Mathematical Foundations of Robust Distributionally Robust Optimization

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www.epfl.ch/labs/rao/

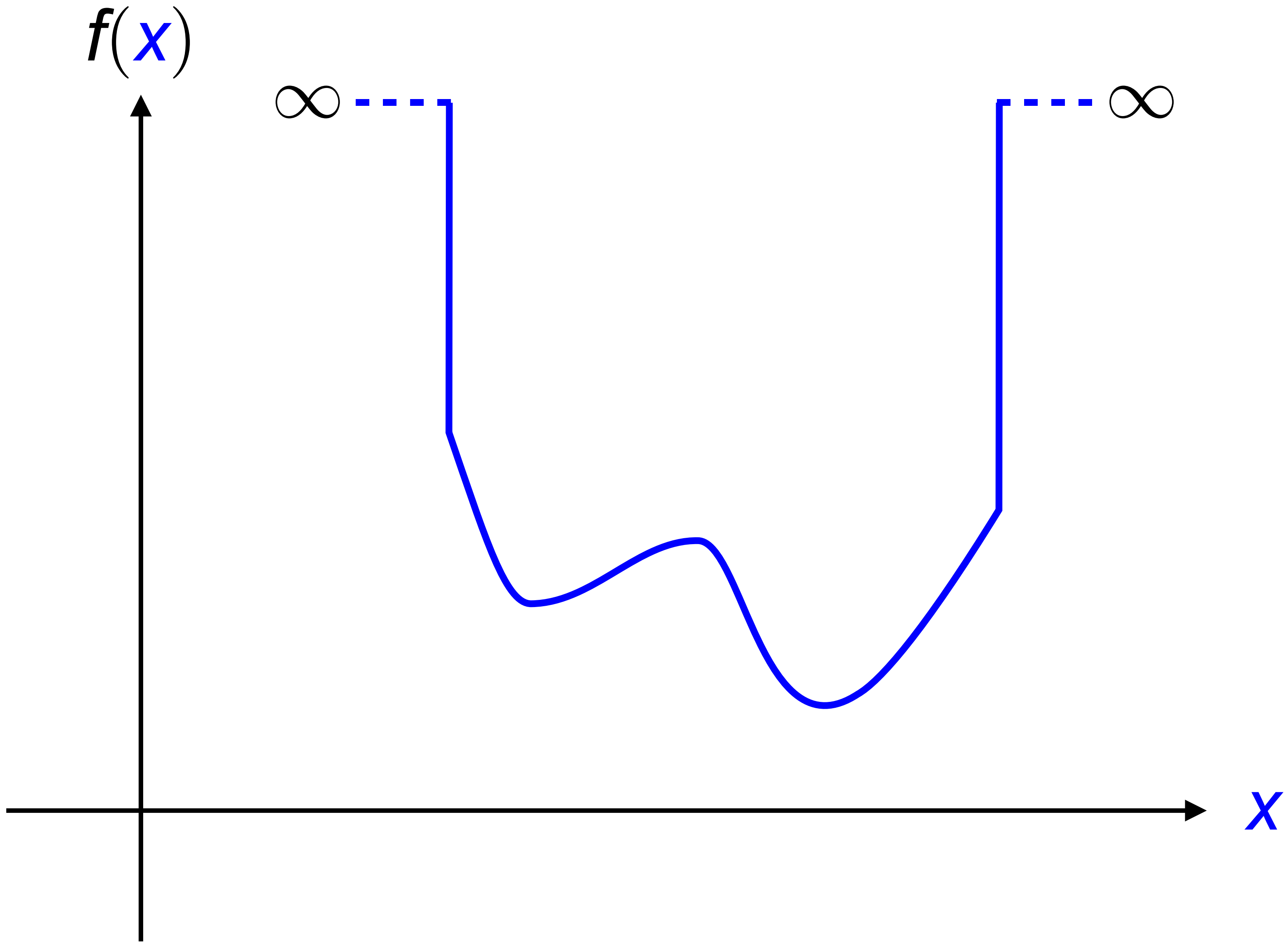
FNSNF Swiss National Science Foundation

NCCR Automation

EPFL

Extended Real-Valued Functions

Proper Functions



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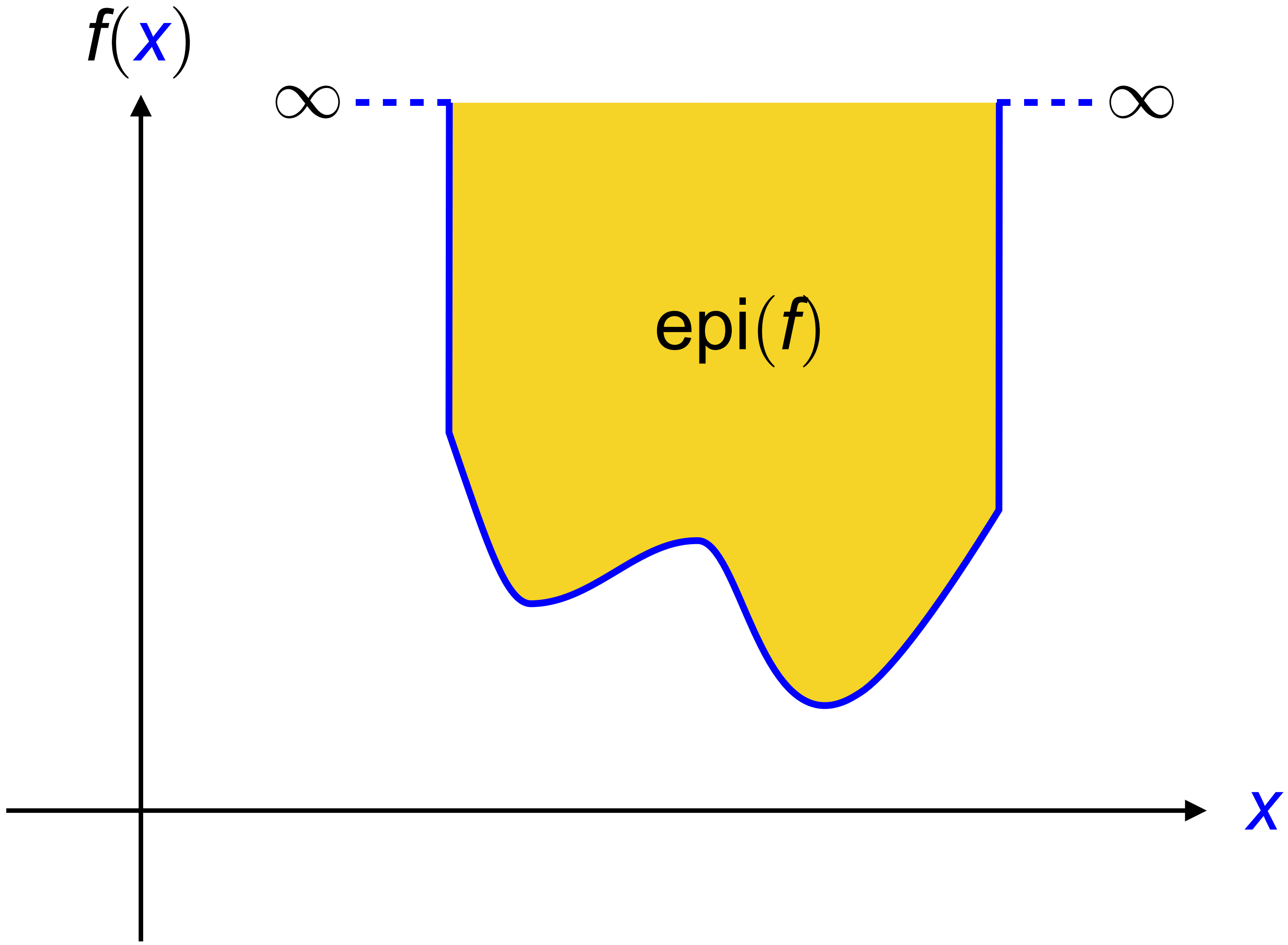

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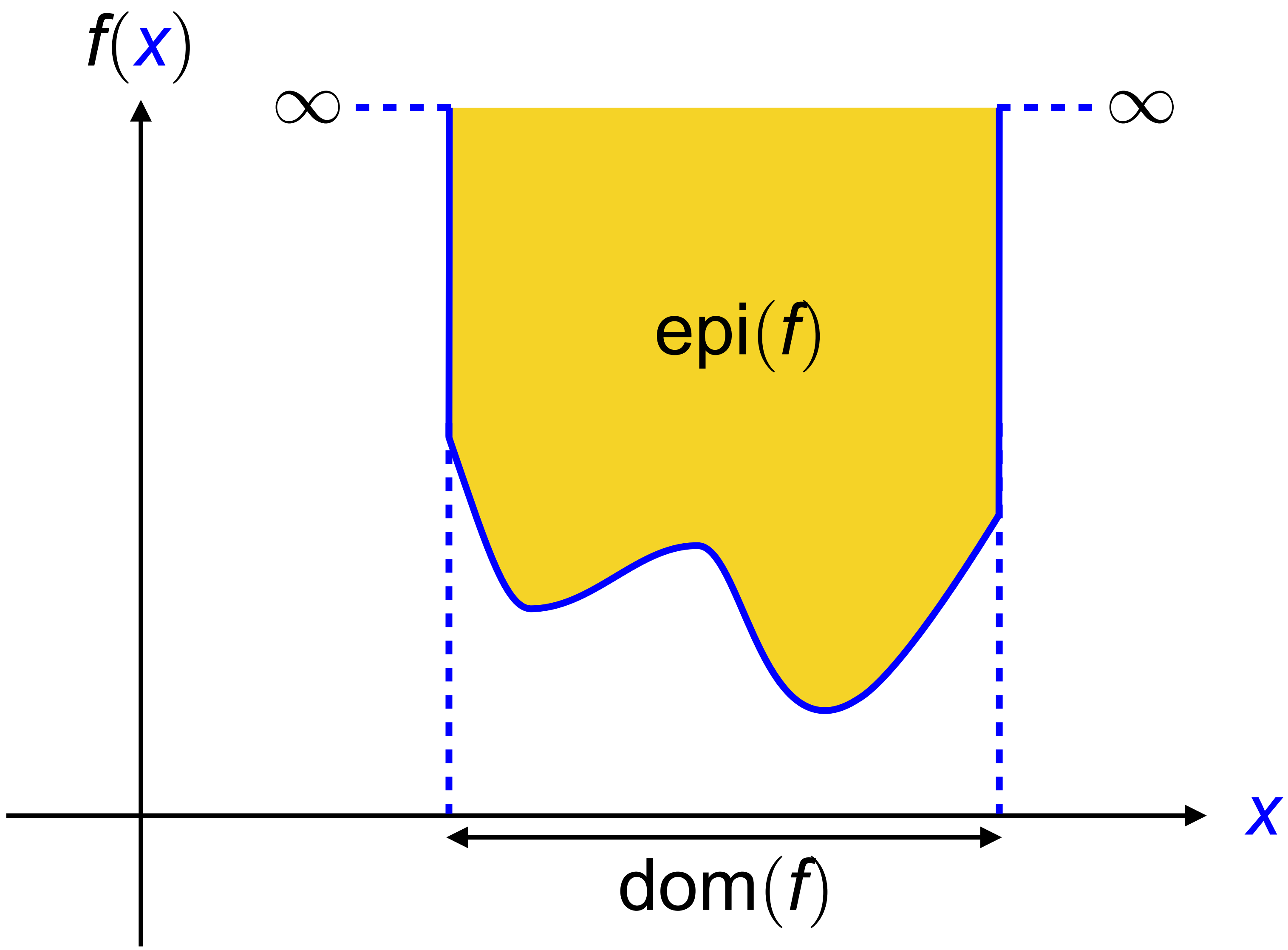
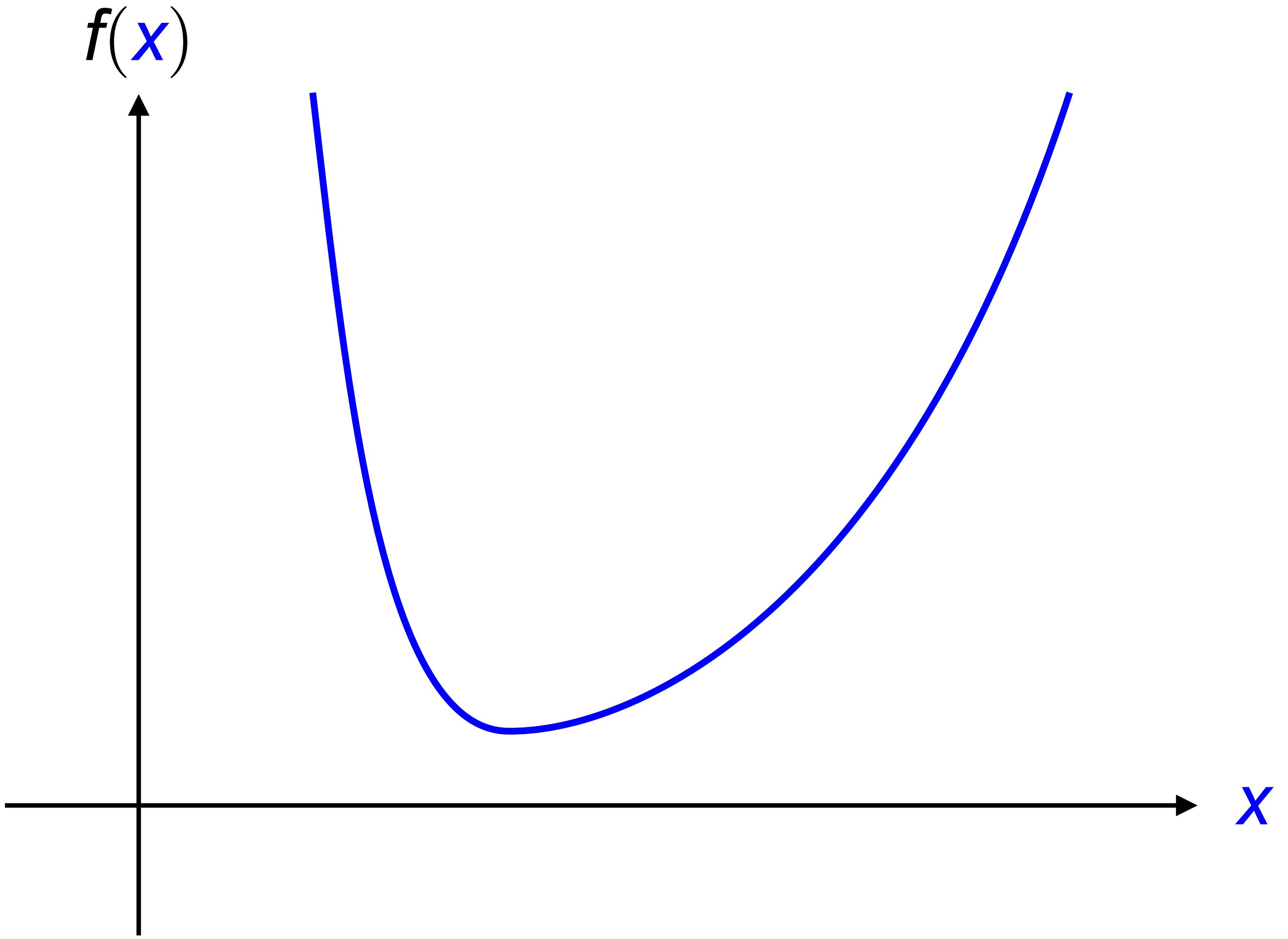


Convex Functions



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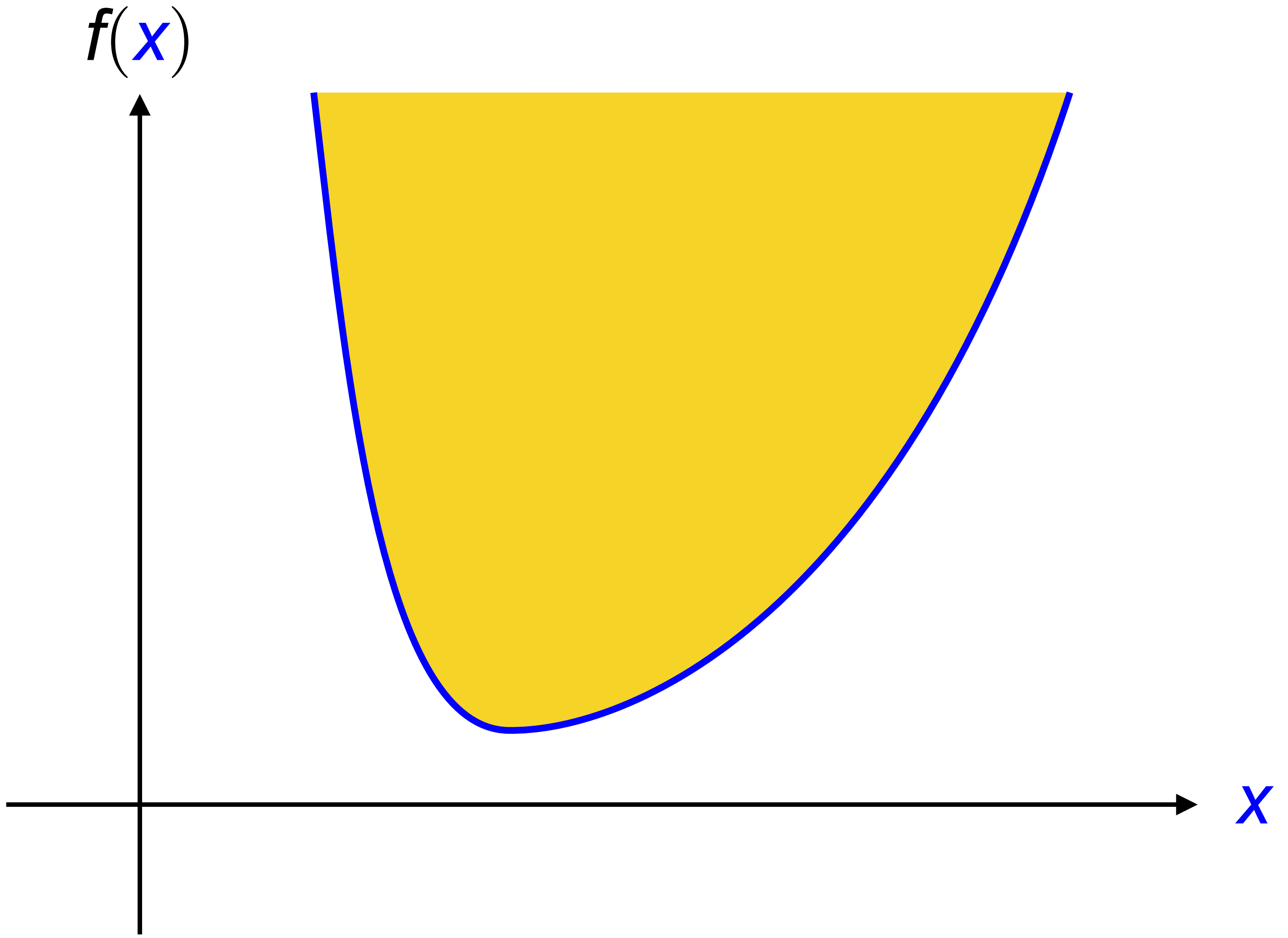

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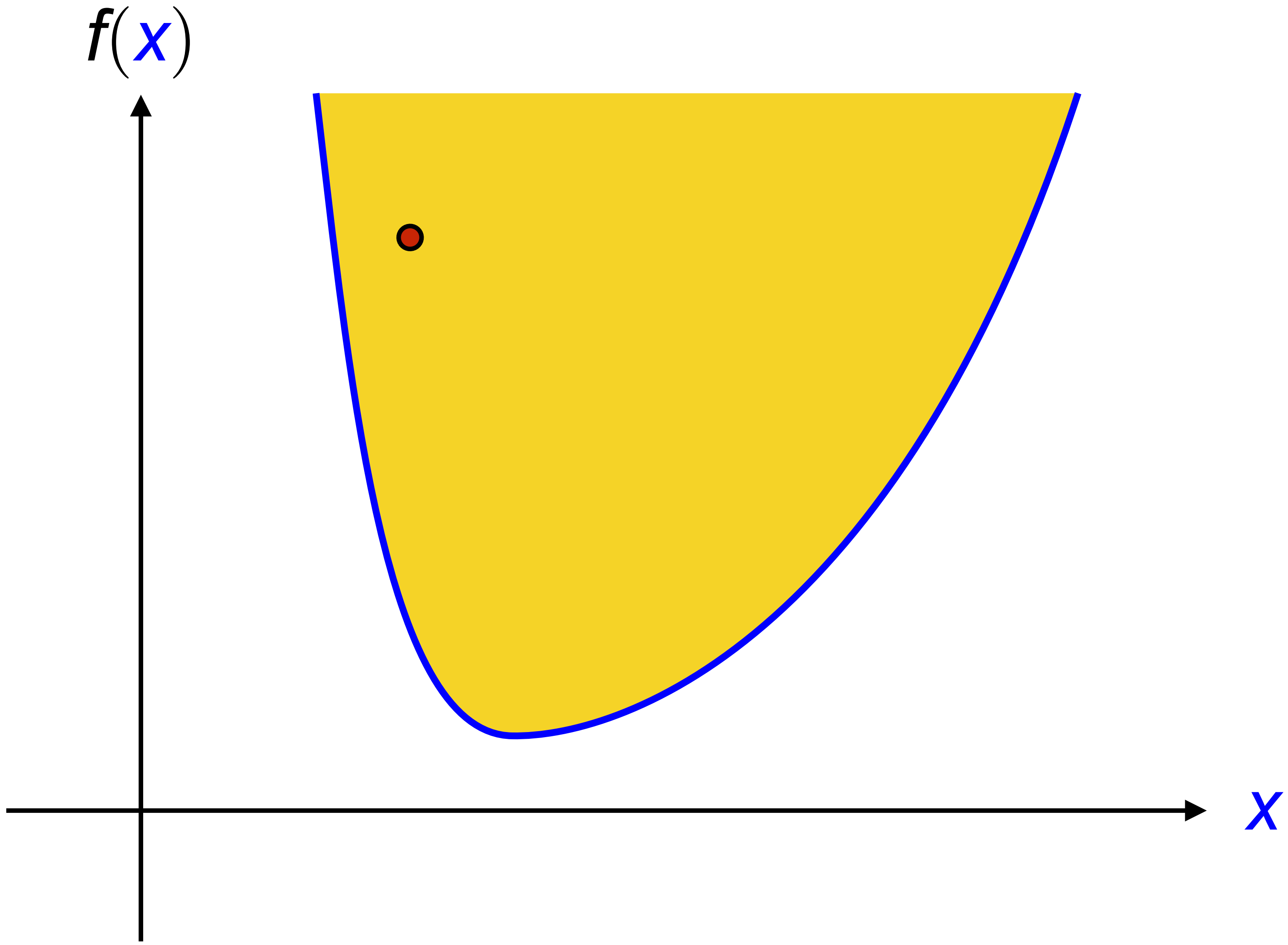

