Two-stage robust optimization with objective uncertainty

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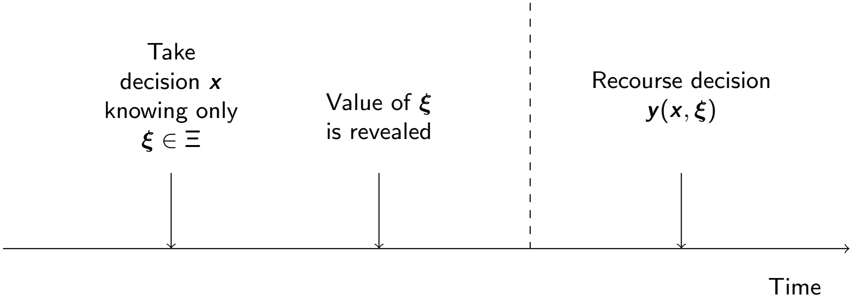
Context / Problem definition

Two-stage mixed integer robust optimization

$$\inf_{\boldsymbol{x} \in \mathcal{X}} \sup_{\boldsymbol{\xi} \in \Xi} \inf_{\boldsymbol{y} \in \mathcal{Y}(\boldsymbol{x}, \xi)} f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$$



Literature review

1. Exact approaches: Often based on dual information or using a facial description of the recourse polyhedron:
   1. Constraint generation: Atamturk and Zeng (2007), Thiele et al. (2009), Bertsimas et al. (2013), Jiang et al. (2014), Zhen et al. (2018)
   2. Constraint-and-Column generation: Ayoub and Poss (2016), Zhao and Zeng (2012), Zeng and Zhao (2013)
   3. Convexification-based: Kämmerling and Kurtz (2020), Arslan and Detienne (2021)
2. Approximate approaches:
   1. Decision rules: Recourse decisions are restricted to be functions of uncertainty: Ben-Tal et al. (2004), Chen and Zhang (2009), Goh and Sim (2010), Kuhn et al. (2011), Vayanos et al. (2011), Georghiou et al. (2015), Bertsimas and Dunning (2016), Bertsimas and Georghiou (2015,2018), Gorissen et al. (2015), Postek and den Hertog (2016)
   2. K-adaptability: K recourse decisions are selected at the first stage, optimization is done over them in the second stage:
      Hanasusanto et al. (2015), Subramanyam et al. (2020), Buchheim and Kurtz (2017), Arslan et al. (2021)

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Based on the work of Arslan and D. (2021)

- Applies to two-stage robust MILP models with objective uncertainty:

$$\min_{\boldsymbol{x} \in \mathcal{X}} \max_{\boldsymbol{\xi} \in \Xi} \min_{\boldsymbol{y} \in \mathcal{Y}(\boldsymbol{x})} f(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\xi})$$

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- The relaxation is exact under the restrictive assumption

$$\mathcal{Y}(\boldsymbol{x}) = \{\boldsymbol{y} \in \mathcal{Y} \mid \boldsymbol{y}_1 \leq \boldsymbol{x}_1\} \text{ with } \boldsymbol{x}_1 \text{ and } \boldsymbol{y}_1 \text{ binary variables}$$

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- otherwise, add *(non-robust)* combinatorial Benders cuts (poor numerical performance).

Contributions

Current work

- Applies to two-stage robust convex mixed integer models with objective uncertainty:

$$\inf_{x \in \mathcal{X}} \sup_{\xi \in \Xi} \inf_{y \in \mathcal{Y}(x)} f(x, y, \xi)$$

  - $\mathcal{X} = \{x \in \{0,1\}^{p_1} \times \mathbb{R}^{n_1 - p_1} \mid g_x(x) \leq 0\}$, g_x convex
  - $\mathcal{Y} = \{y \in \{0,1\}^{p_2} \times \mathbb{R}^{n_2 - p_2} \mid g_y(y) \leq 0\}$, g_y convex
  - $\mathcal{Y}(x) = \{y \in \mathcal{Y} \mid g_{xy}(x, y) \leq 0\}$, g_{xy} convex
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  - f convex in (x, y) and concave in ξ
- Single-stage convex MINLP relaxation with exponentially many variables
- Branch-and-price to solve the problem
- Handles linking constraints with any convex function g_{xy}.

