Min-sup-min robust combinatorial optimization with few recourse solutions

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What is the min-max-min problem?

- It is an extension of the classical robust combinatorial optimization problem:

$$\min_{\boldsymbol{x} \in X} \max_{\xi \in \Xi} \boldsymbol{c}(\xi)^\top \boldsymbol{x} \tag{C-RO}$$

- $X \subseteq \{0,1\}^n$: combinatorial feasibility set
- $\Xi \subset \mathbb{R}^n$: uncertainty set
- $\boldsymbol{c}(\xi)$: cost vector affine in ξ

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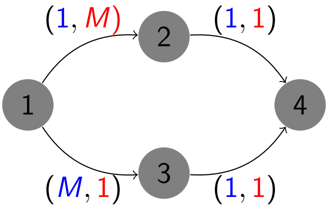
- $[K] = \{1, \ldots, K\}$

$$\min_{\boldsymbol{x}^1,\ldots,\boldsymbol{x}^K \in X} \; \max_{\xi \in \Xi} \; \min_{k \in [K]} \; \boldsymbol{c}(\xi)^\top \mathbf{x}^k \qquad \text{(MMM)}$$

1. prepare K solutions
2. nature chooses worst $\xi \in \Xi$
3. choose the best among the K solutions prepared

Example: Shortest path problem

- Ξ contains two scenarios



- Solving the robust combinatorial problem leads to a solution of cost $M + 1$.
- Setting $K = 2$ we obtain $\mathbf{x}^1 = \{(1,2),(2,4)\}$, $\mathbf{x}^2 = \{(1,3),(3,4)\}$.
- Under any scenario realization, there remains an alternative route of cost 2.

Extension: min-sup-min problem

- $X_0 \subseteq \{0,1\}^n$: combinatorial feasibility set
- $\Xi \subset \mathbb{R}^n$: uncertainty set
- $[K] = \{1, \ldots, K\}$
- $X_1(\xi) = \{x \in X_0 \mid g_\ell(\xi, x) \leq b_\ell,\ \ell \in [L]\}$

$$\min_{\mathbf{x}^1, \ldots, \mathbf{x}^K \in X_0} \ \sup_{\xi \in \Xi} \ \min_{k \in [K] \,:\, \mathbf{x}^k \in X_1(\xi)} f(\xi, \mathbf{x}^k) \qquad \text{(MSM)}$$

1. prepare K solutions
2. nature chooses worst $\xi \in \Xi$
3. choose the best feasible solution among the K solutions prepared

Why the supremum?

- Define the function $F_{\mathbf{x}} : \Xi \to \mathbb{R} \cup \{+\infty\}$ as:

$$F_{\mathbf{x}}(\xi) := \begin{cases} \min\limits_{k \in [K] \,:\, \mathbf{x}^k \in X_1(\xi)} f(\xi, \mathbf{x}^k) & \text{if } \exists k \in [K] \;:\; \mathbf{x}^k \in X_1(\xi) \\ +\infty & \text{otherwise.} \end{cases}$$

- We need to solve $\sup_{\xi \in \Xi} F_{\mathbf{x}}(\xi)$.
- !!! $F_{\mathbf{x}}$ is not continuous in $\xi \in \Xi$ in general !!!.

min-max-min

- Buchheim and Kurtz (17, 18) - coin min-max-min
- Chassein et al. (19), Goerigk et al. (20) - algorithmic improvements, focus on budgeted uncertainty
- Chassein and Goerigk(20), Goerigk et al. (20) - complexity results for budgeted uncertainty

K-adaptability

$$\min_{\substack{\boldsymbol{y}\in Y\\ \boldsymbol{x}^k\in X, k\in[K]}} \max_{\xi\in\Xi} f(\boldsymbol{y},\xi)+\min_{k\in[K]}\left\{g(\mathbf{x}^k,\xi) : T(\xi)\boldsymbol{y} + W\mathbf{x}^k \le h,\ \forall\xi\in\Xi\right\}$$

References

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K-adaptability

- Bertsimas and Caramanis (11) - introduction of finite adaptability
- Hanasusanto et al. (15) - $K-$adaptability in binary setting, dualization, approximation, ...
- Subramanyam et al. (19) - $K-$adaptability in mixed-integer setting, b&b algorithm, ...

