with uncertain capacities

Same 4 complexity results

Proactive / Reactive comparison

$\|\mathcal{S}\|$	Solution	Solution cost d_{val}
	Proactive	**50**
	Reactive (x^A, f^A)	50
2	Reactive (x^B, f^B)	57 (+14%)
	Reactive (x^C, f^C)	62 (+24%)
	Reactive (x^D, f^D)	56 (+12%)
	Proactive	**120**
	Reactive (x^A, f^A)	120
6	Reactive (x^B, f^B)	123 (+2%)
	Reactive (x^C, f^C)	133 (+11%)
	Reactive (x^D, f^D)	121 (+1%)
	Proactive	**196**
	Reactive (x^A, f^A)	197 (+1%)
10	Reactive (x^B, f^B)	211 (+8%)
	Reactive (x^C, f^C)	220 (+12%)
	Reactive (x^D, f^D)	204 (+4%)

$\|\mathcal{S}\|$	Solution	Solution costs d_{struct}
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	Proactive	**23**
	Reactive (x^A, f^A)	26 (+13%)
6	Reactive (x^B, f^B)	28 (+22%)
	Reactive (x^C, f^C)	24 (+4%)
	Reactive (x^D, f^D)	23
	Proactive	**37**
	Reactive (x^A, f^A)	38 (+3%)
10	Reactive (x^B, f^B)	43 (+16%)
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