Assumptions

Current work

- Applies to two-stage robust convex mixed integer models with objective uncertainty:

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We also assume:

- Relatively complete recourse
- f is separable: $f(\boldsymbol{x},\boldsymbol{y},\boldsymbol{\xi}) = \sum_{i\in Q} w_i(\boldsymbol{\xi}) f_i(\boldsymbol{x},\boldsymbol{y})$
- Wlog., $\mathcal{X}$ is contained in a box: $\mathcal{X} \subseteq \times_{j=1}^{n_1} [\ell_j, u_j]$

Reformulation

Extreme optimal recourse solutions

In the next slide, we need the inner inf problem to have extreme optimal solutions.
$\rightarrow$ we make the objective function concave in recourse variables

Wlog., we can assume $w_i(\cdot) \geq 0$ and use an epigraph formulation:

$$\inf_{\boldsymbol{x}\in\mathcal{X}} \sup_{\boldsymbol{\xi}\in\Xi} \inf_{\boldsymbol{y}\in\mathcal{Y}(\boldsymbol{x})} \sum_{i\in Q} w_i(\boldsymbol{\xi}) f_i(\boldsymbol{x},\boldsymbol{y}) = \inf_{\boldsymbol{x}\in\mathcal{X}} \sup_{\boldsymbol{\xi}\in\Xi} \inf_{\boldsymbol{y}\in\mathcal{Y}'(\boldsymbol{x})} \sum_{i\in Q} w_i(\boldsymbol{\xi}) t_i$$

with $\mathcal{Y}'(\boldsymbol{x}) = \{(\boldsymbol{t},\boldsymbol{y}) | \boldsymbol{y} \in \mathcal{Y}(\boldsymbol{x}), t_i \geq f_i(\boldsymbol{x},\boldsymbol{y}) \ \forall i \in Q\}$

From inf-sup-inf to inf-sup

$$\inf_{\boldsymbol{x}\in\mathcal{X}} \sup_{\boldsymbol{\xi}\in\Xi} \inf_{(\boldsymbol{t},\boldsymbol{y})\in\mathcal{Y}'(\boldsymbol{x})} \sum_{i\in Q} w_i(\boldsymbol{\xi}) t_i \quad (1)$$

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By linearity, in $(\boldsymbol{t},\boldsymbol{y})$, of the objective function,

$$= \inf_{\boldsymbol{x}\in\mathcal{X}} \sup_{\boldsymbol{\xi}\in\Xi} \inf_{(\boldsymbol{t},\boldsymbol{y})\in\mathrm{conv}(\mathcal{Y}'(\boldsymbol{x}))} \sum_{i\in Q} w_i(\boldsymbol{\xi}) t_i \tag{2}$$

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Ξ and $\mathrm{conv}\left(\mathcal{Y}'(\boldsymbol{x})\right)$ are convex, $\sum_{i\in Q} w_i(\boldsymbol{\xi}) t_i$ is convex-concave, so by minimax theorem:

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We now have a static robust problem, for which we can write a deterministic formulation.

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$$= \inf_{\boldsymbol{x}\in\mathcal{X},(\boldsymbol{t},\boldsymbol{y})\in\mathrm{conv}(\mathcal{Y}'(\boldsymbol{x}))} f^+(\boldsymbol{t}) \tag{5}$$

Deterministic reformulation

Following [Ben Tal et al., 2009], using Fenchel duality and some conjugate calculus:

Proposition

Problem (2SRO-P) is equivalent to the following problem, in the sense that they have the same optimal objective value:

$$\begin{aligned} &\inf f^+(\boldsymbol{t}) \\ \text{subject to } &\boldsymbol{x} \in \mathcal{X} \\ &(\boldsymbol{t}, \boldsymbol{y}) \in \operatorname{conv}\left(\mathcal{Y}'(\boldsymbol{x})\right) \end{aligned} \quad = \quad \begin{aligned} &\inf \delta^*(\boldsymbol{\xi}|\Xi) - \sum_{i\in Q}(t_i w_i)_*\left(\boldsymbol{v}^i\right) \\ \text{subject to } &\boldsymbol{x} \in \mathcal{X} \\ &(\boldsymbol{t}, \boldsymbol{y}) \in \operatorname{conv}\left(\mathcal{Y}'(\boldsymbol{x})\right) \\ &\sum_{i\in Q} \boldsymbol{v}^i = \boldsymbol{\xi} \\ &\boldsymbol{v}^i \in \mathbb{R}^{|U|} \qquad \forall i \in Q \end{aligned}$$

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$$\begin{aligned} \inf\, & \delta^*(\boldsymbol{\xi}|\Xi) - \sum_{i \in Q} (t_i w_i)_* \left(\boldsymbol{v}^i\right) \\ \text{subject to } & \boldsymbol{x} \in \mathcal{X} \\ & (\boldsymbol{t}, \boldsymbol{y}) \in \operatorname{conv}\left(\mathcal{Y}'(\boldsymbol{x})\right) \\ & \sum_{i \in Q} \boldsymbol{v}^i = \boldsymbol{\xi} \\ & \boldsymbol{v}^i \in \mathbb{R}^{|U|} \qquad \forall i \in Q \end{aligned}$$

- *f^+ takes various forms depending on Ξ and f.*
- *MINLP model*

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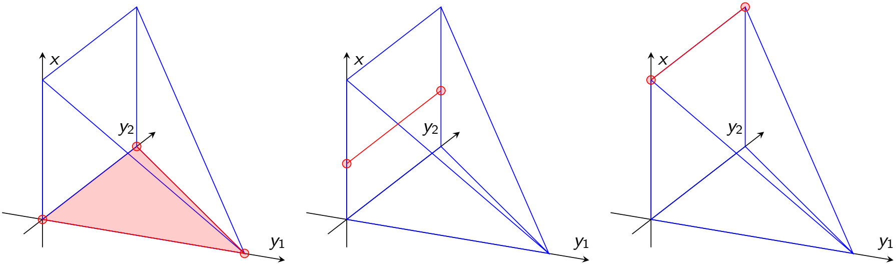
$$\begin{aligned} &\inf \delta^{*}(\boldsymbol{\xi} | \Xi) - \sum_{i \in Q} (t_i w_i)_{*} \left(\boldsymbol{v}^i\right) \\ &\text{subject to } \boldsymbol{x} \in \mathcal{X} \\ &\qquad (\boldsymbol{t}, \boldsymbol{y}) \in \operatorname{conv}\left(\mathcal{Y}'(\boldsymbol{x})\right) \\ &\qquad \sum_{i \in Q} \boldsymbol{v}^i = \boldsymbol{\xi} \\ &\qquad \boldsymbol{v}^i \in \mathbb{R}^{|U|} \qquad \forall i \in Q \end{aligned}$$

- *f^{+} takes various forms depending on Ξ and f.*
- *MINLP model*
- *Disjunctive constraints*

Disjunctive constraints

- $(\boldsymbol{t}, \boldsymbol{y})$ belongs to the convex hull of a set of points that depends on decision variables.
- In red, $\mathrm{conv}\left(\mathcal{Y}(\bar{\boldsymbol{x}})\right)$ for $\bar{\boldsymbol{x}} \in \{0, 0.4, 1\}$.



$$\mathcal{X} = [0,1] \text{ and } \mathcal{Y}(x) = \left\{ \boldsymbol{y} \in \{0,1\}^2 \middle| \begin{array}{rl} y_1 + y_2 & \leq 1 \\ y_1 & \leq 1 - x \end{array} \right\}$$

A Branch-and-Price algorithm

Hull relaxation [H. D. Sherali and X. Zhu, 2006]

Consider $S = \left\{ (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) : \begin{array}{ll} \ell_j \leq x_j \leq u_j & \forall j \in I \\ (\boldsymbol{t}, \boldsymbol{y}) \in \mathcal{Y}'(\boldsymbol{x}) & \end{array} \right\}$,
i.e. $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) \in S$ iff $(\boldsymbol{t}, \boldsymbol{y})$ is a feasible second stage solution for $\boldsymbol{x}$ (in a box).