Discrete relaxation

- Let

$$v(X_0, \Xi) := \min_{\mathbf{x}^1,\ldots,\mathbf{x}^K \in X_0} \; \sup_{\xi \in \Xi} \; \min_{k \in [K] \,:\, \mathbf{x}^k \in X_1(\xi)} f(\xi, \mathbf{x}^k) \tag{MSM}$$

- Consider the relaxation with $\tilde{\Xi} \subseteq \Xi$

$$v(X_0, \tilde{\Xi}) := \min_{\mathbf{x}^1,\ldots,\mathbf{x}^K \in X_0} \; \max_{\xi \in \tilde{\Xi}} \; \min_{k \in [K] \,:\, \mathbf{x}^k \in X_1(\xi)} f(\xi, \mathbf{x}^k) \tag{MSM-R}$$

- We have that $v(X_0, \tilde{\Xi}) \leq v(X_0, \Xi)$.

(MSM-R): *p*-center formulation

- $\tilde{\Xi}$ is finite
- X_0 is finite

parameters

$X_0 = \{\mathbf{x}_s, \forall s \in [r]\}$
$\tilde{\Xi} = \{\xi^j, \forall j \in [m]\}$
$\mathcal{I} = \{s \in [r], j \in [m] : \exists \ell \in L \text{ s.t. } g(\xi^j, \mathbf{x}_s) > b_\ell\}$

variables

ω: epigraph formulation
z_s: solution s is used
y_{sj}: solution s is assigned to scenario j

(MSM-R): *p*-center formulation

$$
\begin{aligned}
\min \quad & \omega \\
\text{s.t.} \quad & \omega \geq \sum_{s \in [r]} f(\xi^j, \mathbf{x}_s) y_{sj} && \forall j \in [m] \\
& \sum_{s \in [r]} y_{sj} = 1 && \forall j \in [m] \\
& \sum_{s \in [r]} z_s \leq K \\
& y_{sj} \leq z_s && \forall\ s \in [r], j \in [m] \\
& {\color{red} y_{sj} = 0} && {\color{red} \forall (s,j) \in \mathcal{I}} \\
& z \in \{0,1\}^r, y \in \{0,1\}^{r \times m}
\end{aligned}
$$

(MSM-R): radius-based approach

- let ρ be a given cost
- question: is the optimal value less than ρ?

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- $cov_{sj}(\rho) = 1 \iff f(\xi^j, \mathbf{x}_s) \leq \rho$ and $g(\xi^j, \mathbf{x}_s) \leq b_\ell \quad \forall \ell \in L$

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- answer **yes** $\iff$ the following is **feasible**

$$\sum_{s\in[r]} cov_{sj}(\rho)\, y_{sj} = 1 \qquad \forall j \in [m]$$

$$\sum_{s\in[r]} z_s \le K$$

$$y_{sj} \le z_s \qquad \forall s \in [r]\ j \in [m]$$

$$y \in \{0,1\}^{r\times m}, z \in \{0,1\}^r$$

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&y \in \{0,1\}^{r \times m}, z \in \{0,1\}^r &&
\end{aligned}$$

- algorithm: binary search on ρ

- Let $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}}^1, \tilde{\mathbf{x}}^2, \ldots, \tilde{\mathbf{x}}^K)$ be an optimal solution to (MSM-R).
- Is $\tilde{\mathbf{x}}$ feasible/optimal for (MSM)?
- Separation problem:

$$v(\tilde{\mathbf{x}}, \Xi) := \sup_{\xi \in \Xi} F_{\tilde{\mathbf{x}}}(\xi) = \sup_{\xi \in \Xi} \min_{k \in [K] \,:\, \tilde{\mathbf{x}}^k \in X_1(\xi)} f(\xi, \tilde{\mathbf{x}}^k).$$

- We have that $v(X_0, \Xi) \leq v(\tilde{\mathbf{x}}, \Xi)$.