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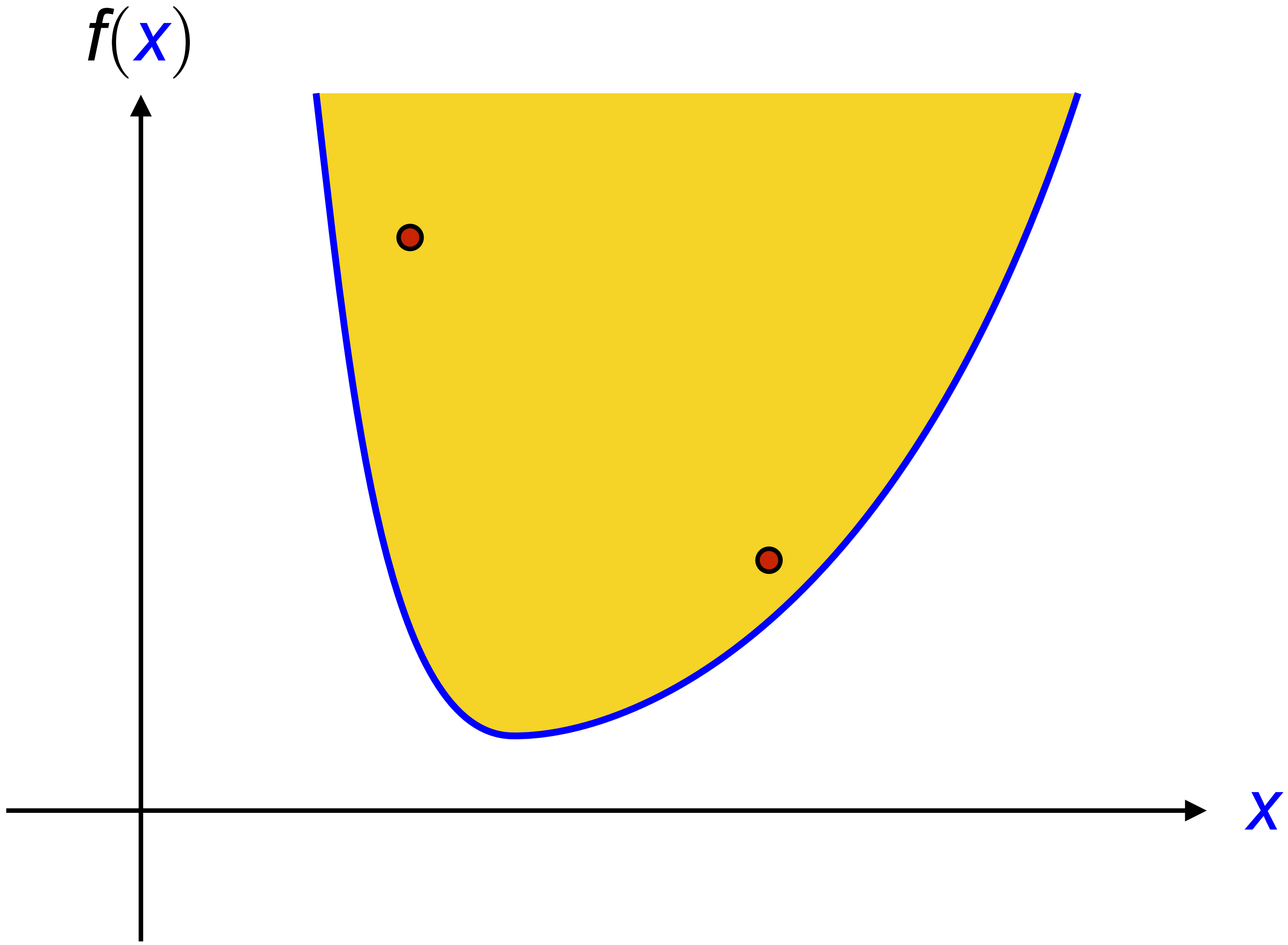

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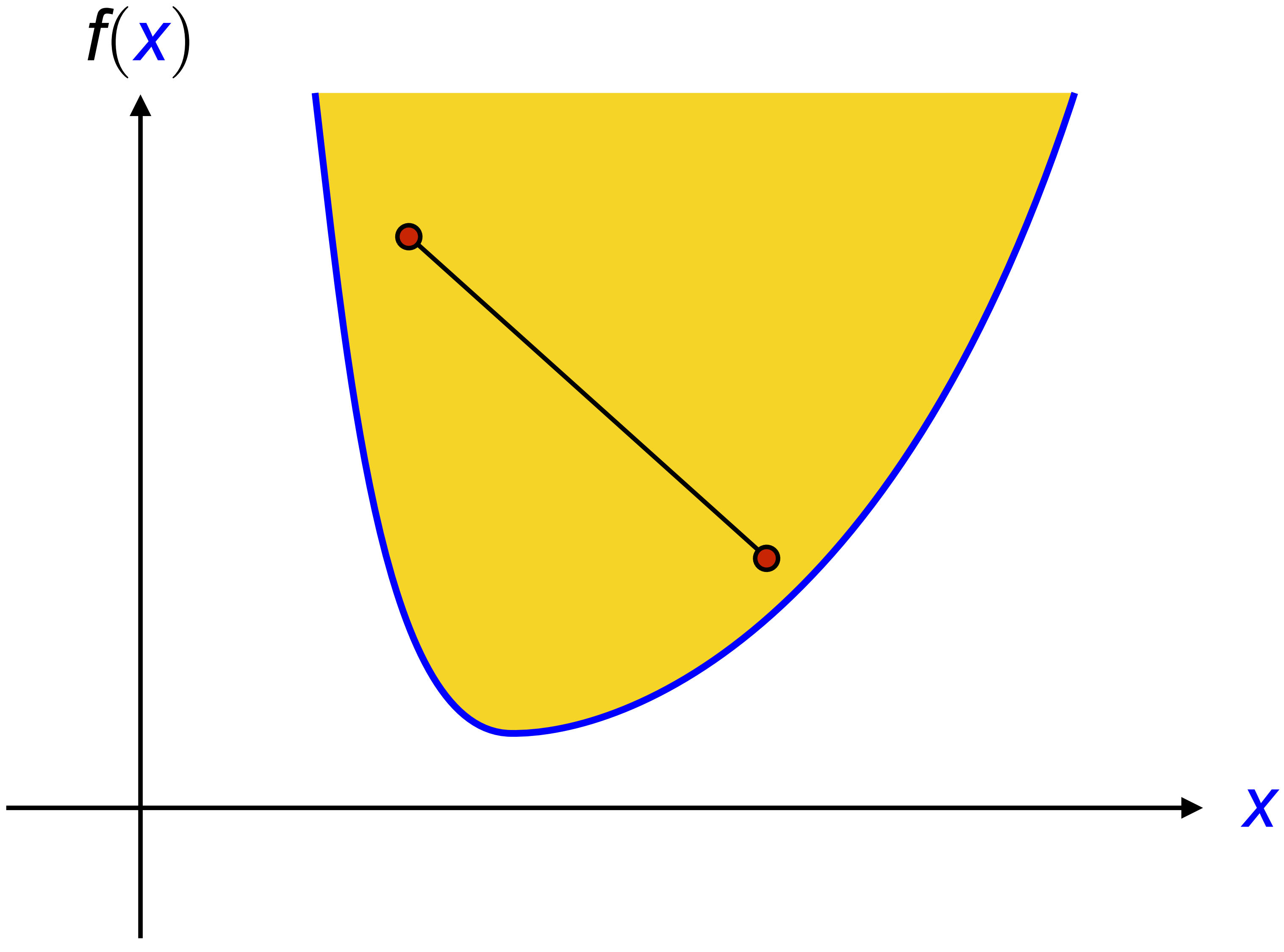

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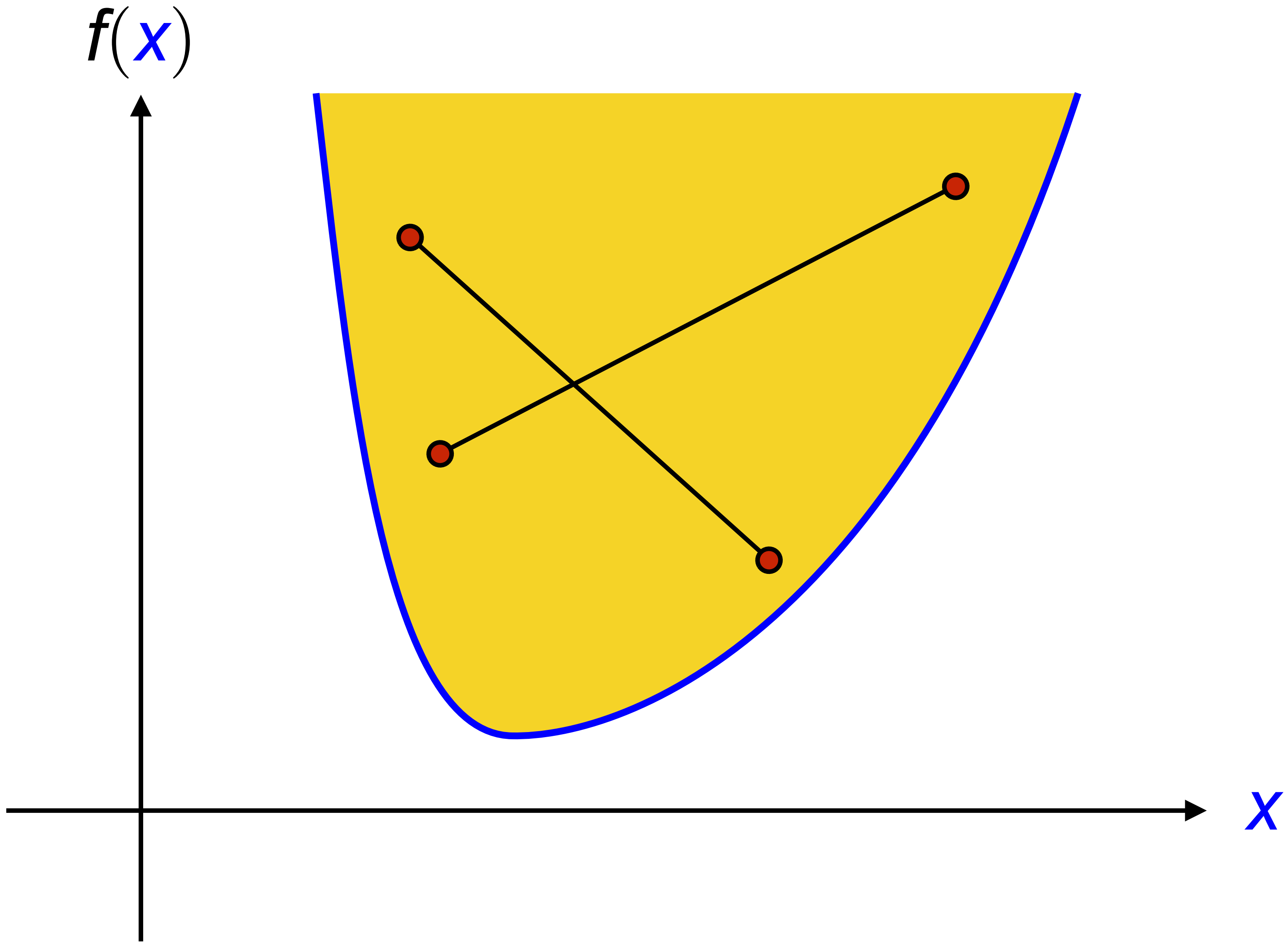

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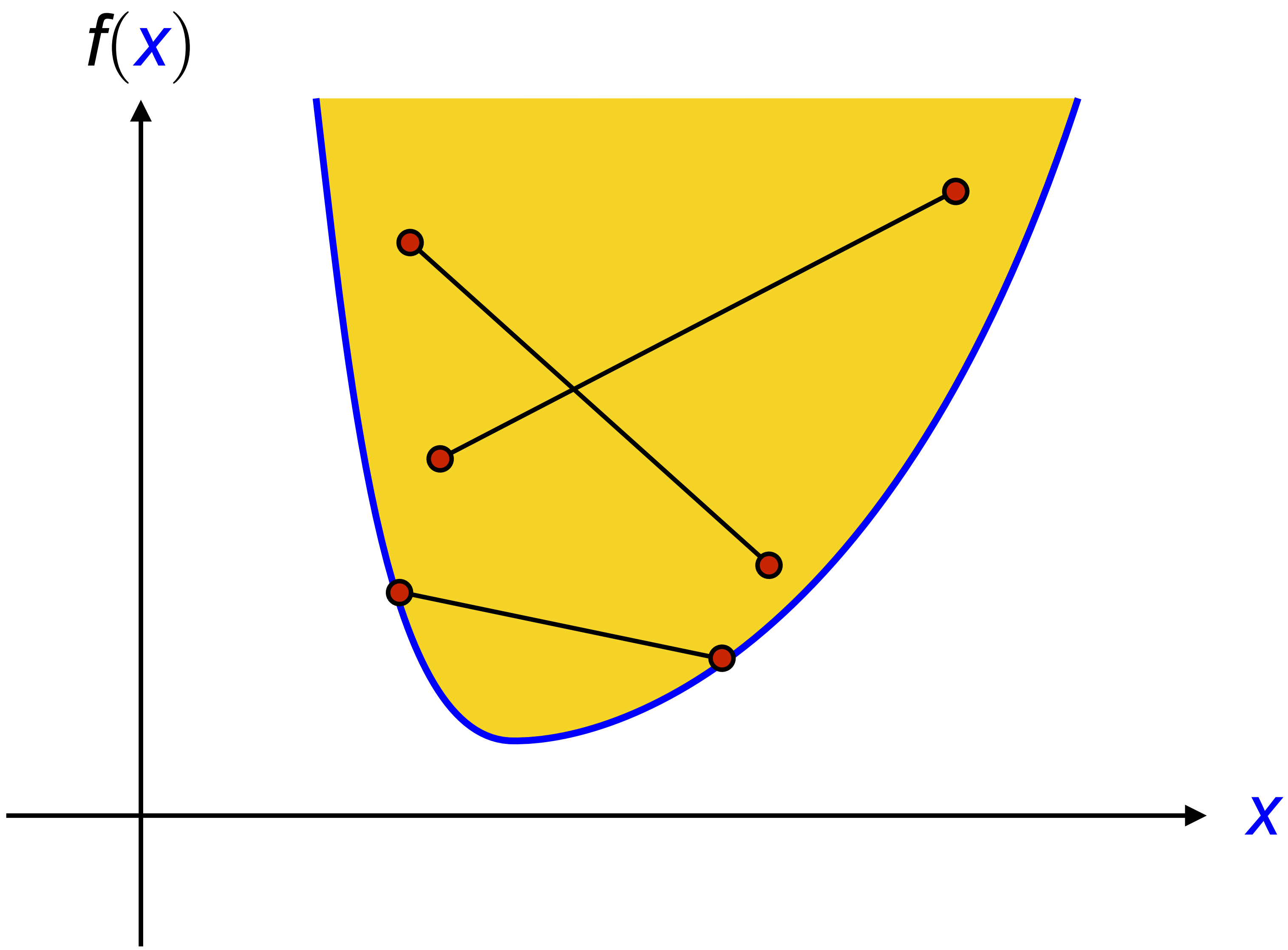

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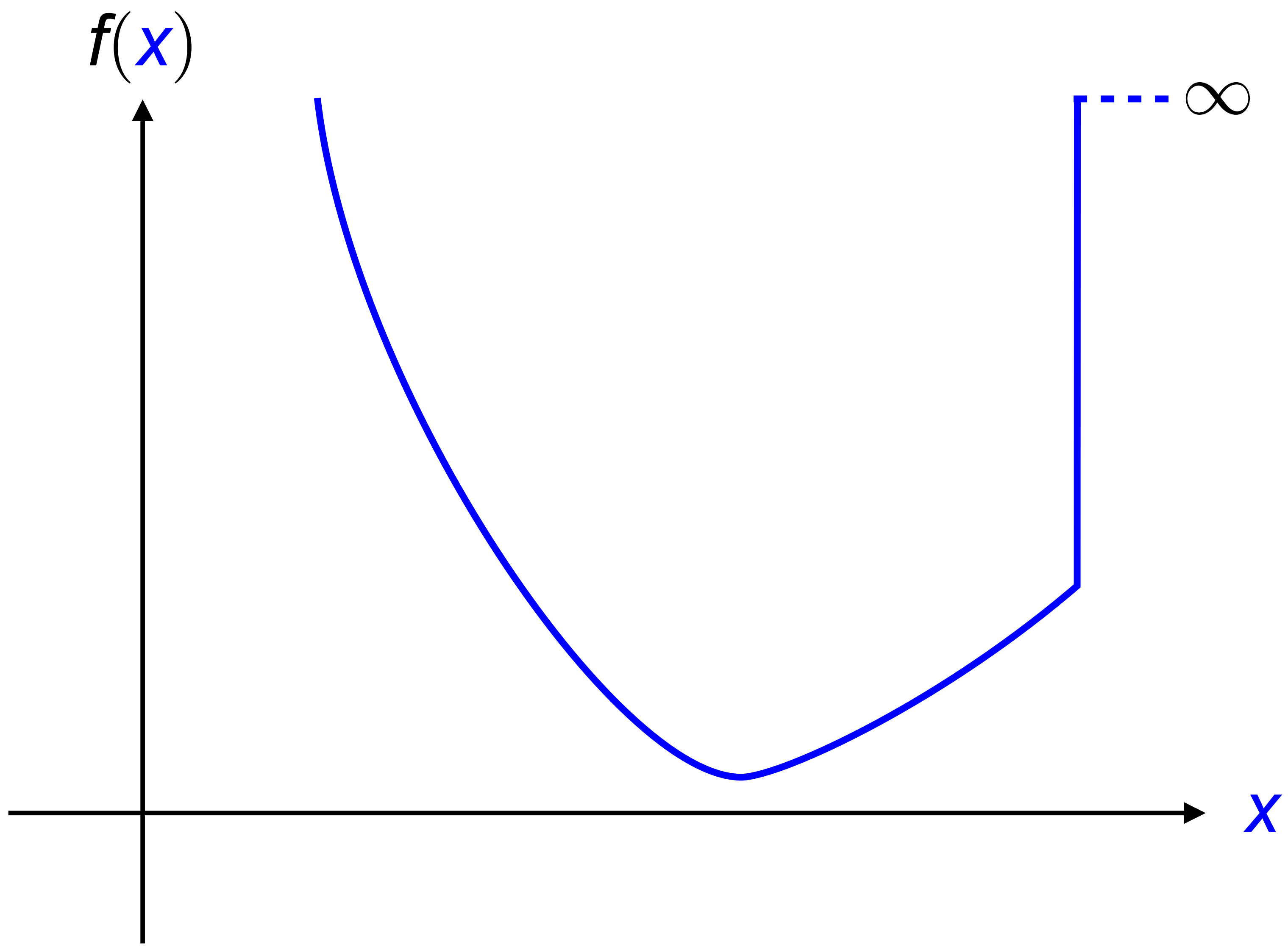