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i.e. $(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) \in S$ iff $(\boldsymbol{t}, \boldsymbol{y})$ is a feasible second stage solution for $\boldsymbol{x}$ (in a box).
Then

$$\mathcal{Y}'(\bar{\boldsymbol{x}}) = \mathrm{proj}_{\boldsymbol{t},\boldsymbol{y}} \left(S \cap \left\{ (\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{t}') : \boldsymbol{x}' = \bar{\boldsymbol{x}} \right\} \right)$$

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$$\mathrm{conv}\left(\mathcal{Y}'(\bar{\boldsymbol{x}})\right) = \mathrm{proj}_{\boldsymbol{t},\boldsymbol{y}}(\mathrm{conv}(S \cap \{(\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{t}') : \boldsymbol{x}' = \bar{\boldsymbol{x}}\}))$$

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$$\subseteq \mathrm{proj}_{\boldsymbol{t},\boldsymbol{y}}(\mathrm{conv}(S) \cap \{ (\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{t}') : \boldsymbol{x}' = \bar{\boldsymbol{x}} \})$$

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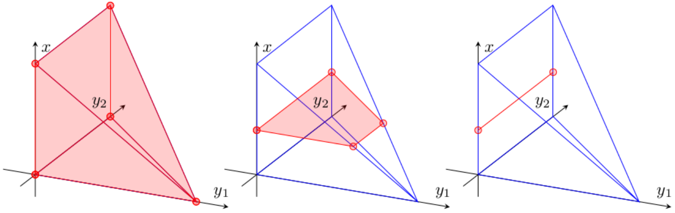

(a) conv (S) (b) conv $(S) \cap \{(\boldsymbol{x}, \boldsymbol{y}) : \boldsymbol{x} = \bar{\boldsymbol{x}}\}$ (c) $\{\bar{\boldsymbol{x}}\} \times$ conv $(\mathcal{Y}(\bar{\boldsymbol{x}}))$

Hull relaxation [H. D. Sherali and X. Zhu, 2006]

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$$\subseteq \mathrm{proj}_{\boldsymbol{t},\boldsymbol{y}} (\mathrm{conv}(S) \cap \{ (\boldsymbol{x}', \boldsymbol{y}', \boldsymbol{t}') : \boldsymbol{x}' = \bar{\boldsymbol{x}} \})$$

Lower bounding problem

$$\inf \left\{ f^+(\boldsymbol{t}) \;\middle|\; \boldsymbol{x} \in \mathcal{X}, (\boldsymbol{t}, \boldsymbol{y}) \in \mathrm{conv} \left(\mathcal{Y}'(\boldsymbol{x}) \right) \right\}$$

$$\geq \inf \left\{ f^+(\boldsymbol{t}) \;\middle|\; \boldsymbol{x} \in \mathcal{X}, (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) \in \mathrm{conv} \left(S \right) \right\}$$

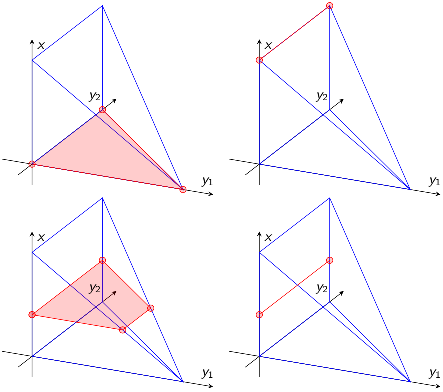
Advantage: $\mathrm{conv}(S)$ does not depend on the value of $\boldsymbol{x}$.
Hence, the relaxation can more simply be solved by (non-linear) column generation.