Alternative separation scheme[1]

- Let $\tilde{v} = v(X_0, \tilde{\tilde{\Xi}})$
- Define the maximum violation of $\tilde{\mathbf{x}}$ by $\xi \in \Xi$:

$$S(\xi, \tilde{v}, \tilde{\mathbf{x}}) = \min_{k \in [K]} \max \left\{ f(\xi, \tilde{\mathbf{x}}^k) - \tilde{v}, \max_{\ell \in [L]} \left\{ g_\ell(\xi, \tilde{\mathbf{x}}^k) - b_\ell \right\} \right\}.$$

- $S(\xi, \tilde{v}, \tilde{\mathbf{x}})$ is continuous in $\xi \in \Xi$.
- We may define the separation problem:

$$S(\tilde{v}, \tilde{\mathbf{x}}) = \max_{\xi \in \Xi} \; S(\xi, \tilde{v}, \tilde{\mathbf{x}}) \qquad \text{(Sep)}$$

- $\tilde{\mathbf{x}} \in X_0^K$ is optimal with value $\tilde{v}$ if and only if $S(\tilde{v}, \tilde{\mathbf{x}}) \leq 0$.

[1] Subramanyam et al. (19)

Solving (Sep): MIP formulation

$$
\begin{aligned}
S(\tilde{v}, \tilde{\mathbf{x}}) = \max \quad & \eta \\
\text{s.t.} \quad & \sum_{\ell=0}^{L} u_{k\ell} = 1, && \forall k \in K \\
& u_{k0} = 1 \implies \eta \le f(\xi, \tilde{\mathbf{x}}^k) - \tilde{v}, && \forall k \in K \\
& u_{k\ell} = 1 \implies \eta \le g_\ell(\xi, \tilde{\mathbf{x}}^k) - b_\ell, && \forall k \in K, \ell \in L \\
& \xi \in \Xi \\
& u \in \{0,1\}^{K \times (L+1)}.
\end{aligned}
$$

Scenario generation algorithm

Input: Enumerated set of solutions X_0;
Initialization: $\tilde{\Xi} = \{\xi^0\}$, $\mathbf{x}^* = null$;
repeat
    Compute $\tilde{v} = v(X_0, \tilde{\Xi})$ and $\tilde{\mathbf{x}}$ by solving (MSM-R), set $\tilde{v} = +\infty$ and $\tilde{\mathbf{x}} = null$ if the problem is infeasible ;
    if $\tilde{\mathbf{x}} \neq null$ **then**
        Compute $S(\tilde{v}, \tilde{\mathbf{x}})$ and $\tilde{\xi} \in \arg\max_{\xi \in \Xi} S(\xi, \tilde{v}, \tilde{x})$ by solving (Sep);
        if $S(\tilde{v}, \tilde{\mathbf{x}}) > 0$ **then**
            $\tilde{\Xi} \leftarrow \tilde{\Xi} \cup \{\tilde{\xi}\}$;
            scenario_added = **true**;
        else
            scenario_added = **false**;
            $\mathbf{x}^* = \tilde{\mathbf{x}}$;
        end
    end
    else **break** ;
until *scenario_added* = **false**;
Return: $\mathbf{x}^*$

Reducing $|X_0|$

- only *good* solutions matter
  example:
  - consider an UB of 10
  - consider $\tilde{\mathbf{x}} \in X_0$ such that $\min_{\xi \in \Xi} f(\xi, \tilde{\mathbf{x}}) > 10$
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- $X_0(UB) = \left\{ \mathbf{x} \in X_0 : \min_{\xi \in \Xi} f(\xi, \mathbf{x}) \leq UB \right\}$

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computing $X_0(UB)$ recursively

- define $X_0(\bar{\mathbf{x}})$ as possible extensions of the partial solution $\bar{\mathbf{x}}$

$$X_0(\bar{\mathbf{x}}) = \{\mathbf{x} \in X_0 : \mathbf{x}_i = \bar{\mathbf{x}}_i, \ \forall i \in I\}$$

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- pruning of $\bar{\mathbf{x}}$ if

$$\min_{\xi \in \Xi, \boldsymbol{x} \in X_0(\bar{\mathbf{x}})} f(\xi, \boldsymbol{x}) > UB$$

Comparison of solution methods

Reference	Abbr.	Ξ	f	$X_1(\xi)$
Hanasusanto et al. (15)	M	P	lin.	Y
Subramanyam et al. (19)	BB	Any	lin.	Y
Chassein et al. (19)	EN	B	lin.	N
This work	SG	Any	nonlin.	Y

results: SP instances from Hanasusanto et al. (2015)

$$\min_{\mathbf{x}^1,\dots,\mathbf{x}^K \in X_0} \quad \max_{\xi \in \Xi} \quad \min_{k \in [K]} \sum_{(i,j)\in\mathcal{A}} (\bar{c}_{ij} + \hat{c}_{ij}\xi_{ij}) {x_{ij}}^k$$