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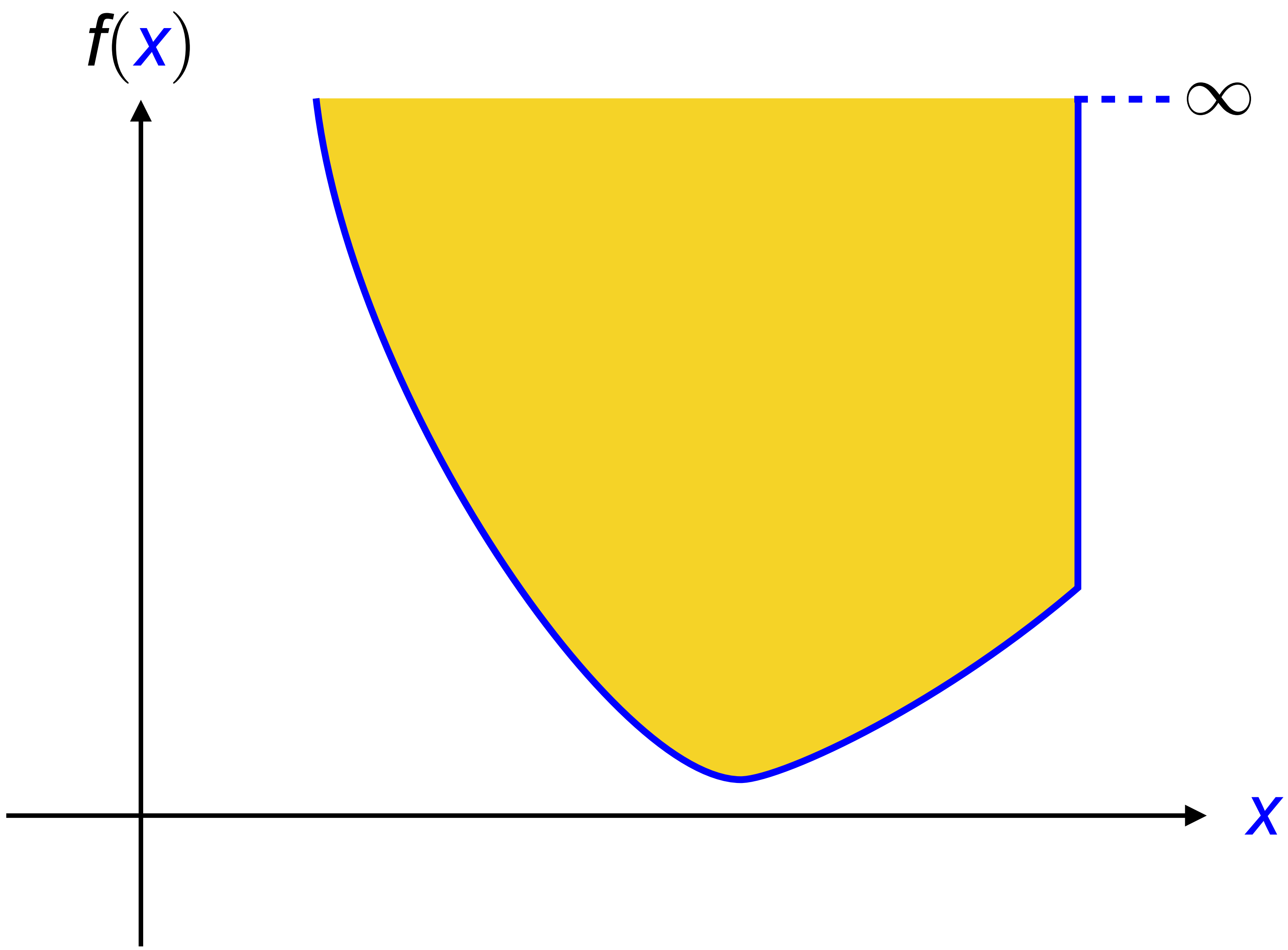

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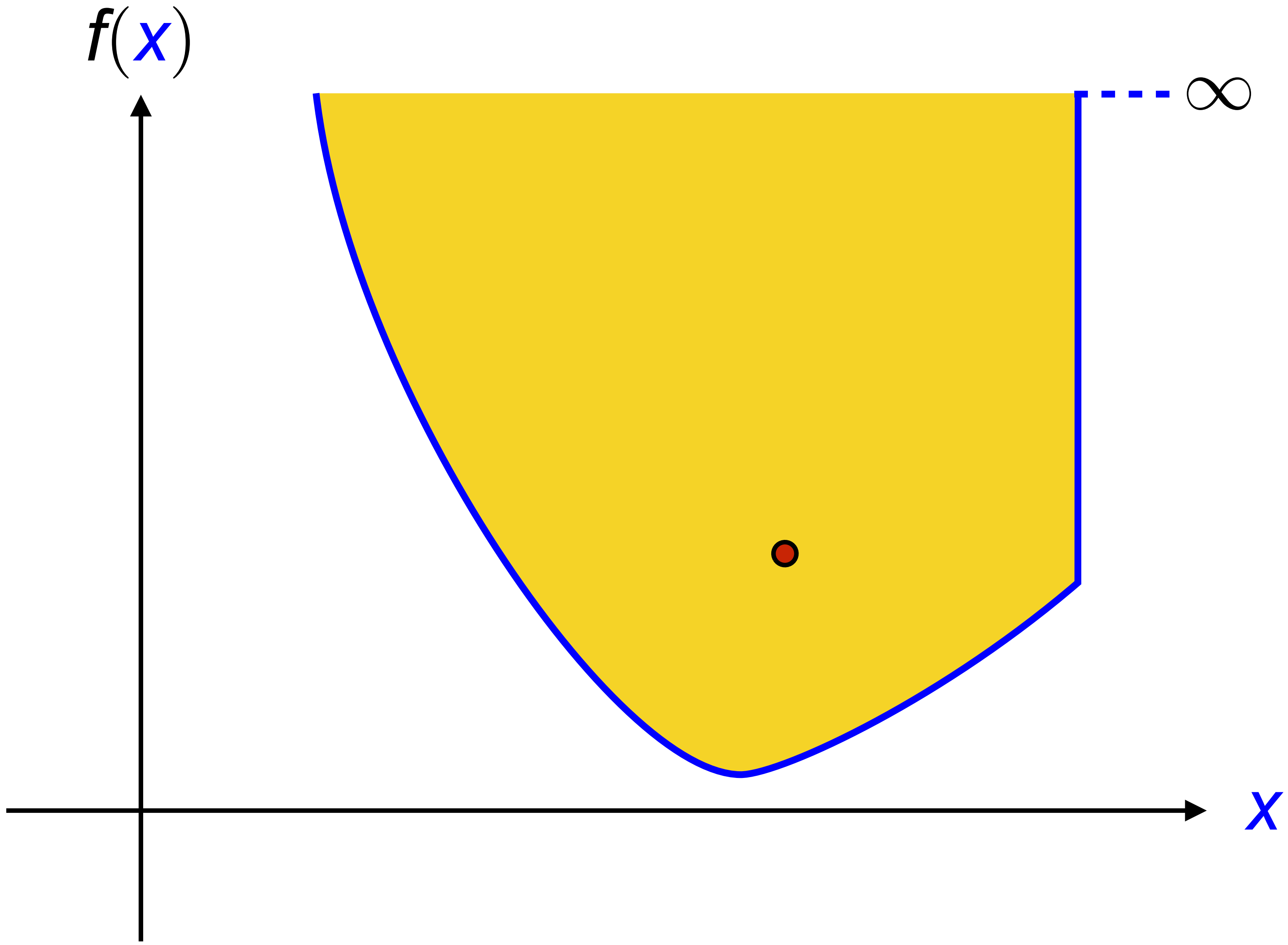

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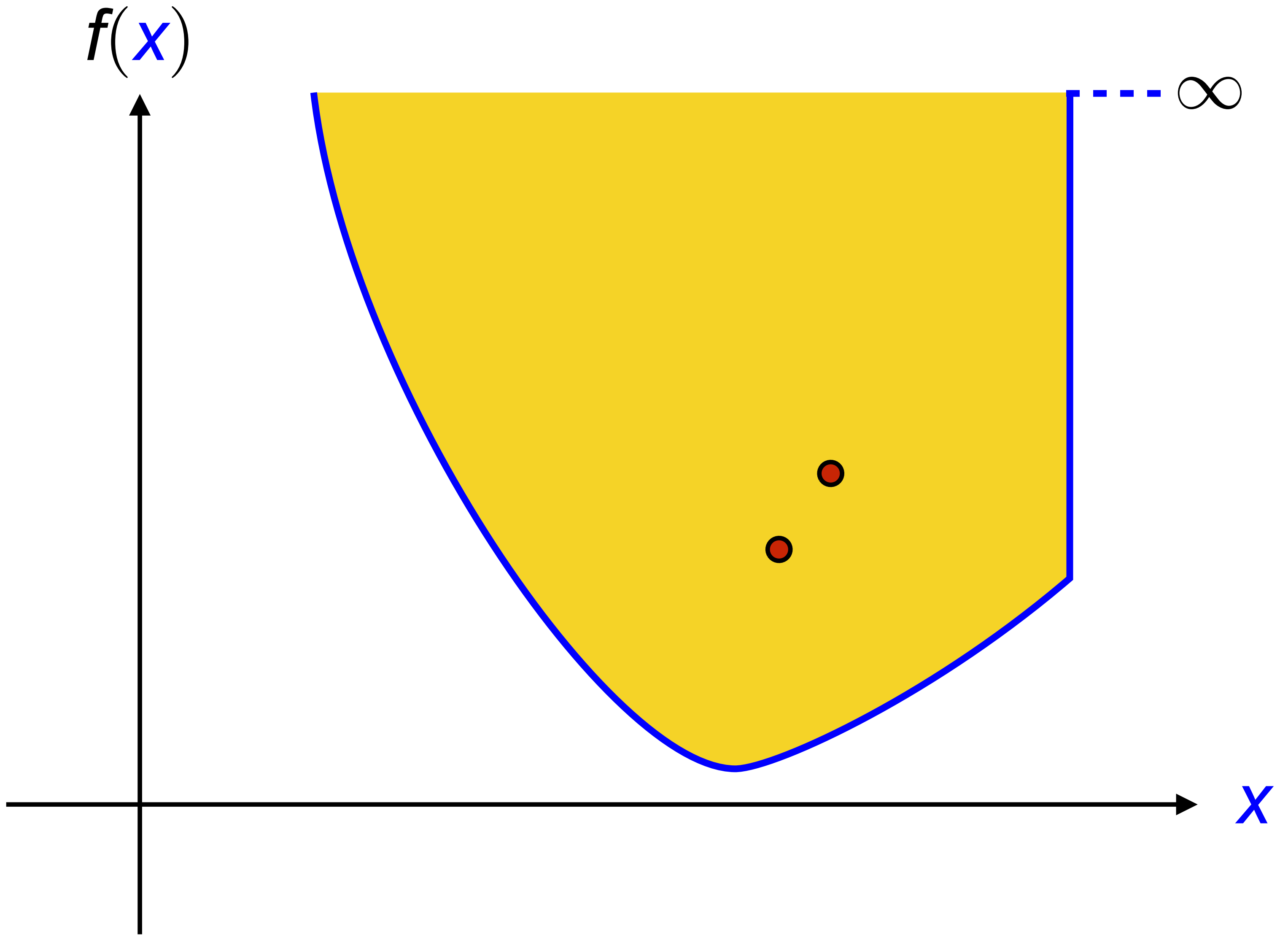

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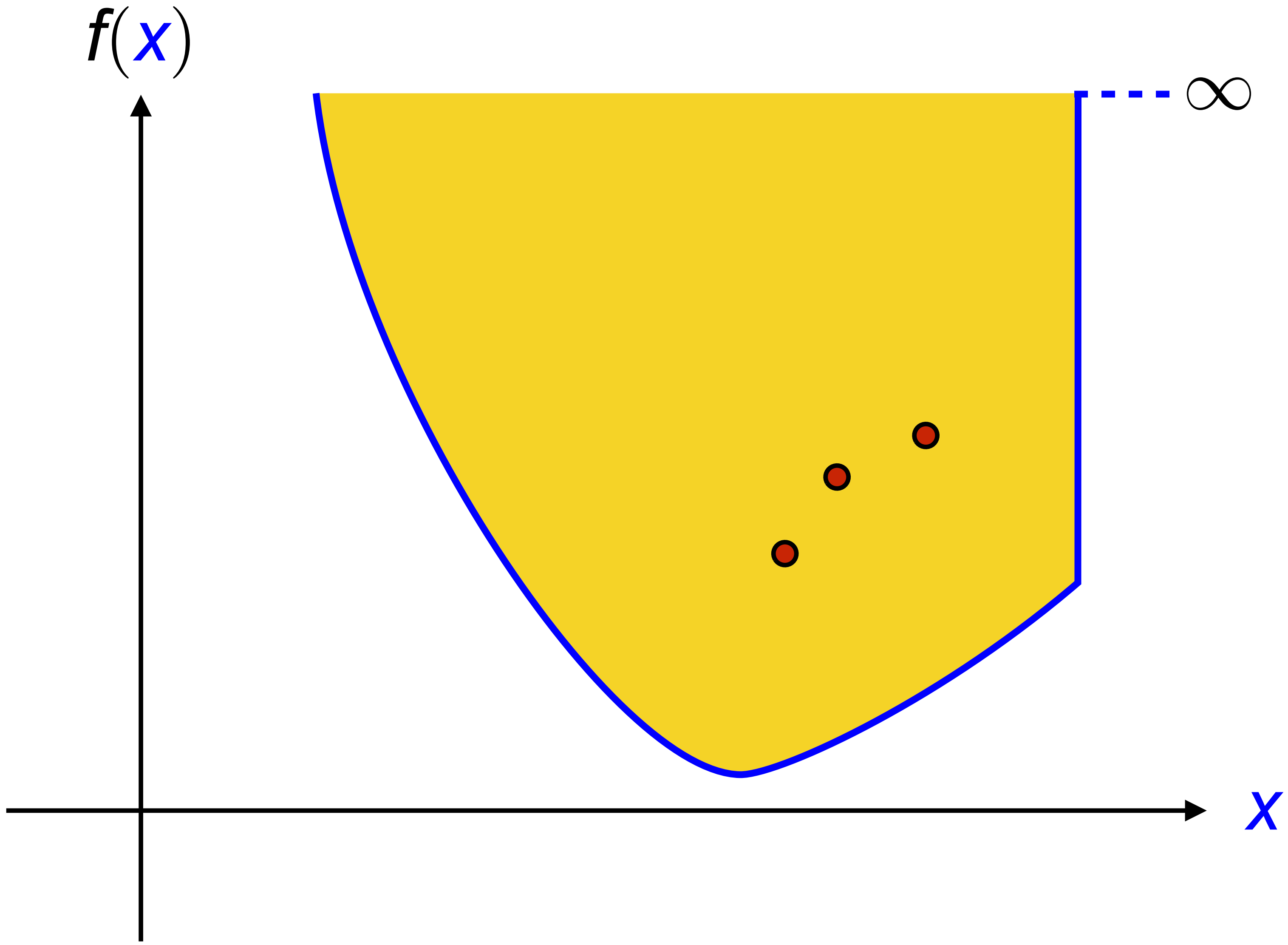

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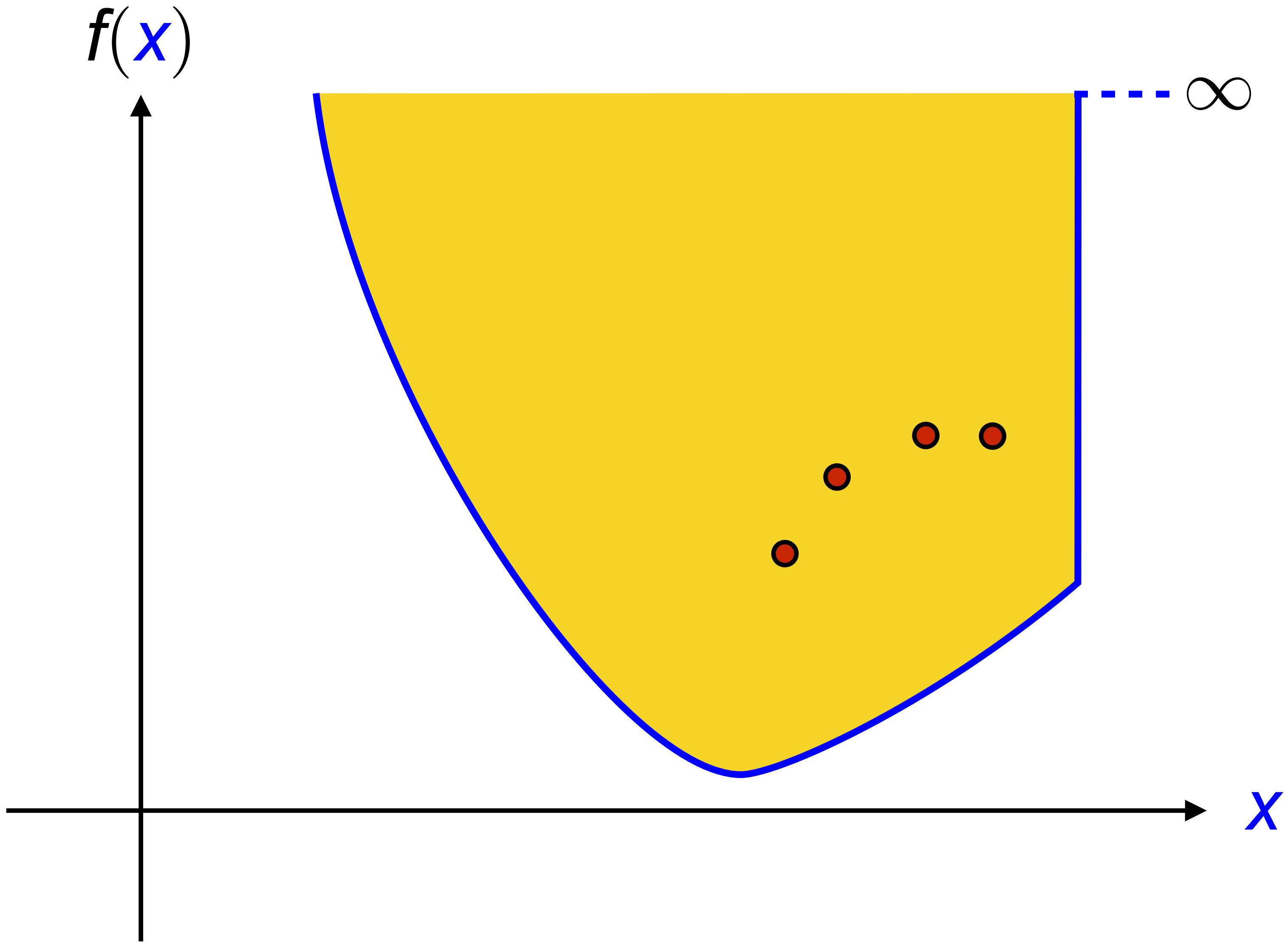

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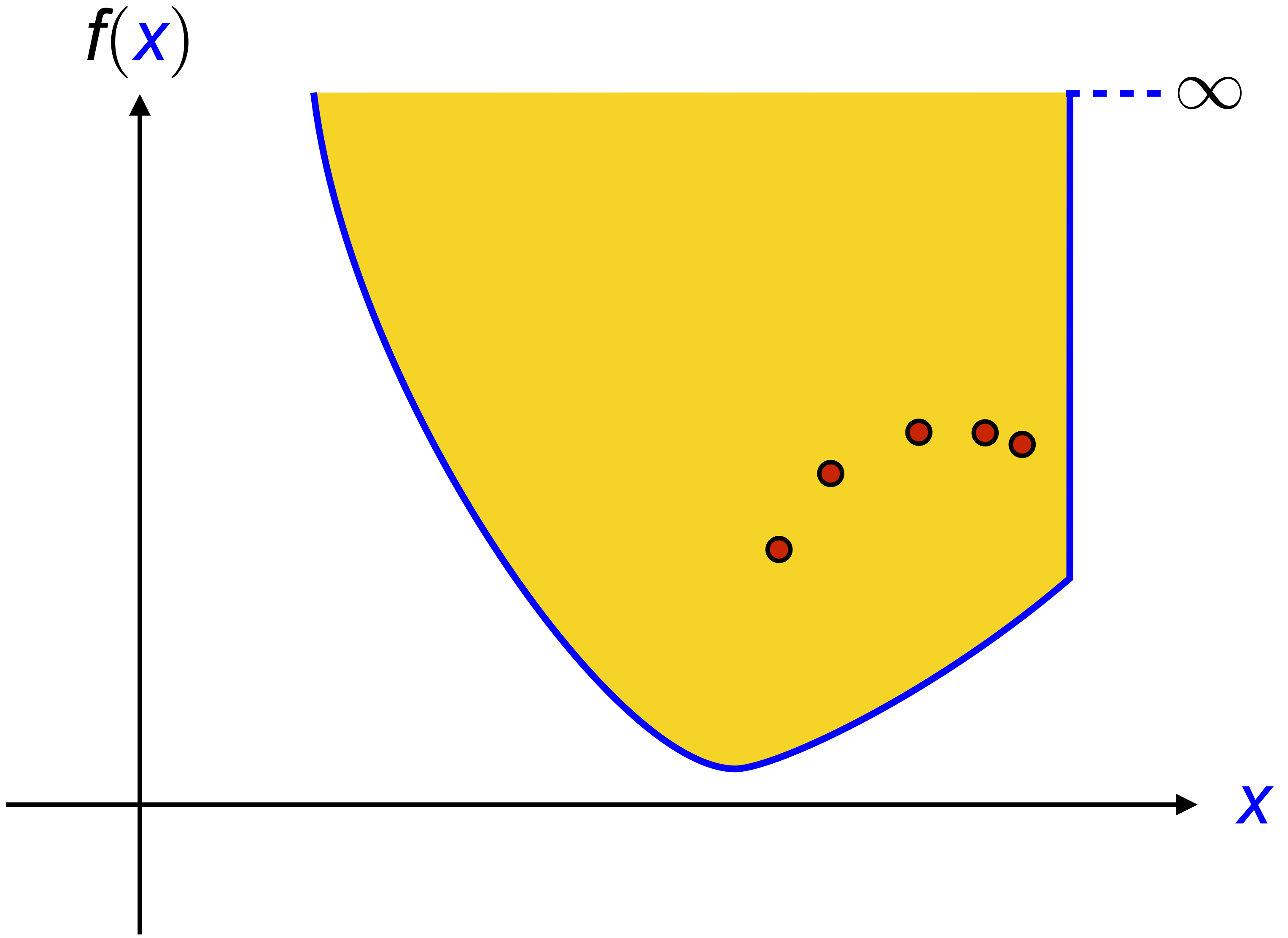