Hull relaxation [H. D. Sherali and X. Zhu, 2006]

Equality condition

If $\bar{\boldsymbol{x}} \in \mathrm{vert}\left(\times_{j=1}^{n_1}[\ell_j, u_j]\right)$, then

$$\mathrm{conv}\left(\mathcal{Y}'(\bar{\boldsymbol{x}})\right) = \mathrm{proj}_{\boldsymbol{t},\boldsymbol{y}}\left(\mathrm{conv}\left(S\right) \cap \{(\boldsymbol{x},\boldsymbol{t},\boldsymbol{y}) : \boldsymbol{x} = \bar{\boldsymbol{x}}\}\right)$$



Equality holds for $x = 0$ or $x = 1$

Inclusion is strict for $x = 0.4$

Hull relaxation [H. D. Sherali and X. Zhu, 2006]

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Corollary 1

Let $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{t}^*)$ be an optimal solution of the relaxation.
If $\boldsymbol{x}^* \in \mathrm{vert}\left(\times_{j=1}^{n_1}[\ell_j, u_j]\right)$, then it is feasible and optimal for the original problem.

Hull relaxation [H. D. Sherali and X. Zhu, 2006]

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Corollary 2

If all first-stage variables involved in $\mathcal{Y}'(x)$ are binary variables, then the relaxation is a reformulation.

Spatial branching

At node q of the branch-and-price algorithm

- the box of first-stage solutions is $\boldsymbol{x} \in \times_{j=1}^{n_1}[\ell_j^q, u_j^q]$,
- the hull is refined: $S^q = \left\{(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) \;\middle|\; \boldsymbol{x} \in \times_{j=1}^{n_1}[\ell_j^q, u_j^q],\ (\boldsymbol{t}, \boldsymbol{y}) \in \mathcal{Y}'(\boldsymbol{x})\right\}$
- the lower bounding problem is $(R^q) := \inf\left\{f^+(\boldsymbol{t}) \;\middle|\; \boldsymbol{x} \in \mathcal{X}, (\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{t}) \in \operatorname{conv}(S^q)\right\}$.

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Let $(\boldsymbol{x}^q, \boldsymbol{y}^q, \boldsymbol{t}^q)$ be an optimal solution of (R^q).

- If $\boldsymbol{x}^q \in \operatorname{vert}\left(\times_{j=1}^{n_1}[\ell_j^q, u_j^q]\right)$, the node is solved optimally.

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Let $(\boldsymbol{x}^q, \boldsymbol{y}^q, \boldsymbol{t}^q)$ be an optimal solution of (R^q).

- If some binary first-stage variable have fractional value, we branch on the most fractional one first.

Spatial branching

At node q of the branch-and-price algorithm

- the box of first-stage solutions is $\boldsymbol{x} \in \times_{j=1}^{n_1}[\ell_j^q, u_j^q]$,
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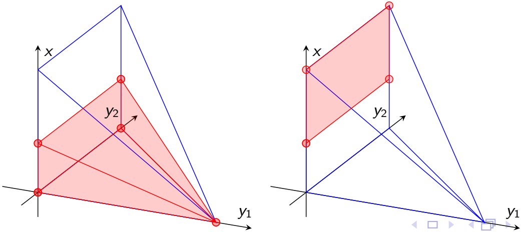
Let $(\boldsymbol{x}^q, \boldsymbol{y}^q, \boldsymbol{t}^q)$ be an optimal solution of (R^q).

- Otherwise, we branch on the variable x_j which is the furthest from its bounds:

$$j \in \arg\sup\left\{\inf\{x_i^q - \ell_i^q, u_i^q - x_i^q\} \mid i = 1, \ldots, n_1\right\}$$

and we create two child nodes with $x_j \in [\ell_j^q, x_j^q]$ and $x_j \in [x_j^q, u_j^q]$.



Convergence

Proposition

Our branch-and-price algorithm either

- *terminates in finite time and returns an optimal solution*
- *or enters an infinite sequence P of nodes with $(\boldsymbol{x}^p, \boldsymbol{y}^p, \boldsymbol{t}^p)_{p \in P} \to (\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{t}^*)$, with $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{t}^*)$ an optimal solution*
- *or enters an infinite sequence P of nodes with $(\boldsymbol{x}^p, \boldsymbol{y}^p, \boldsymbol{t}^p)_{p \in P} \to (\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{t}^*)$, such that f^+ is not lower semi-continuous at $\boldsymbol{t}^*$.*