- $X_0 = \{\mathbf{x} \in \{0,1\}^{|A|} : \sum_{(i,j)\in\delta^+(i)} x_{ij} - \sum_{(j,i)\in\delta^-(i)} x_{ij} = b_i, \forall i \in V\}$
- $\Xi = \{\xi \in [0,1]^{|A|} : \sum_{(i,j)\in\mathcal{A}} \xi_{ij} \leq \Gamma\}$

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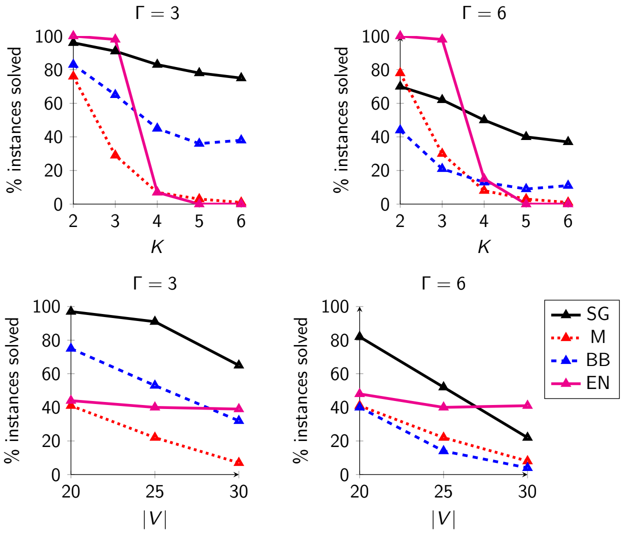


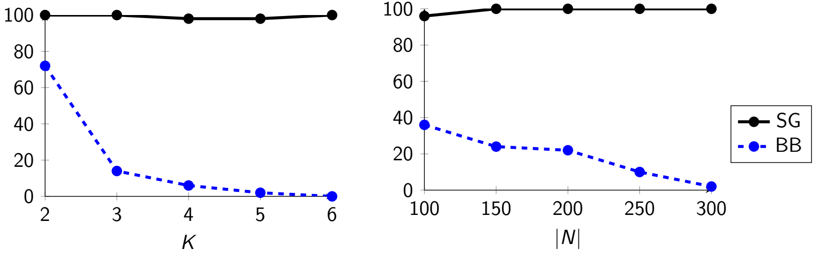
Figure: Percentage of instances solved in 2 hours.

results: Knapsack with conflicts

$$\min_{\mathbf{x}^1,\dots,\mathbf{x}^K \in X_0} \max_{\xi \in \Xi} \min_{k \in [K]\,:\,\mathbf{x}^k \in X_1(\xi)} \sum_{i \in N} \Big(1 + \sum_{j \in M} \frac{\Phi_{ij}\xi_j}{2}\Big) \bar{p}_i {x_i}^k$$

- $X_0 = \left\{ x \in \{0,1\}^{|N|} : x_i + x_j \leq 1 \quad \forall (i,j) \in C \right\}$
- $X_1(\xi) = \left\{ x \in X_0 : \sum_{i \in N} \Big(1 + \sum_{j \in M} \frac{\Psi_{ij}\xi_j}{2}\Big) \bar{w}_i x_i \leq B \right\}$
- $\Xi = [-1,1]^{|M|}$

results: Knapsack with conflicts



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Conclusions

take-away message

- two-stage RO with integer recourse is hard
- our proposal: assume few “good” recourse decisions are feasible
- heuristic variant for larger problems

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future work

Extensions to K-adaptability

$$\min_{\substack{\mathbf{y}\in Y\\ \mathbf{x}^k\in X, k\in[K]}} \max_{\xi\in\Xi} f(\mathbf{y},\xi)+\min_{k\in[K]}\left\{g(\mathbf{x}^k,\xi) : T(\xi)\mathbf{y}+W\mathbf{x}^k \leq h,\ \forall\xi\in\Xi\right\}$$

Thank you for your attention!