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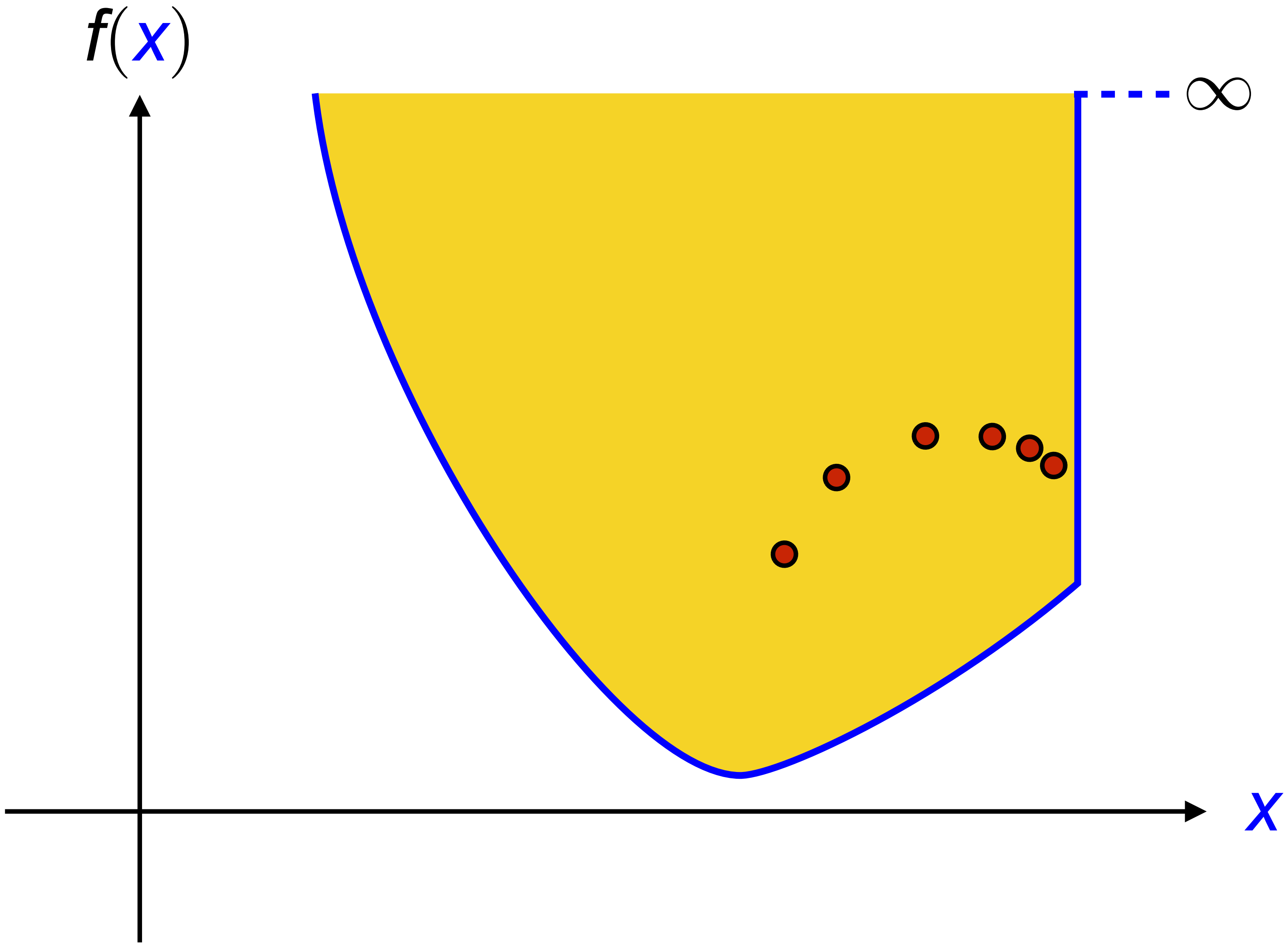

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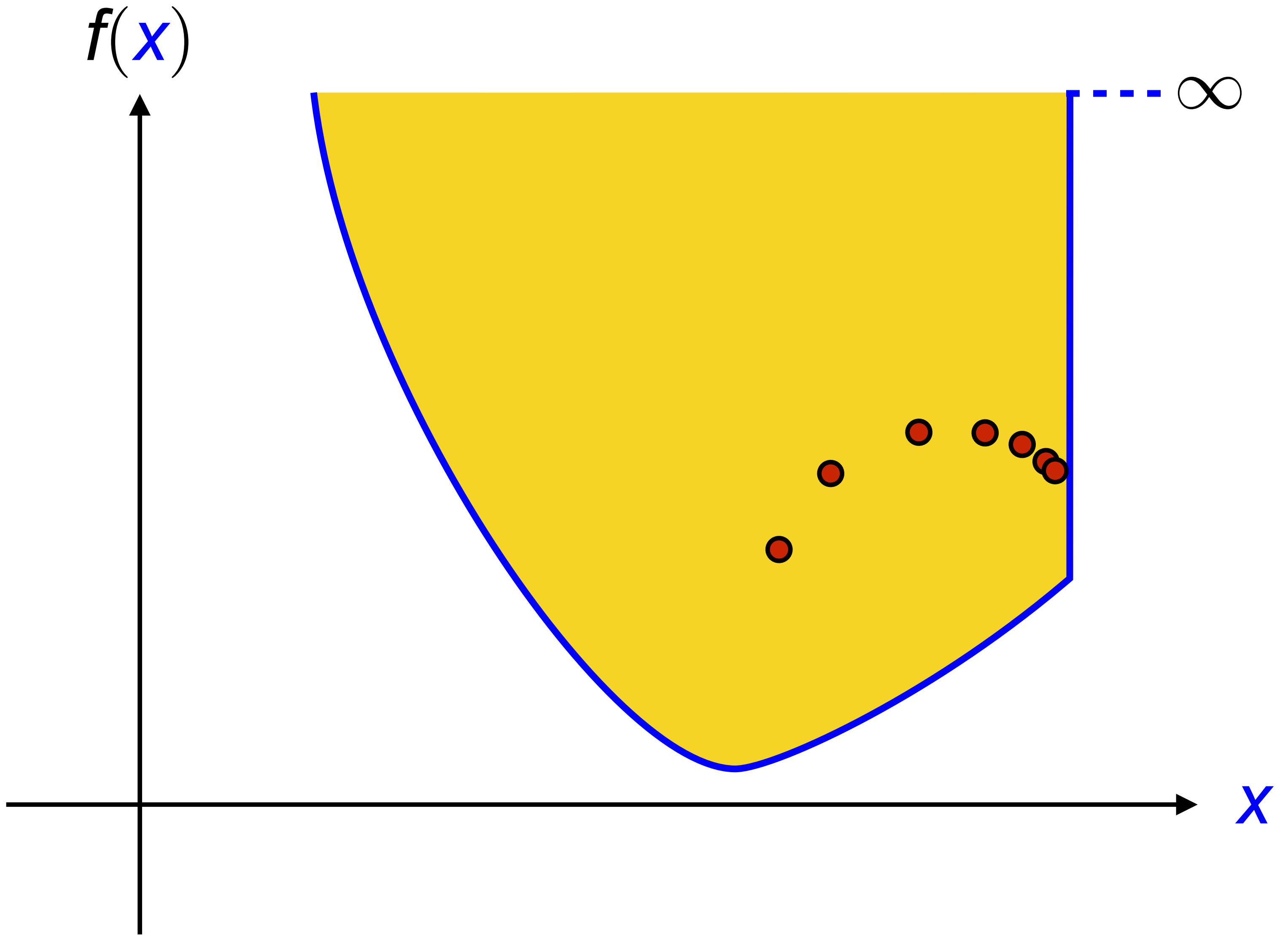

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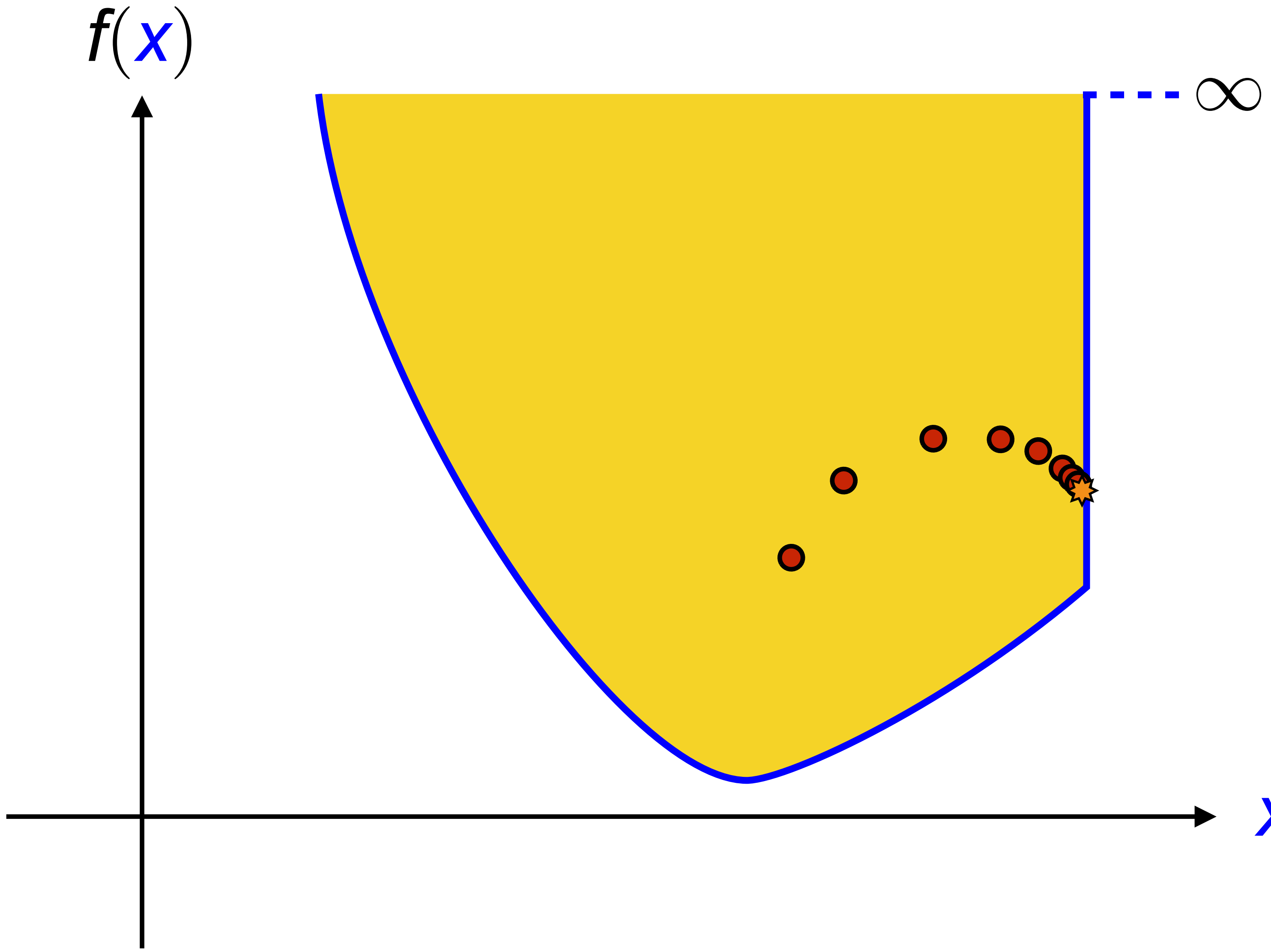
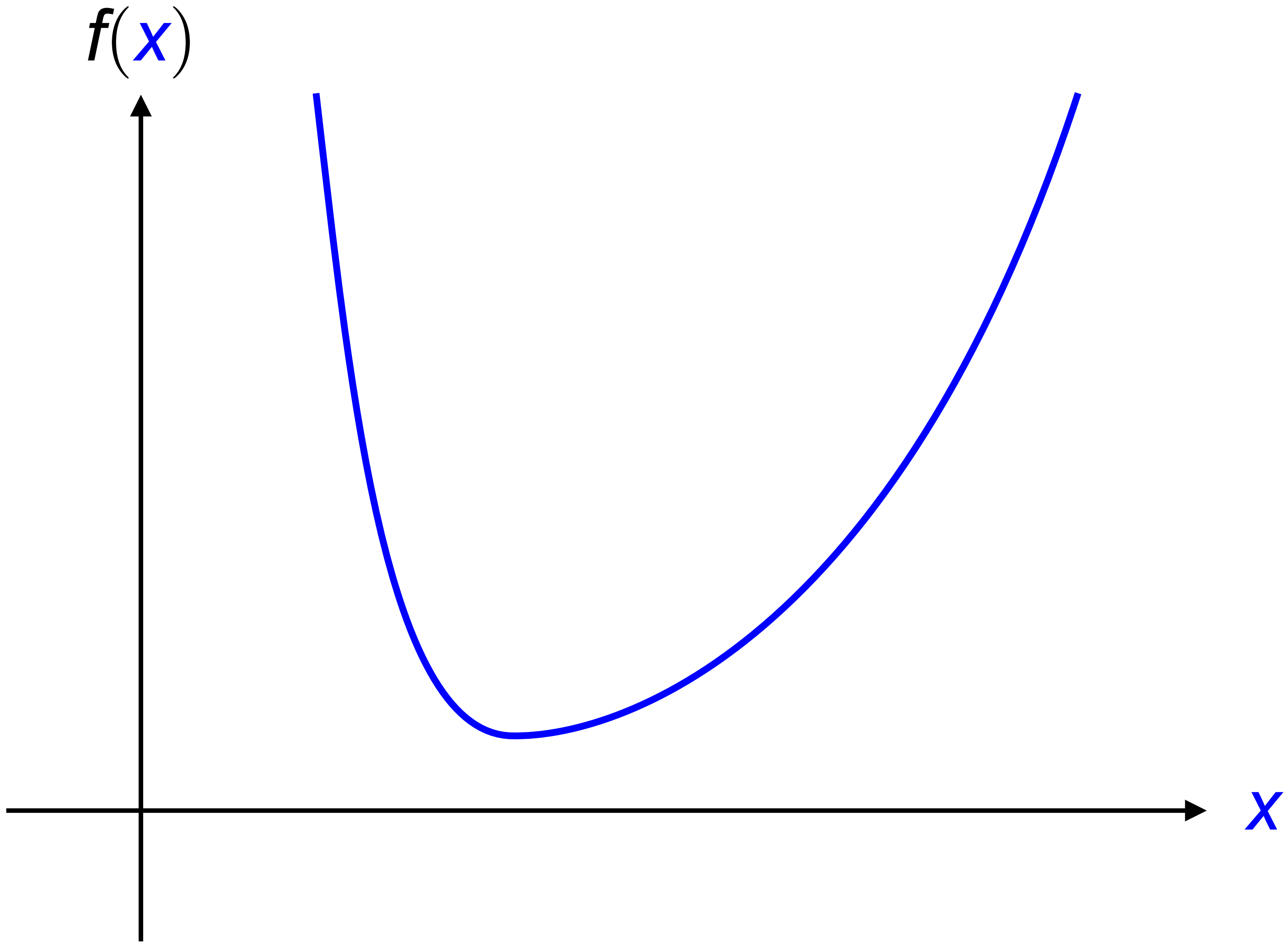


PCC Functions

pcc = proper, convex and closed



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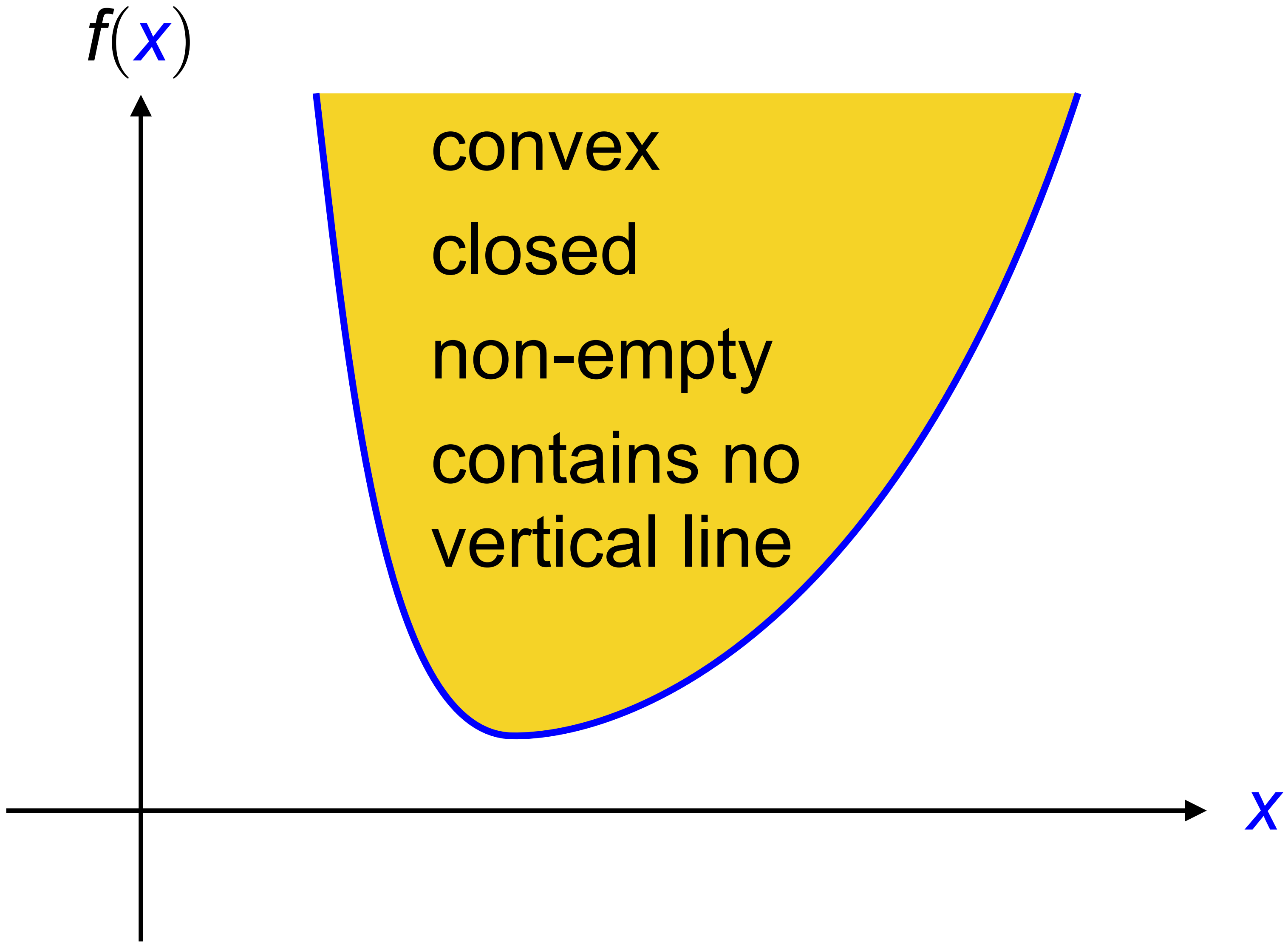

Convexity-Preserving Transformations

Conjugate Functions

$$f(x) \quad \rightsquigarrow \quad f^*(w) = \sup_{x} \left\{ w^\top x - f(x) \right\}$$