Corollary

If f^+ is lower semi-continuous (e.g. case of MILP), our branch-and-price algorithm either

- *terminates in finite time and returns an optimal solution*
- *or enters an infinite sequence P of nodes with $(\boldsymbol{x}^p, \boldsymbol{y}^p, \boldsymbol{t}^p)_{p \in P} \to (\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{t}^*)$, with $(\boldsymbol{x}^*, \boldsymbol{y}^*, \boldsymbol{t}^*)$ an optimal solution.*

Computational experiments

Capital Budgeting Problem [Hanasusanto et al., 2015], [Arslan et D., 2021]

Here-and-now Invest in some assets / possibility to request a loan

Capital Budgeting Problem [Hanasusanto et al., 2015], [Arslan et D., 2021]

Here-and-now Invest in some assets / possibility to request a loan

Uncertainty Unknown risk factors which impacts the profits

Capital Budgeting Problem [Hanasusanto et al., 2015], [Arslan et D., 2021]

Here-and-now	Invest in some assets / possibility to request a loan
Uncertainty	Unknown risk factors which impacts the profits
Wait-and-see	Invest in some assets / possibility to request a loan

Capital Budgeting Problem [Hanasusanto et al., 2015], [Arslan et D., 2021]

Here-and-now Invest in some assets / possibility to request a loan

Uncertainty Unknown risk factors which impacts the profits

Wait-and-see Invest in some assets / possibility to request a loan

Catch: investment incentives

Early investments enjoy a first-mover advantage whereas a postponed investment in asset $i \in \mathcal{N}$ only generates a fraction $f \in [0, 1)$ of the profit $\tilde{p}_i$.

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Risk factors

Profits are given, for asset $i \in \mathcal{N}$, as $\tilde{p}_i(\boldsymbol{\xi}) = \bar{p}_i(1 + \Delta_i(\boldsymbol{\xi}))$ with $\boldsymbol{\xi} \in [-1, 1]^M$

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Independent risk factors $\Delta_i(\boldsymbol{\xi}) = {\boldsymbol{g}^i}^T \boldsymbol{\xi} + b_i$

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Dependent risk factors $\Delta_i(\boldsymbol{\xi}) = \boldsymbol{\xi}^T \boldsymbol{Q}^i \boldsymbol{\xi}/2 + {\boldsymbol{g}^i}^T \boldsymbol{\xi} + b_i$

Capital Budgeting Problem - model

$$\max_{(\boldsymbol{x},x_0)\in\mathcal{X}} \min_{\boldsymbol{\xi}\in\Xi} \max_{(\boldsymbol{y},y_0)\in\mathcal{Y}(\boldsymbol{x},x_0)} \left[\sum_{i\in\mathcal{N}} \tilde{p}_i(\boldsymbol{\xi})(x_i + f y_i) - (1+\lambda)C_1 x_0 - (1+\lambda\mu)C_2 y_0\right]$$

- $\mathcal{X} = \{(\boldsymbol{x}, x_0) \in \{0,1\}^N \times [0,1] : \sum_{i\in\mathcal{N}} c_i x_i \leq B + C_1 x_0\}$
- $\mathcal{Y}(\boldsymbol{x}, x_0) = \{(\boldsymbol{y}, y_0) \in \{0,1\}^N \times [0,1] : \sum_{i\in\mathcal{N}} c_i(x_i + y_i) \leq B + C_1 x_0 + C_2 y_0, y_i + x_i \leq 1 \ \forall i \in \mathcal{N}\}$
- C_1 and C_2 are maximum amounts for loans
- λ and $\lambda\mu$ are interest rates for loans