Hanasusanto et al. (15)

$$\min_{\mathbf{x}^1,\dots,\mathbf{x}^K \in X} \quad \max_{\xi \in \Xi} \quad \min_{k \in [K]} \quad \xi^\top \mathbf{C}\mathbf{x}^k$$

$$\iff \quad \min_{\mathbf{x}^1,\dots,\mathbf{x}^K \in X} \quad \max_{\xi \in \Xi} \quad \max_{\boldsymbol{\omega}} \left\{ \boldsymbol{\omega} \; : \; \boldsymbol{\omega} \le \xi^\top \mathbf{C}\mathbf{x}^k, \; k \in [K] \right\}$$

$$\iff \quad \min_{\mathbf{x}^1,\dots,\mathbf{x}^K \in X} \quad \max_{\xi, \boldsymbol{\omega}} \left\{ \boldsymbol{\omega} \; : \; \xi \in \Xi, \; \boldsymbol{\omega} \le \xi^\top \mathbf{C}\mathbf{x}^k, \; k \in [K] \right\}$$

$$\underset{\Xi = \{D\xi \le d, \xi \ge 0\}}{\overset{Dual}{\iff}} \quad \left\{ \begin{array}{lll} \min_{\mathbf{x}^1,\dots,\mathbf{x}^K \in X} & \min & \sum_{l \in [m]} d_l \boldsymbol{\alpha}_l \\ & s.t. & D_j^\top \boldsymbol{\alpha} \ge \boldsymbol{\gamma}^k \mathbf{C} \sum_{k \in [K]} \mathbf{x}_j^k \quad \forall j \in [n] \\ & & \sum_{k \in [K]} \boldsymbol{\gamma}^k = 1 \\ & & \boldsymbol{\gamma}, \boldsymbol{\alpha} \ge 0 \end{array} \right.$$

Subramanyam et al. (19)

- Branch-and-bound based on disjunctive formulation:

$$\begin{aligned} \min\ & \theta \\ \text{s.t.}\ & \theta \in \mathbb{R}, \mathbf{x}^1, \ldots, \mathbf{x}^k \in X \\ & \bigvee_{k\in[K]} \begin{bmatrix} c(\xi)^\top \mathbf{x}^k \leq \theta \\ W(\xi)\mathbf{x}^k \leq h(\xi) \end{bmatrix} \qquad \forall \xi \in \Xi \end{aligned}$$

- Solve scenario-based relaxation:

$$\begin{aligned} \min\ & \theta \\ \text{s.t.}\ & \theta \in \mathbb{R}, \mathbf{x}^1, \ldots, \mathbf{x}^k \in X \\ & \left. \begin{array}{l} c(\xi)^\top \mathbf{x}^k \leq \theta \\ W(\xi)\mathbf{x}^k \leq h(\xi) \end{array} \right\} \qquad \forall \xi \in \Xi_k, \forall k \in [K] \end{aligned}$$

- Separate and branch: $\{\Xi_1, \ldots, \Xi_k \cup \{\xi^*\}, \ldots, \Xi_K\}$ for $k \in [K]$

Discontinuous example

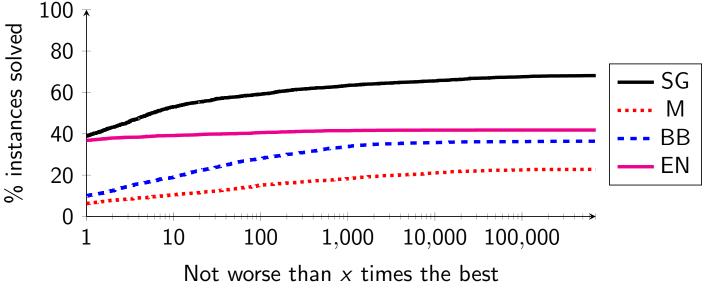
- Consider[2]:

$$\min_{\mathbf{x}^1,\mathbf{x}^2\in\{0,1\}} \sup_{\xi\in[0,1]} \min_{k\in\{1,2\}\,:\,\mathbf{x}^k\geq\xi} (\xi-1)(1-2\mathbf{x}^k)$$

- Let $\mathbf{x}^1 = 1$ and $\mathbf{x}^2 = 0$.
- If $\xi = 0$ then $\mathbf{x}^2$ is optimal with value -1.
- If $\xi > 0$ then $\mathbf{x}^1$ is optimal with value $1-\xi$.
- The function $F_{\mathbf{x}}(\xi)$ is discontinuous at $\xi = 0$.

[2] Subramanyam et al. (19)

Performance SP



Performance KP