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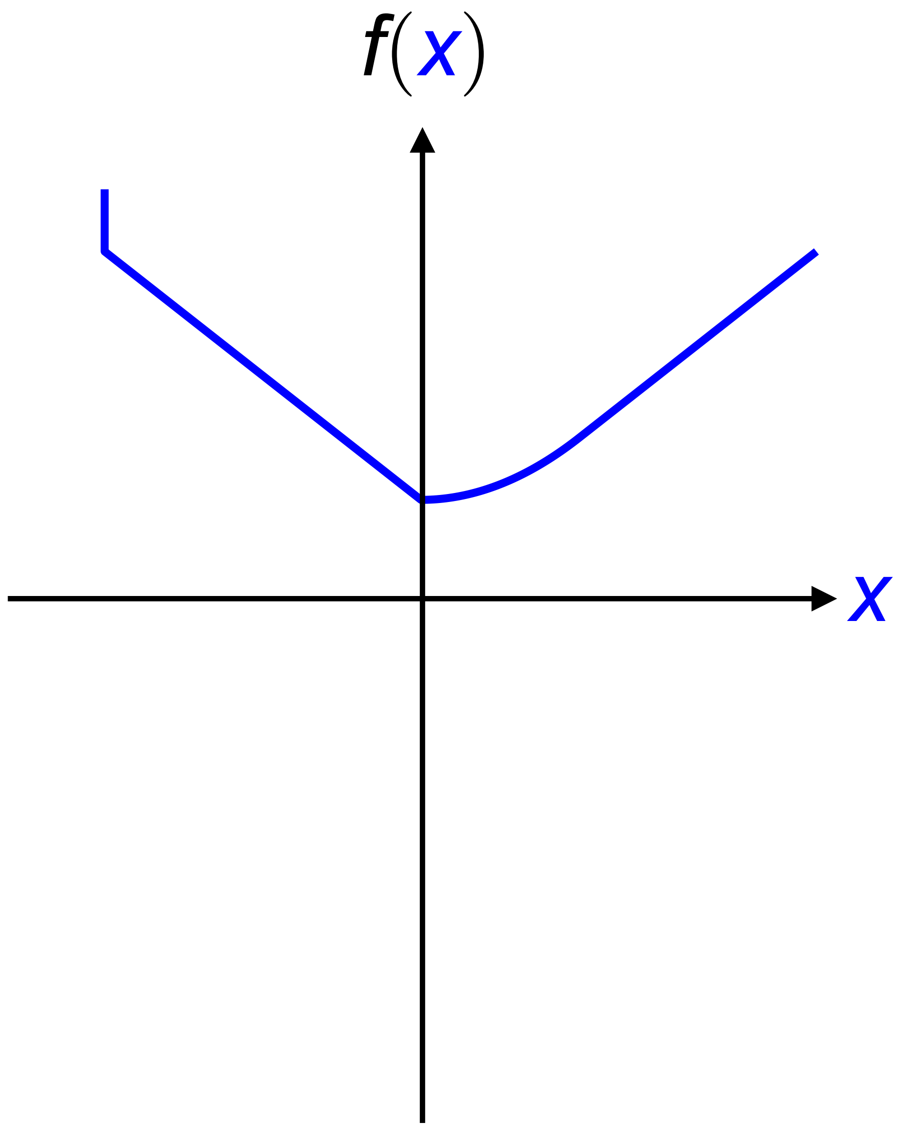

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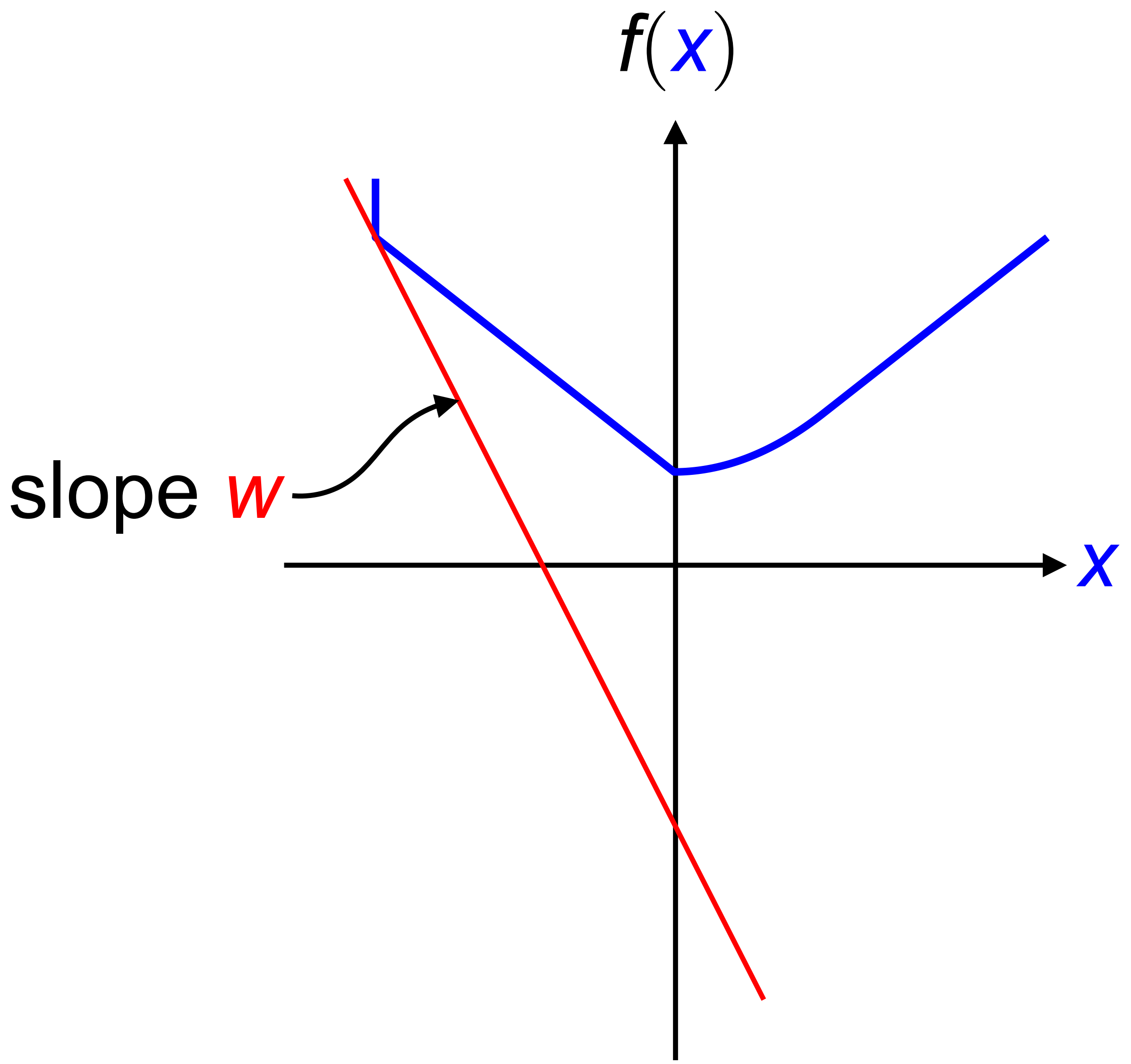

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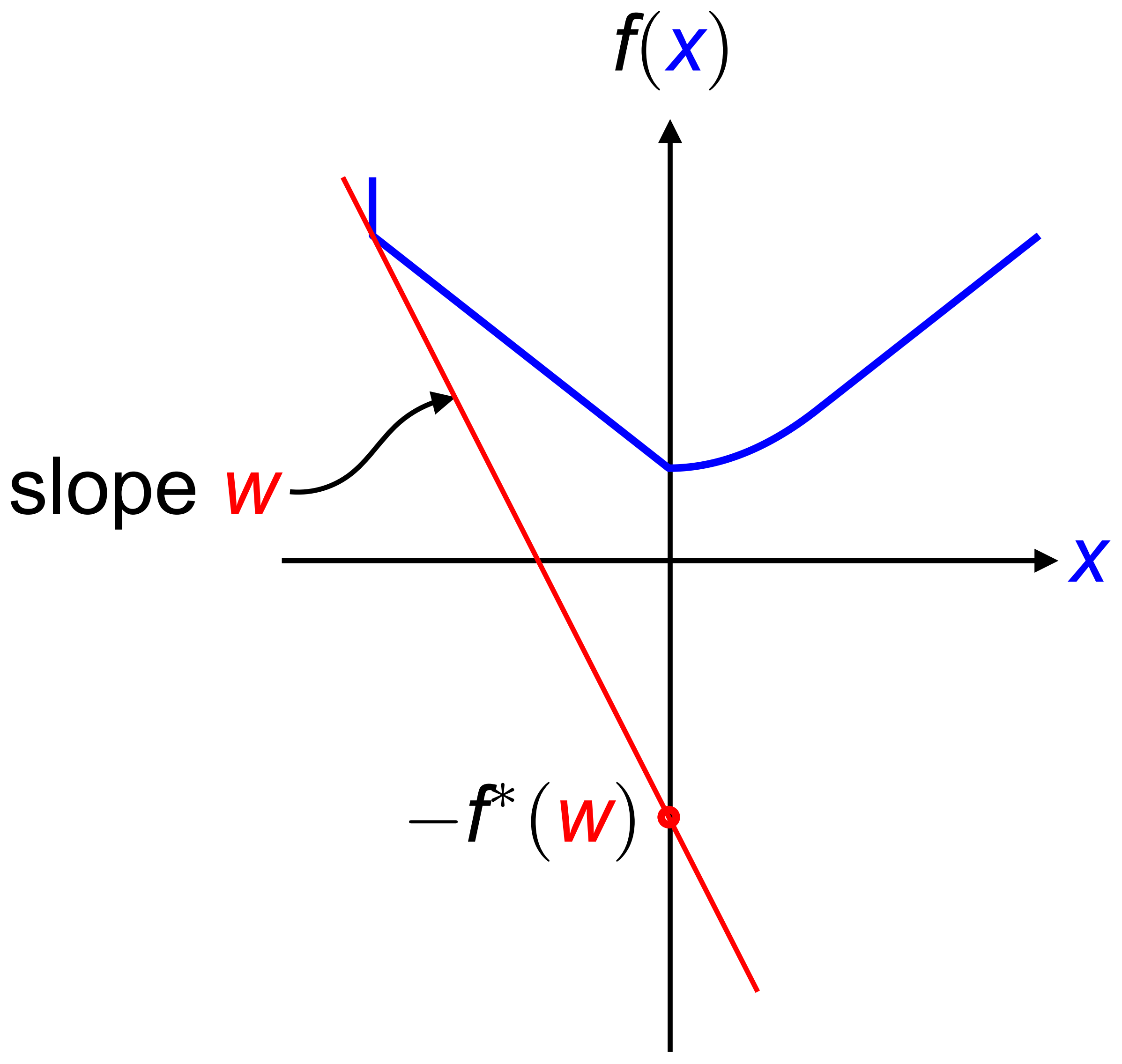

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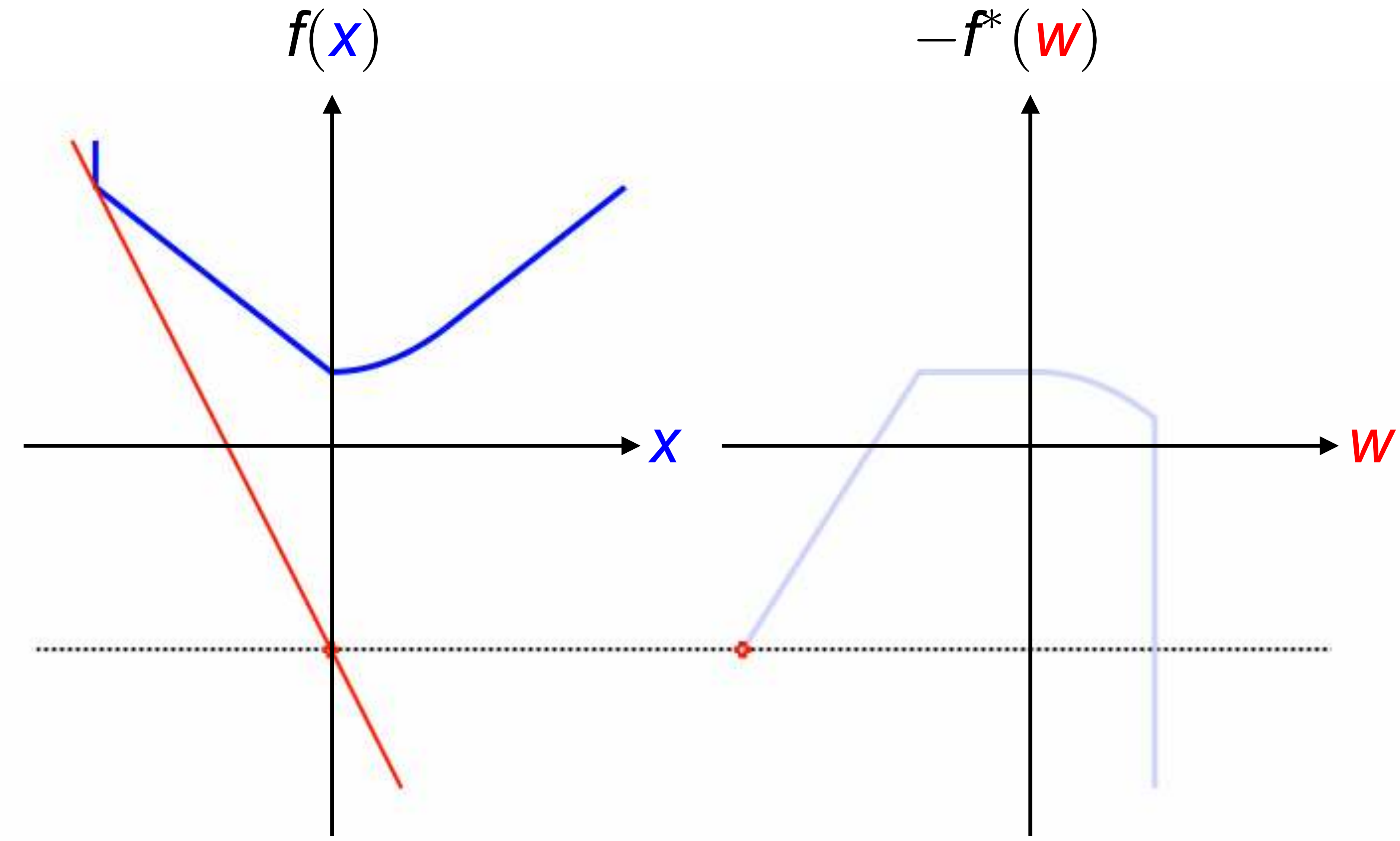

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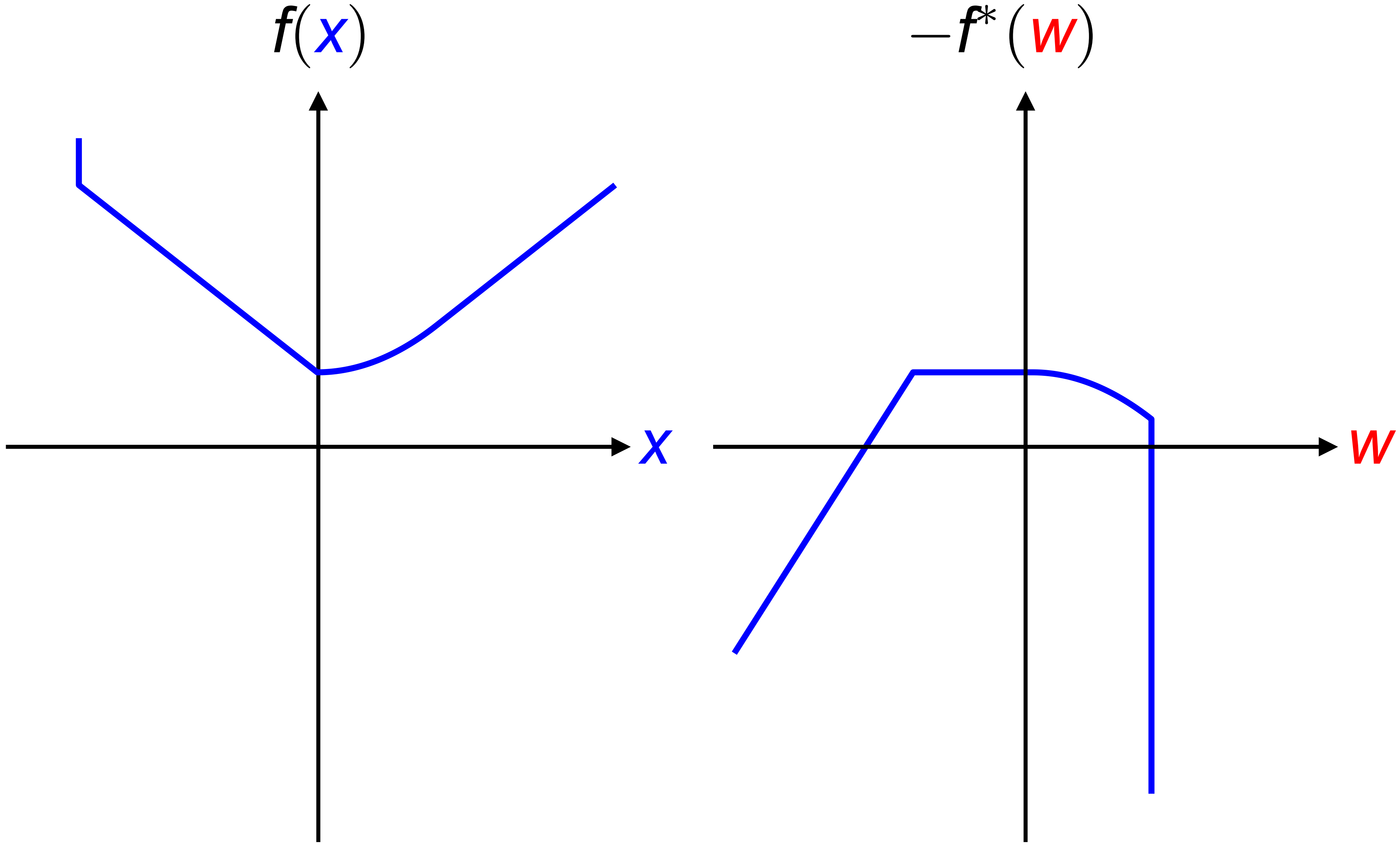

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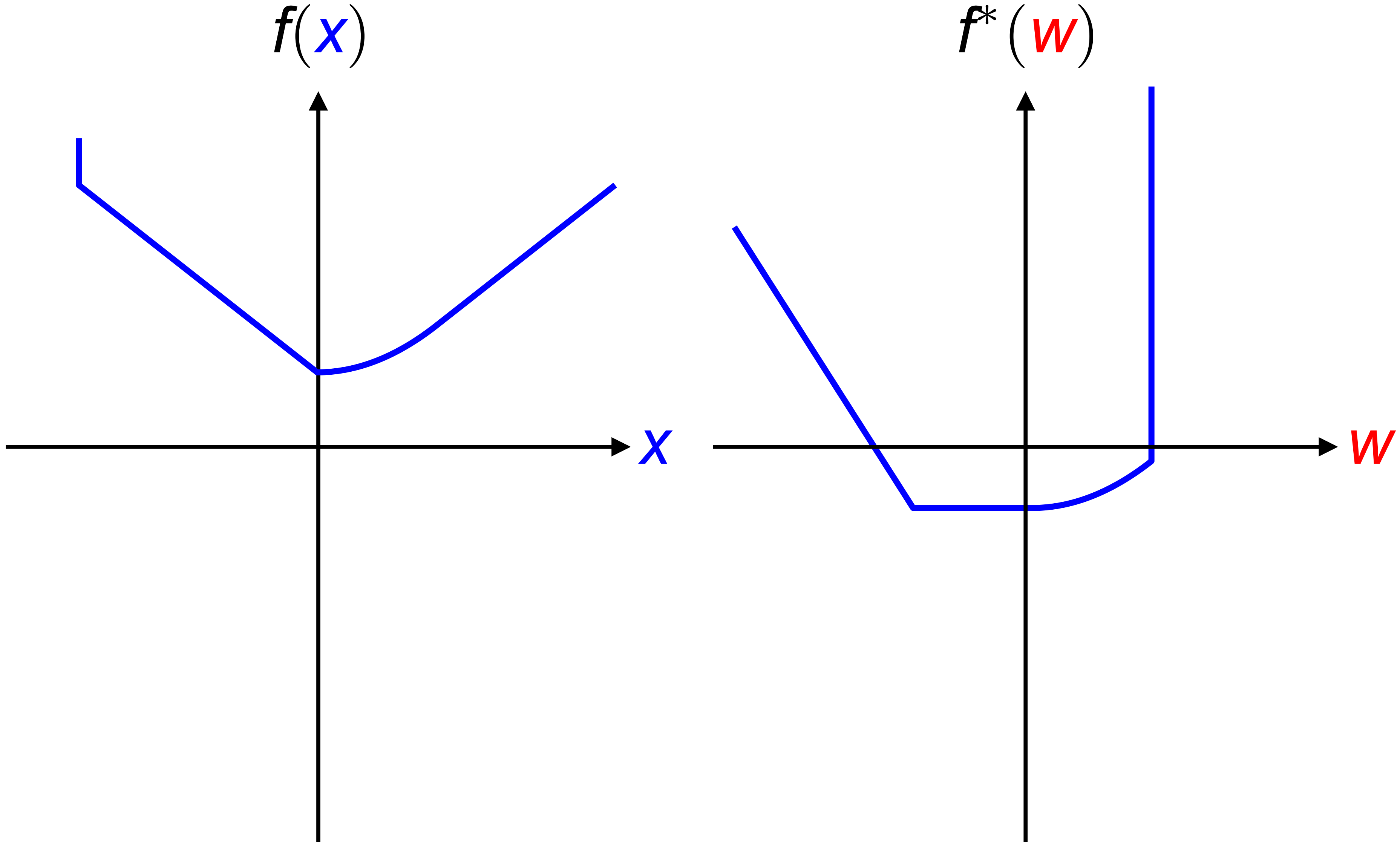