Capital Budgeting Problem - after reformulation

$$\max \; -(1+\lambda)C_1 x_0 - (1+\lambda)\mu C_2 y_0 + \sum_{i\in\mathcal{N}} \Bigg[\bar{p}_i(x_i + f y_i) - |\xi_i| - r_i + ({\boldsymbol{g}^i}^T {\boldsymbol{Q}^i}^{-1})^T \boldsymbol{s}^i - \left(\frac{{\boldsymbol{g}^i}^T {\boldsymbol{Q}^i}^{-1} \boldsymbol{g}^i}{2} \right) t_i \Bigg]$$

subject to

$$(\boldsymbol{x}, x_0) \in \mathcal{X}$$

$$(\boldsymbol{y}, y_0) \in \operatorname{conv}(\mathcal{Y}(\boldsymbol{x}, x_0))$$

$$t_i = (x_i + f y_i)\bar{p}_i \qquad \forall i \in \mathcal{N}$$

$$2 t_i r_i \geq {\boldsymbol{s}^i}^T {\boldsymbol{Q}^i}^{-1} \boldsymbol{s}^i \qquad \forall i \in \mathcal{N}$$

$$\sum_{i\in\mathcal{N}} \boldsymbol{s}^i = \boldsymbol{\xi}$$

$$\boldsymbol{\xi} \in \mathbb{R}^{|\mathcal{N}|}, \boldsymbol{t} \in \mathbb{R}_+^{|\mathcal{N}|}, \boldsymbol{r} \in \mathbb{R}_+^{|\mathcal{N}|}, \boldsymbol{s}^i \in \mathbb{R}^{|\mathcal{N}|} \quad \forall i \in \mathcal{N}$$

Computational results

	N	M	No loans time	No loans # opt	Binary loans time	Binary loans # opt	Continuous loans time	Continuous loans # opt
Q	30	4	5.5	40	6.0	40	10.7	40
		8	118.4	40	135.8	40	124.3	39
	40	4	8.9	40	9.4	40	15.0	40
		8	169.0	40	97.2	39	117.4	39
	50	4	10.1	40	9.8	40	13.5	40
		8	179.5	40	190.2	40	144.0	39
	60	4	13.8	40	13.2	40	16.2	40
		8	25.1	40	25.7	40	37.5	40
L	30	4	241.1	37	110.7	36	217.3	37
		8	580.0	32	618.9	32	480.9	31
	40	4	223.8	37	195.6	38	193.1	36
		8	447.1	28	276.3	28	247.9	28
	50	4	744.8	22	740.1	24	798.6	25
		8	523.2	39	238.3	40	251.9	40
	60	4	832.9	7	957.4	7	1238.7	8
		8	946.1	32	695.0	33	665.5	32

- Implemented in C++ with Mosek version 9.2
- Time limit is 3600 seconds

Conclusion

- New (exact) solution method for hard optimization problems
- Embeds a large amount of problems (convexity, mixed integer second stage)

A Branch-and-Price algorithm

Restricted Master Problem (RMP)

$$\inf \delta^*(\boldsymbol{\xi}|\Xi) - \sum_{i\in Q}(t_i w_i)_* \left(\boldsymbol{v}^i\right)$$

$$\text{subject to } \boldsymbol{x} \in \overline{\mathcal{X}} \cap [\boldsymbol{l}^q, \boldsymbol{u}^q]$$

$$(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{y}) = \sum_{i\in K} \alpha_i (\bar{\mathbf{x}}^{qi}, \bar{\mathbf{t}}^{qi}, \bar{\mathbf{y}}^{qi})$$

$$\sum_{i\in K} \alpha_i = 1$$

$$\sum_{i\in Q} \boldsymbol{v}^i = \boldsymbol{\xi}$$

$$\boldsymbol{v}^i \in \mathbb{R}^{|U|} \; \forall i \in Q \qquad \boldsymbol{\xi} \in \mathbb{R}^{|U|}$$

$$\alpha_i \geq 0 \qquad \forall i \in K$$

At each node q, we enforce $(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{y}) \in \mathrm{conv}\,(S^q)$ by using column generation.

A Branch-and-Price algorithm

Restricted Master Problem (RMP)

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$$\sum_{i \in K} \alpha_i = 1$$

$$\sum_{i \in Q} \boldsymbol{v}^i = \boldsymbol{\xi}$$

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Pricing Problem (PP)

by solving the following problem (iter. k)

$$\inf -\boldsymbol{\lambda}^{qk*^T}(\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{y}) - \eta^{qk*}$$

$$\text{subject to } (\boldsymbol{x}, \boldsymbol{t}, \boldsymbol{y}) \in S^q$$