Perspective Functions

$$f(x) \quad \rightsquigarrow \quad \underline{f}(x, t) = t \cdot f\,(x/t)$$

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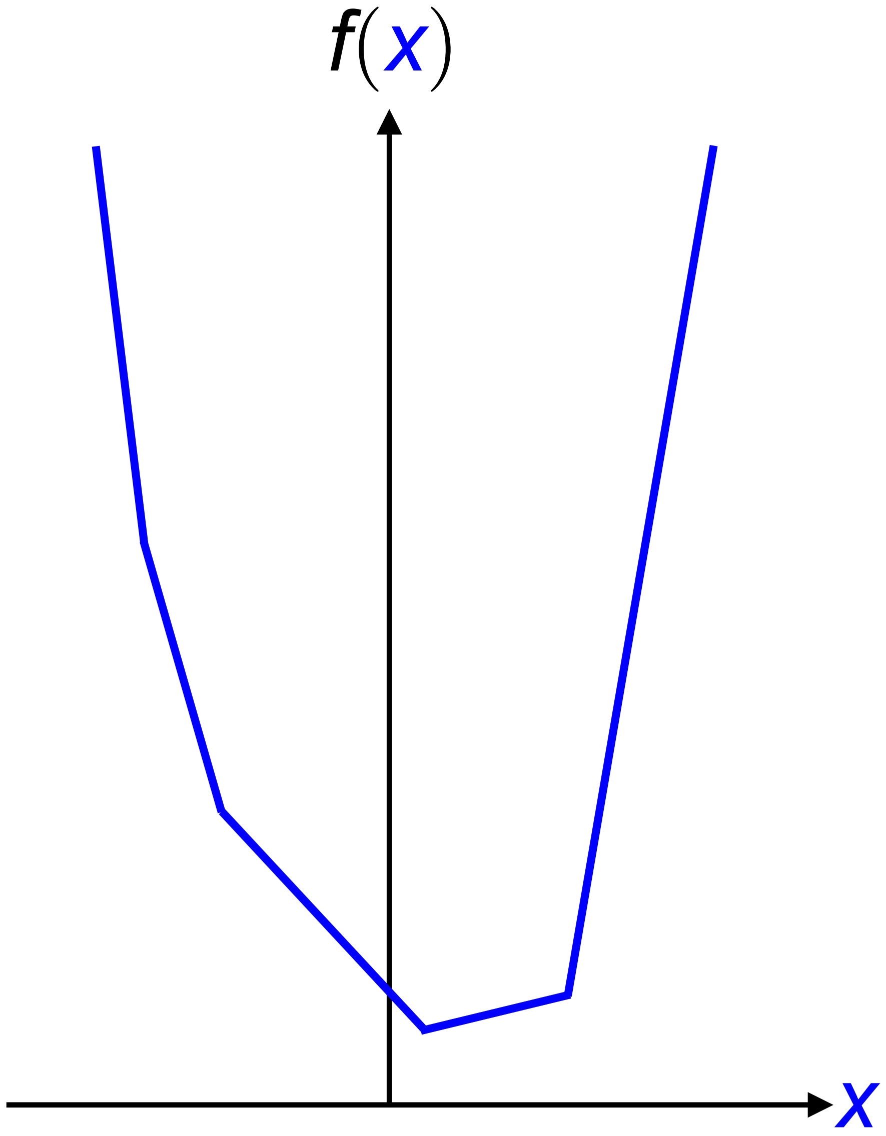

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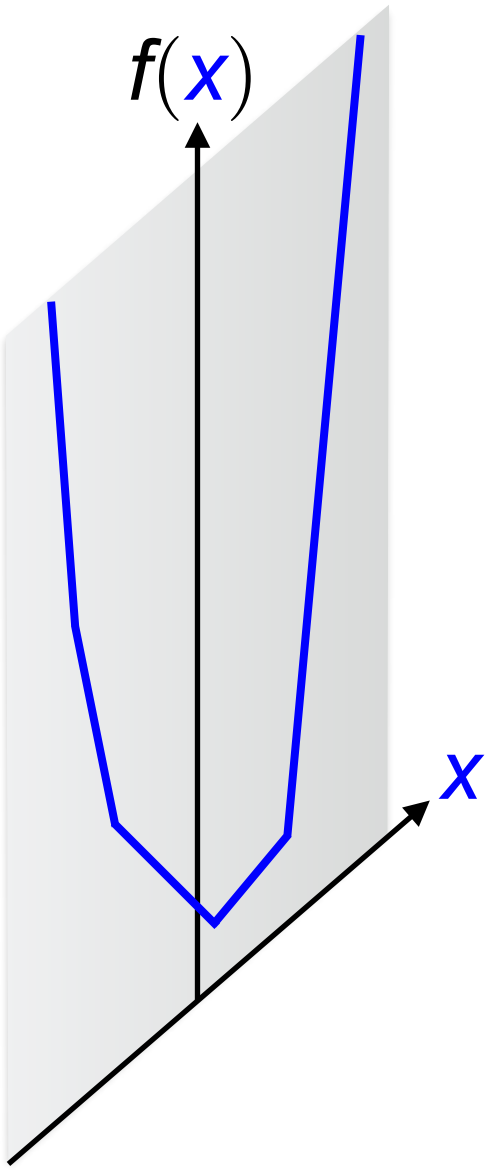

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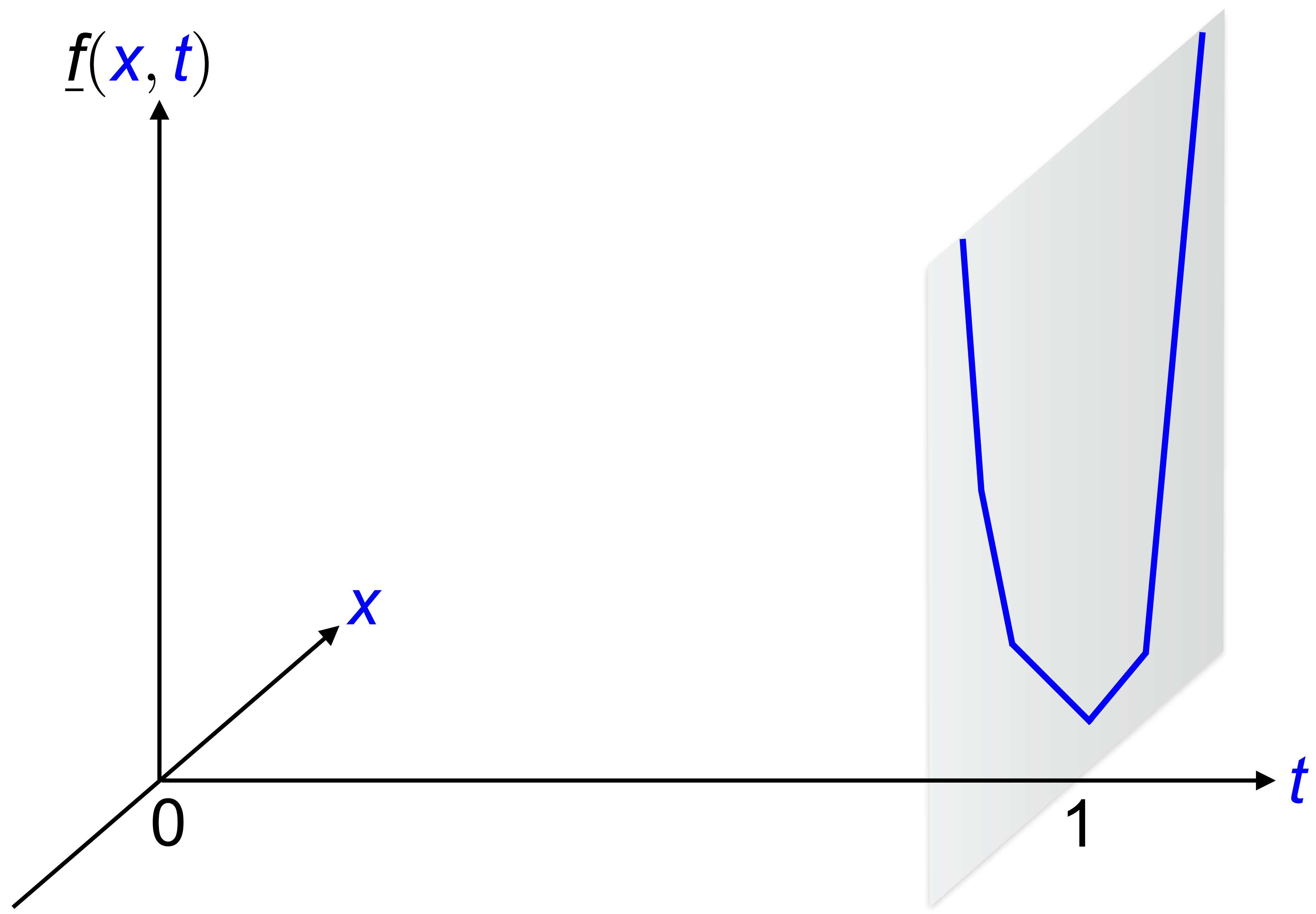

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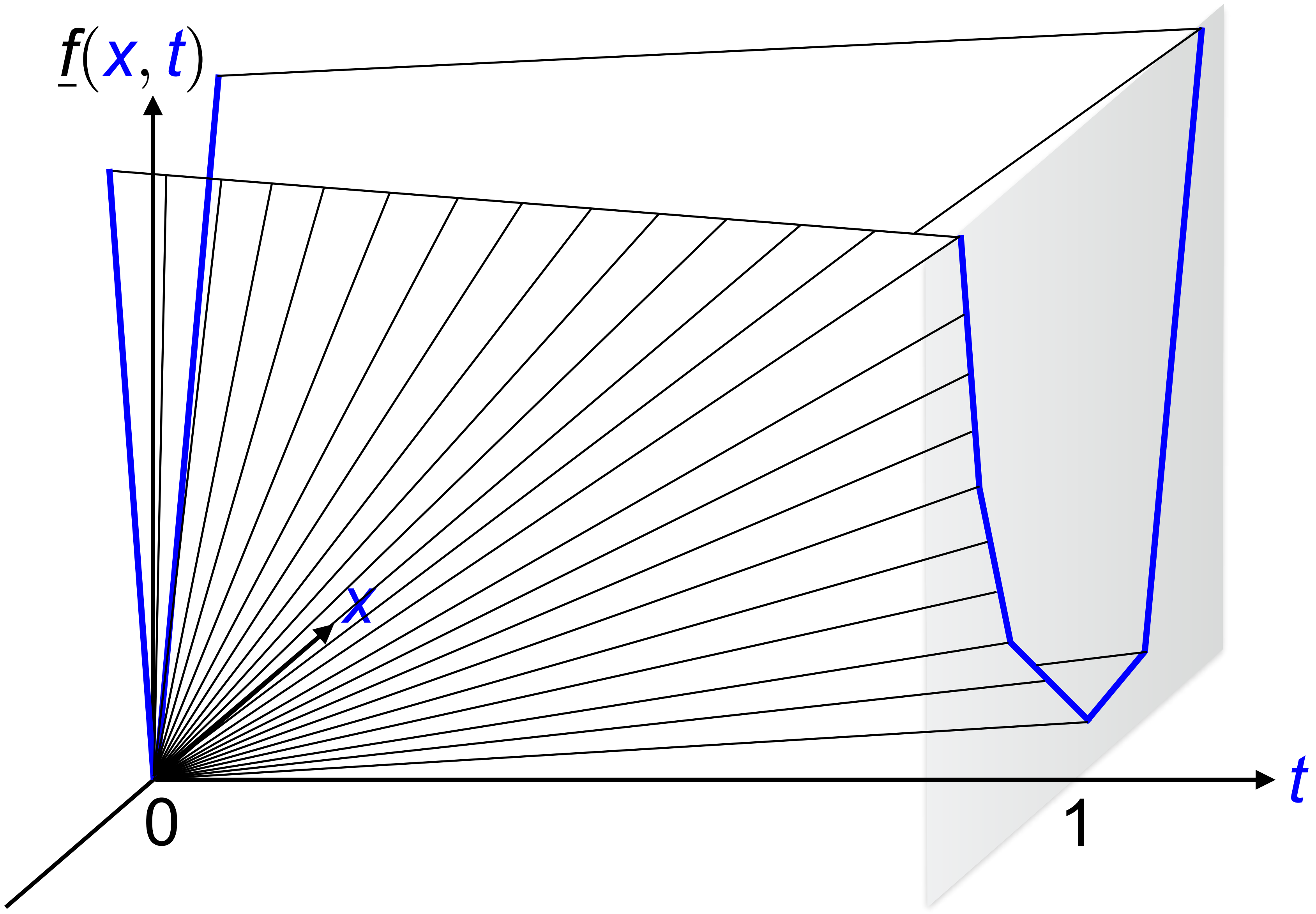

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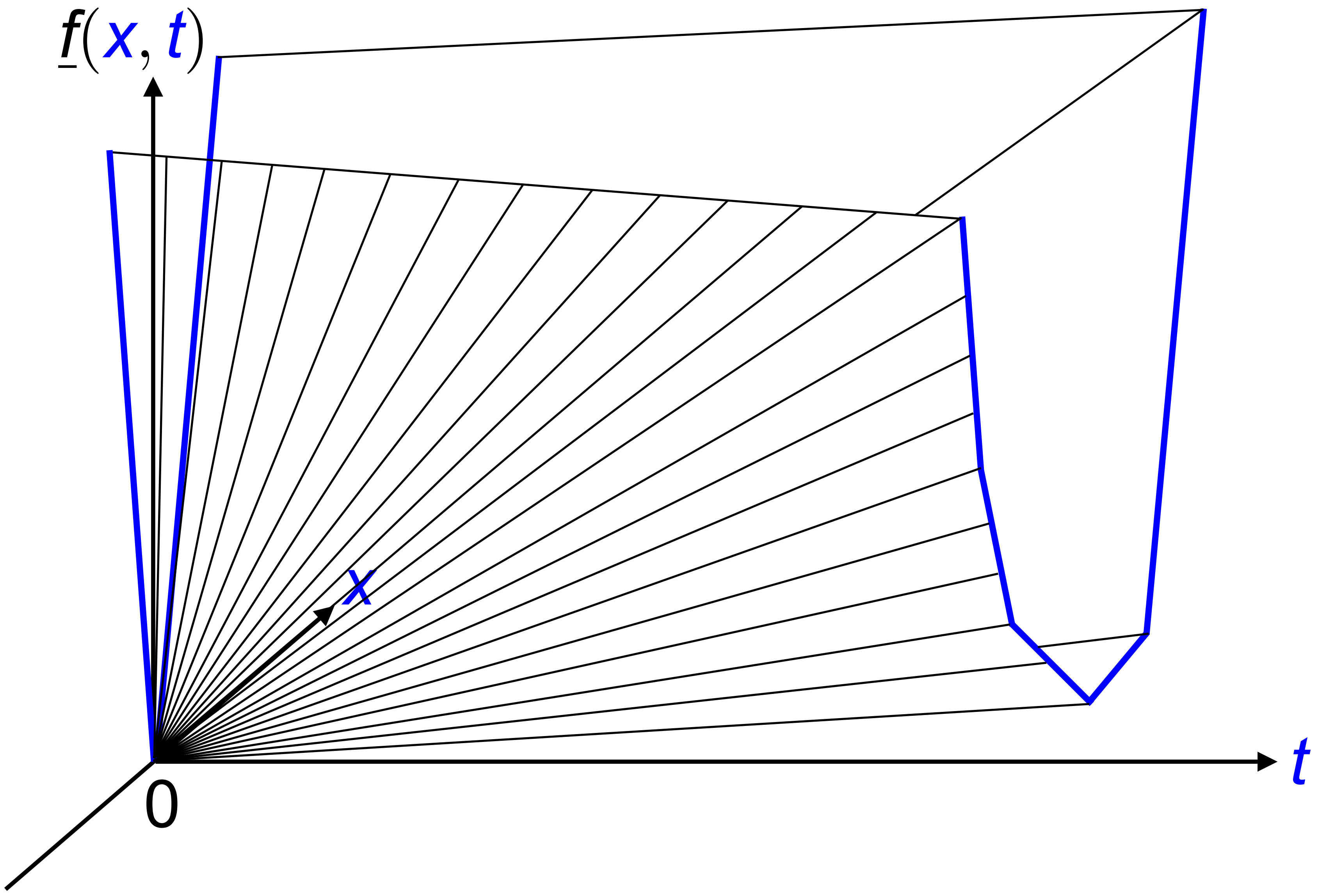

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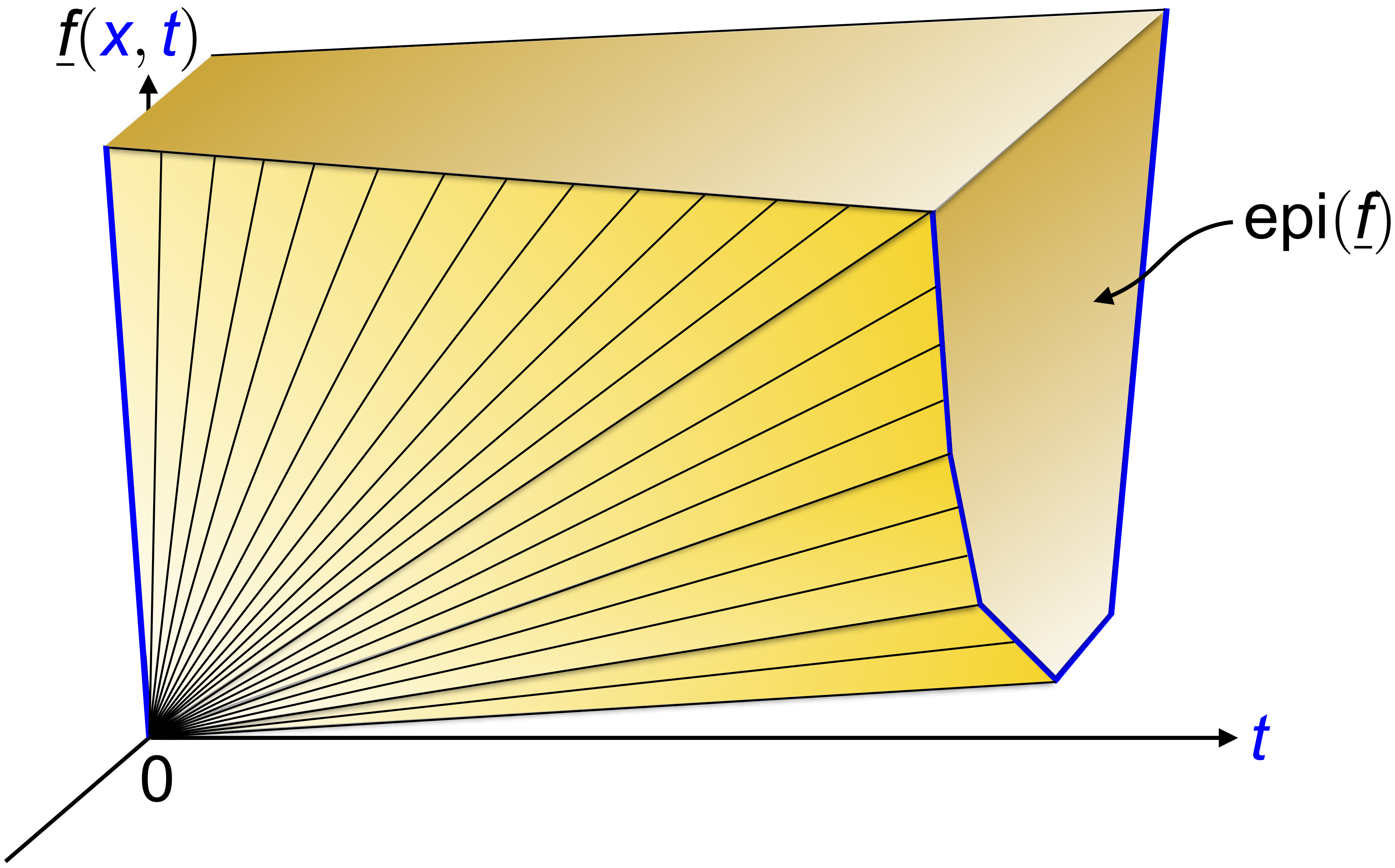

Minkowski Sums

$$C_1, C_2 \quad \rightsquigarrow \quad C_1 + C_2 = \{x_1 + x_2 : x_1 \in C_1,\ x_2 \in C_2\}$$

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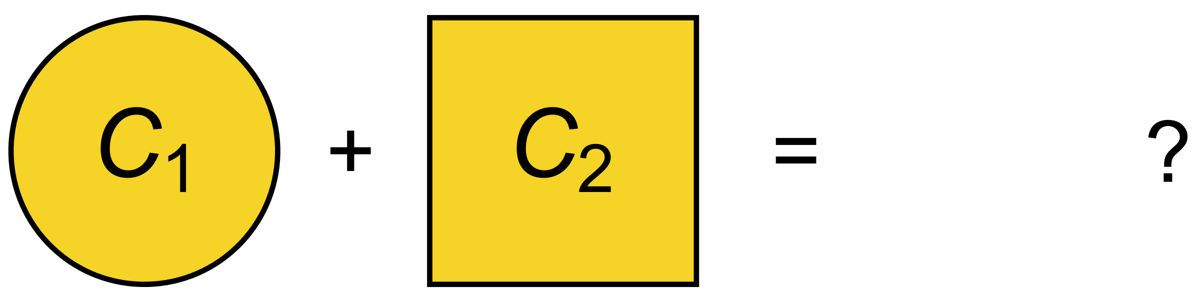

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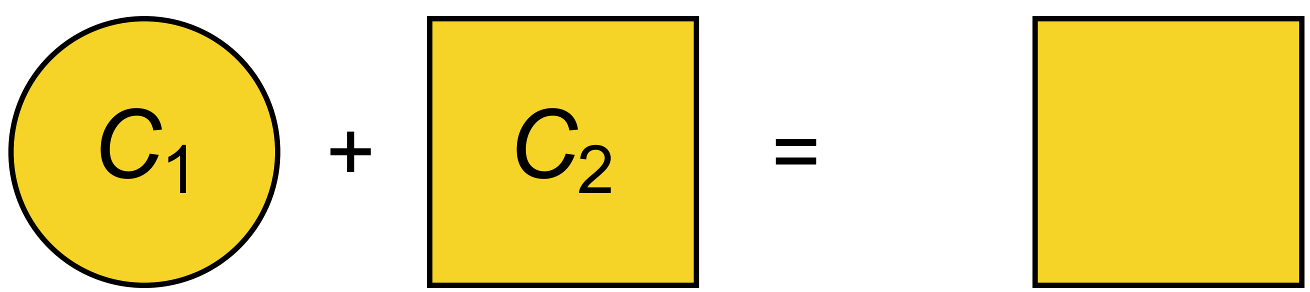

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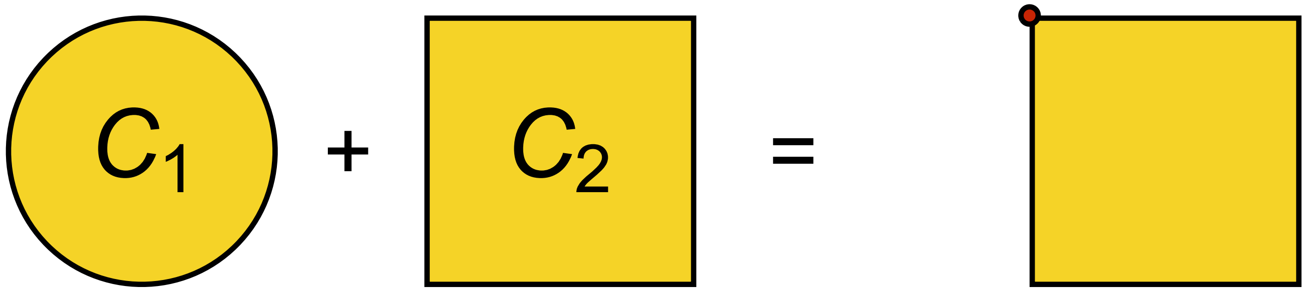

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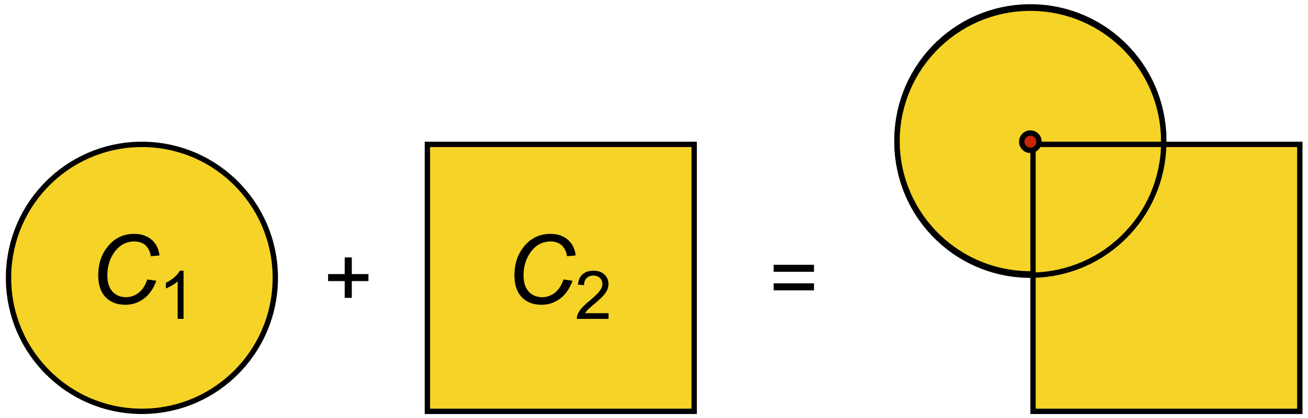

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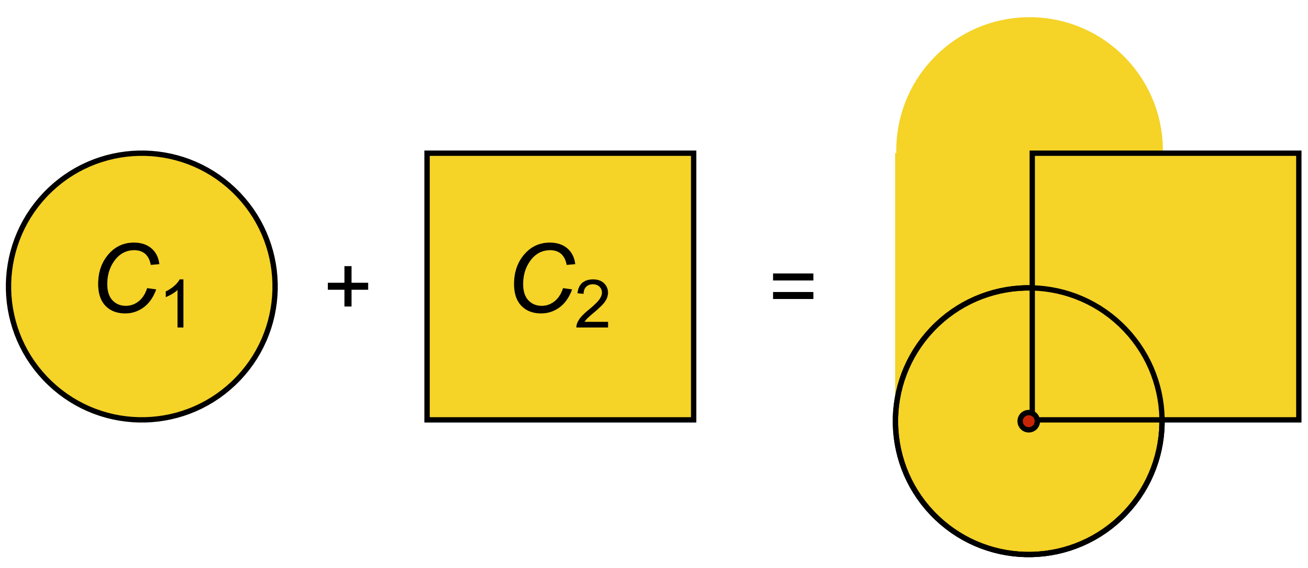

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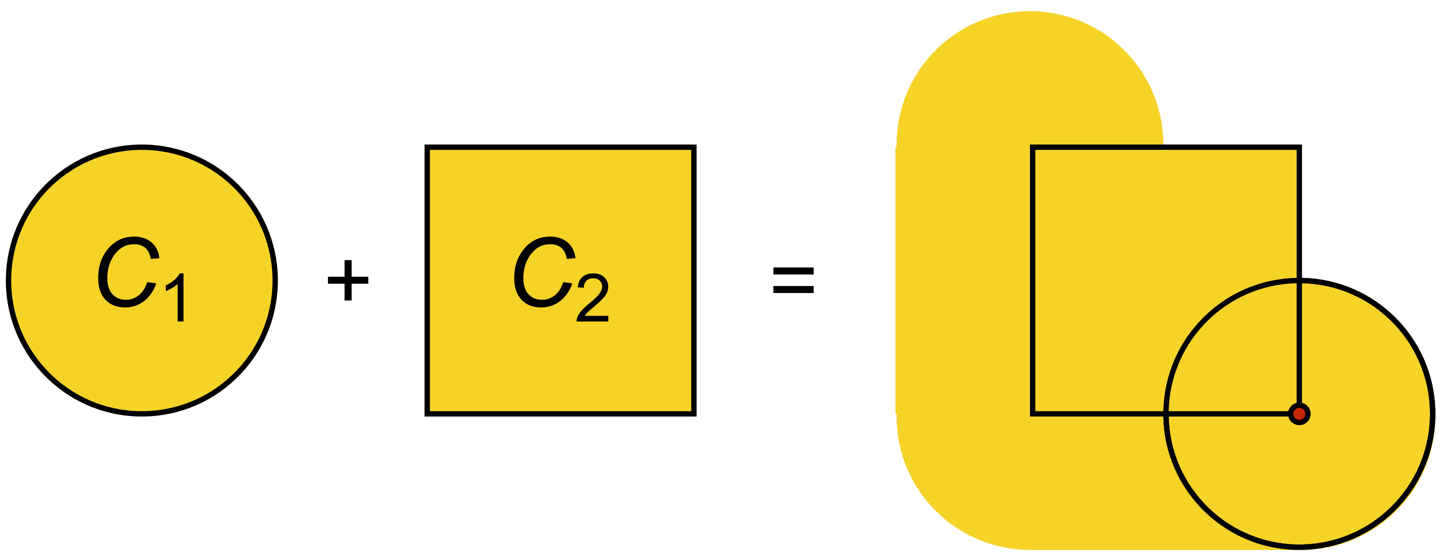

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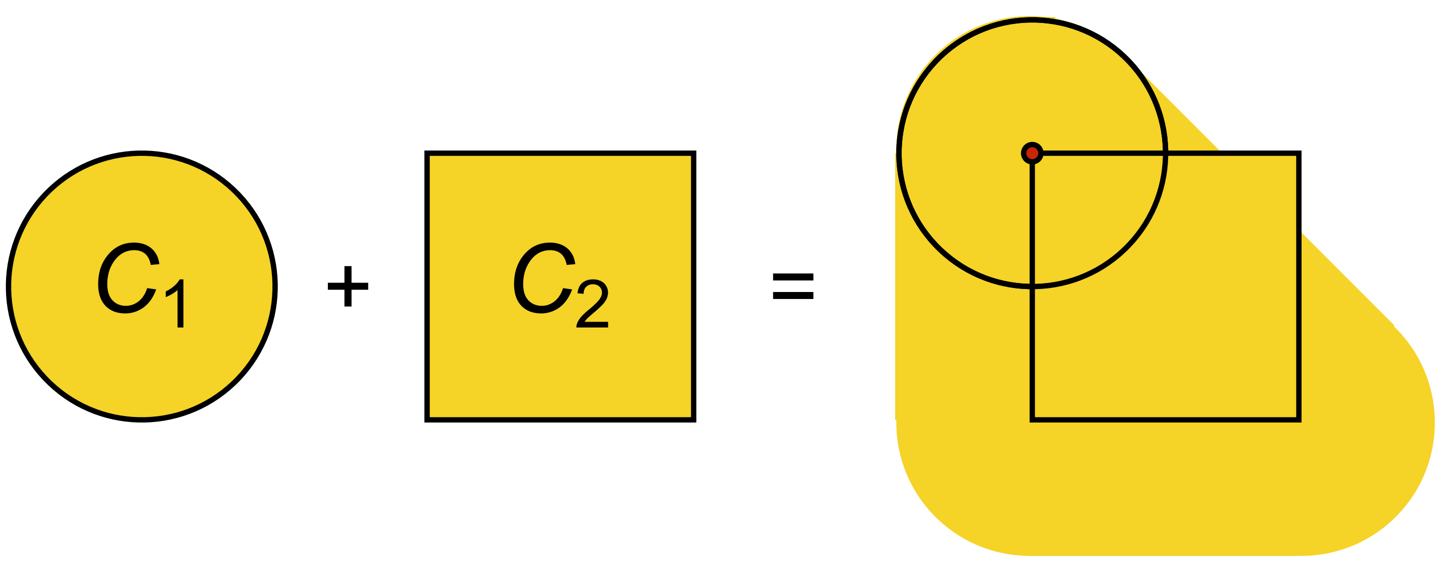

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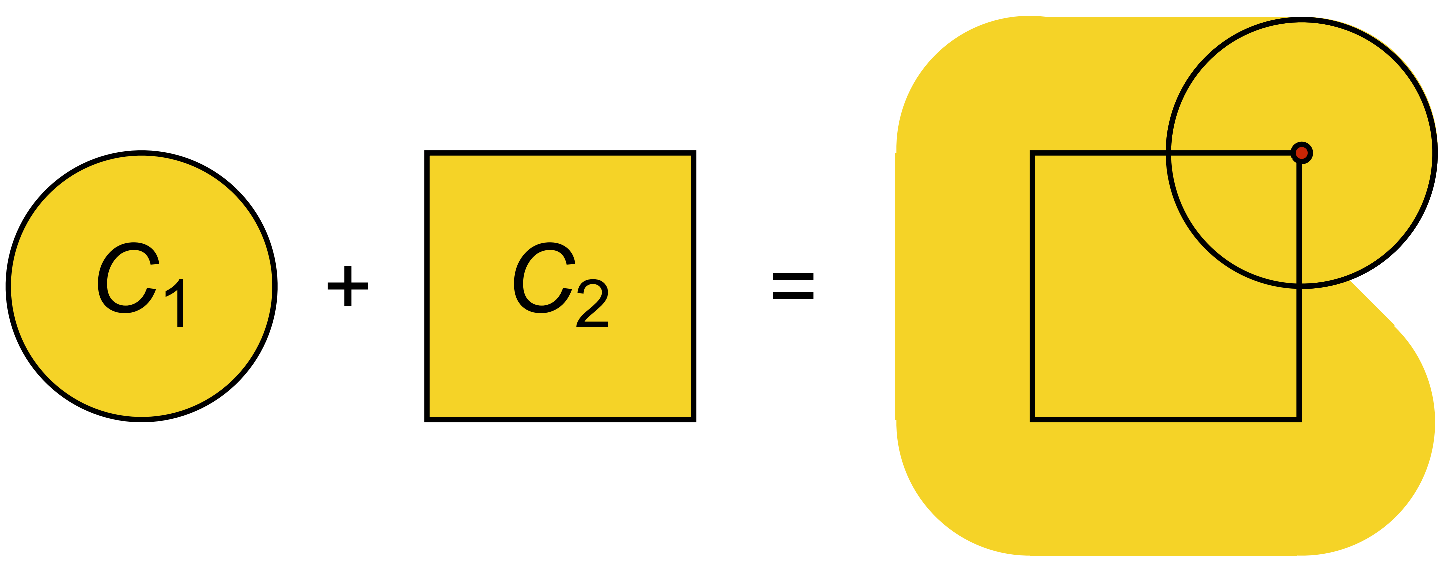

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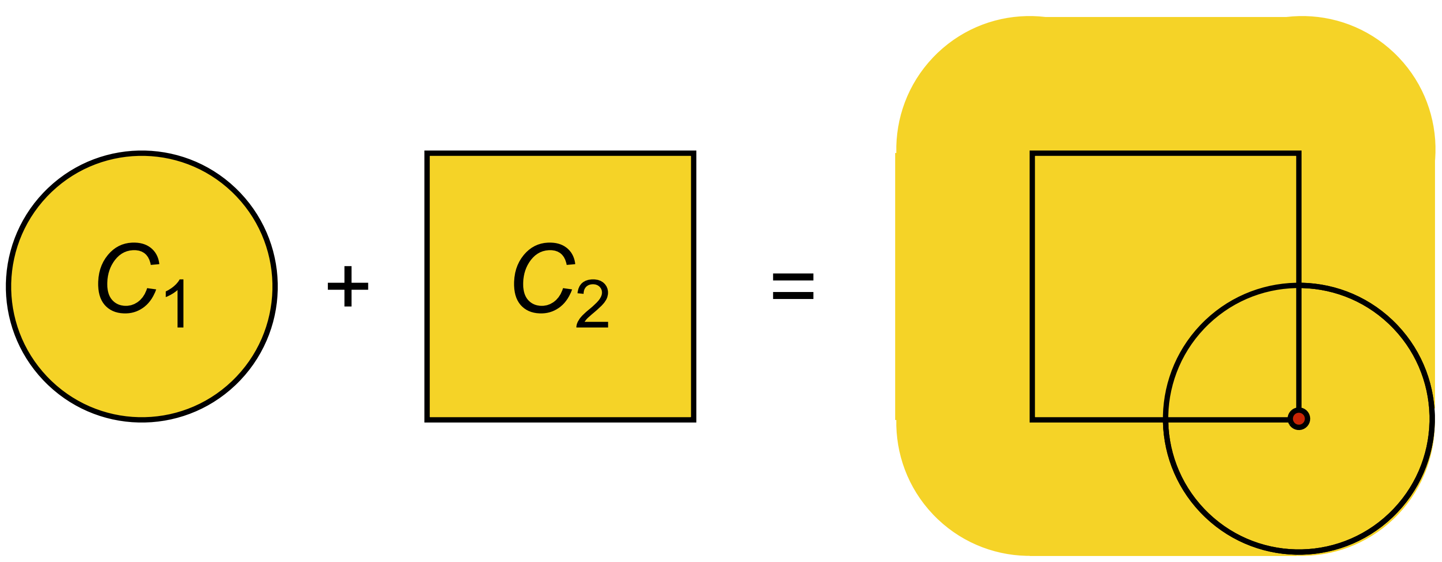

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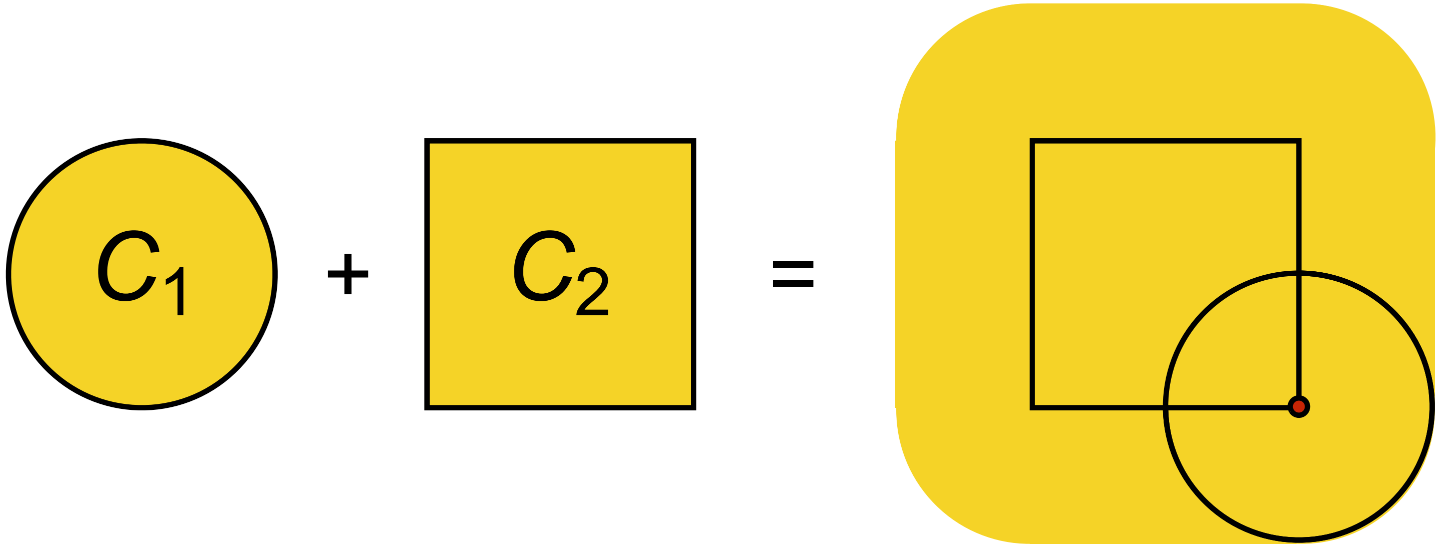

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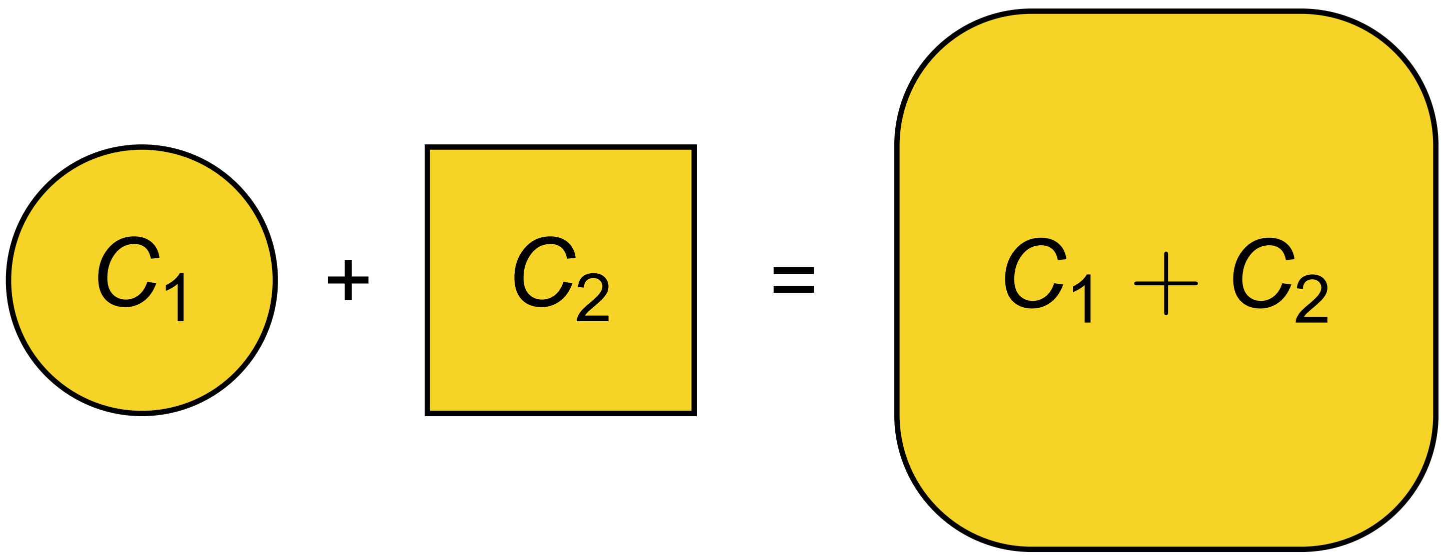

Infimal Convolution

$$f_1(x_1),\, f_2(x_2) \quad \rightsquigarrow \quad (f_1 \,\Box\, f_2)(x) = \inf_{x_1+x_2=x} f_1(x_1) + f_2(x_2)$$

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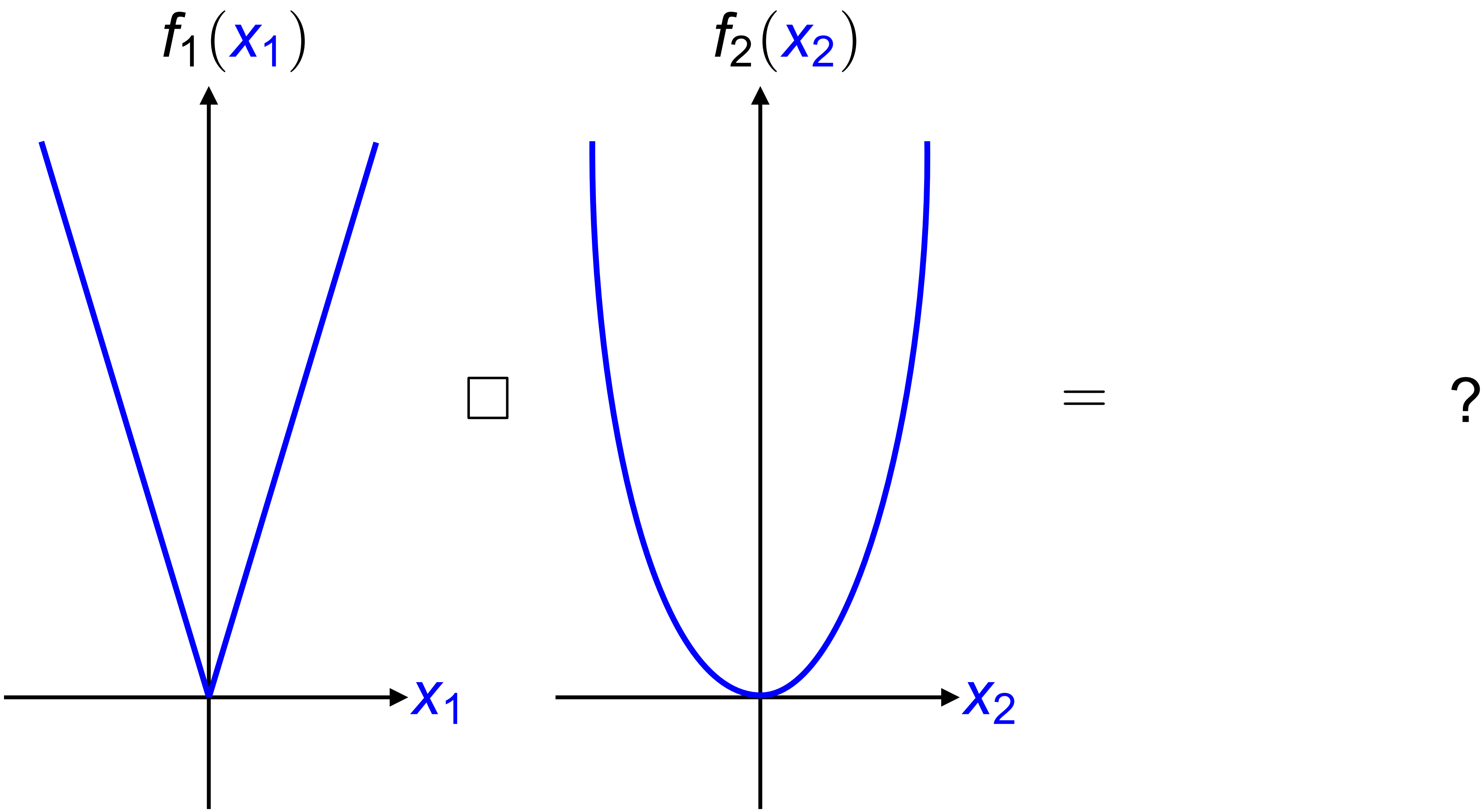

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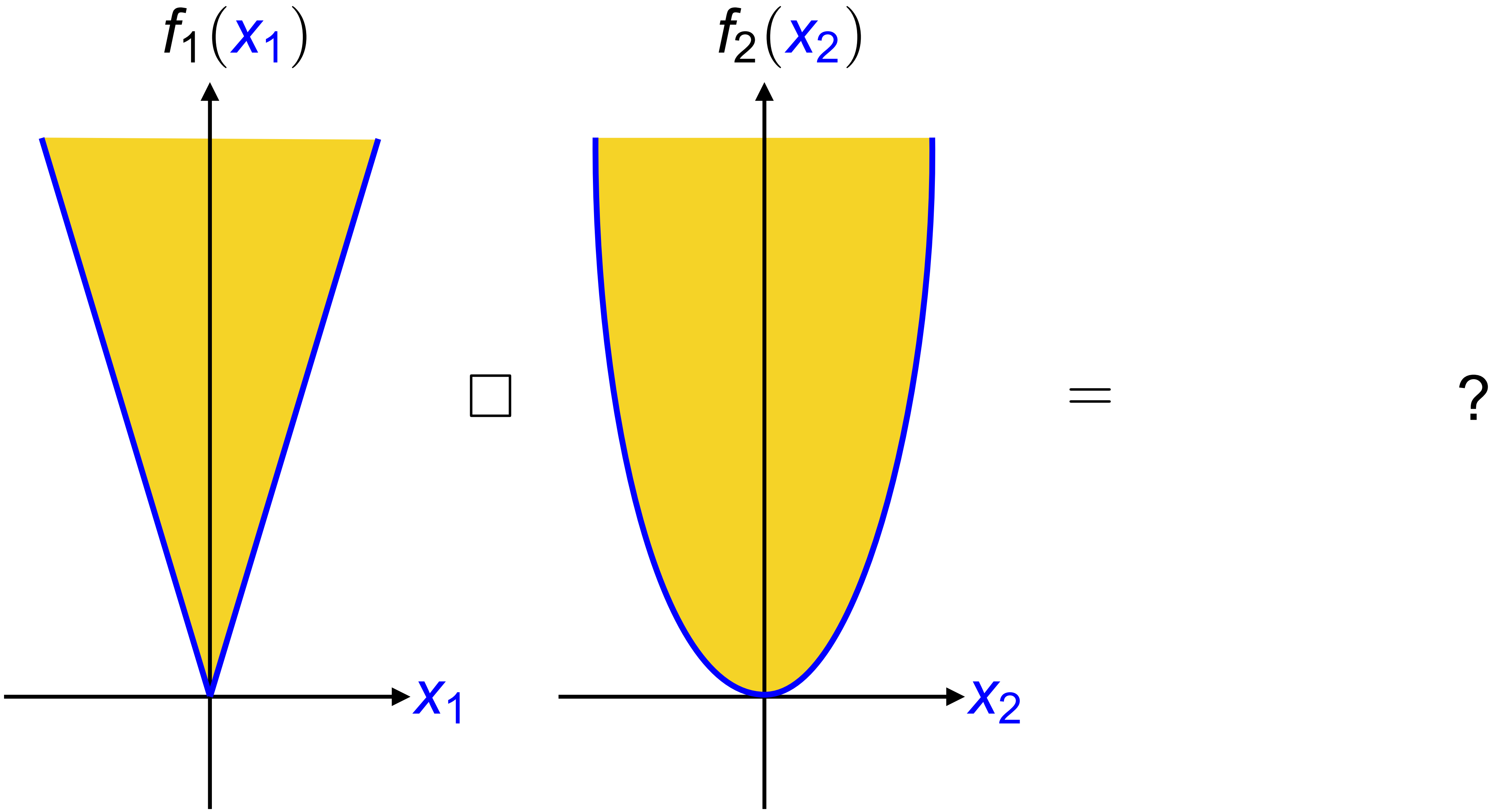

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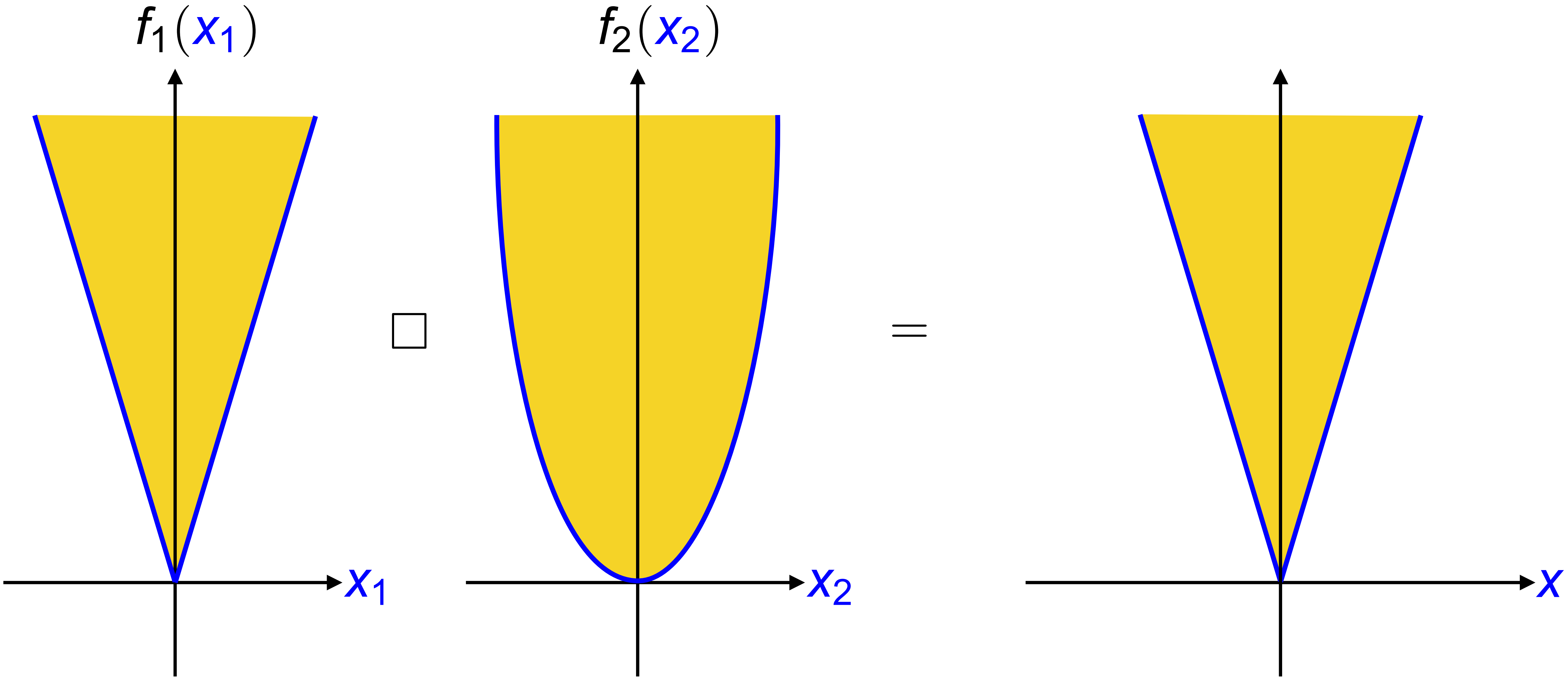

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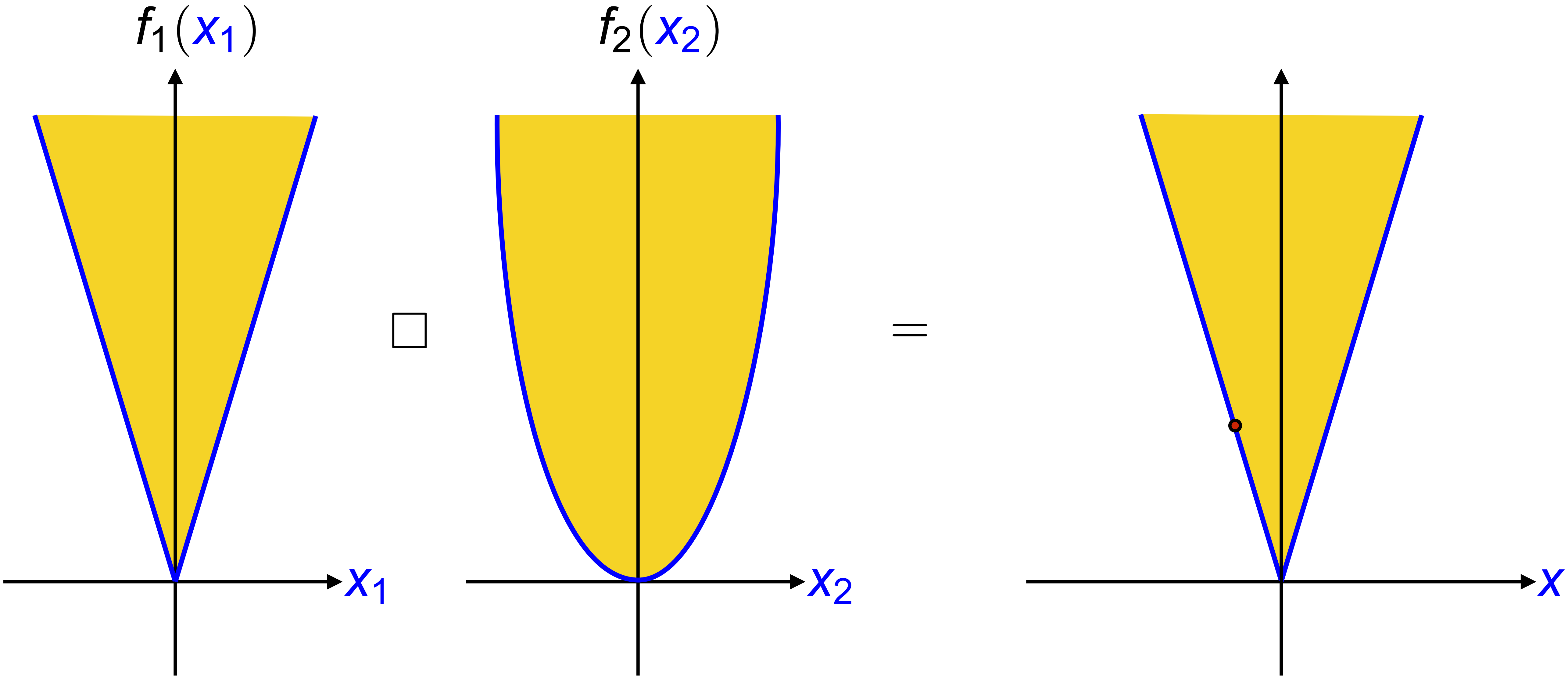

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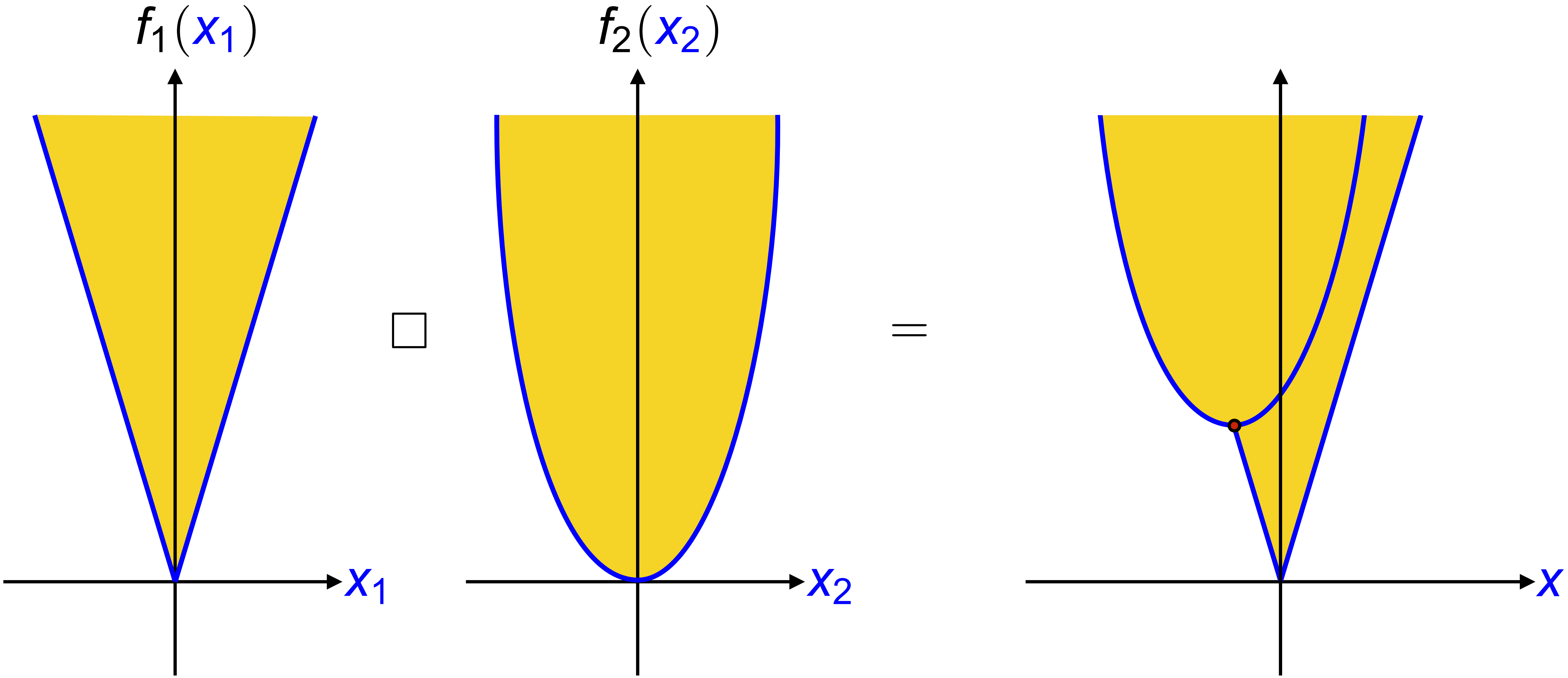

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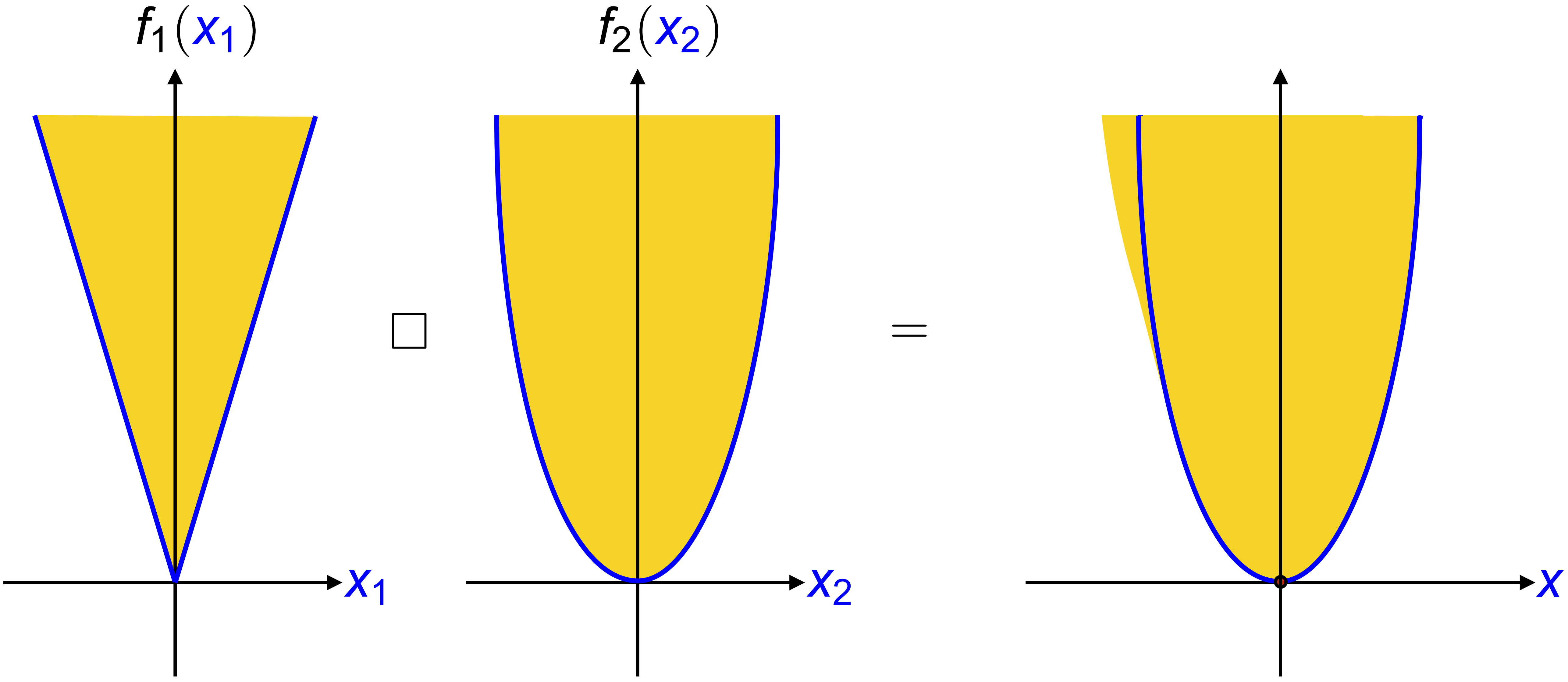

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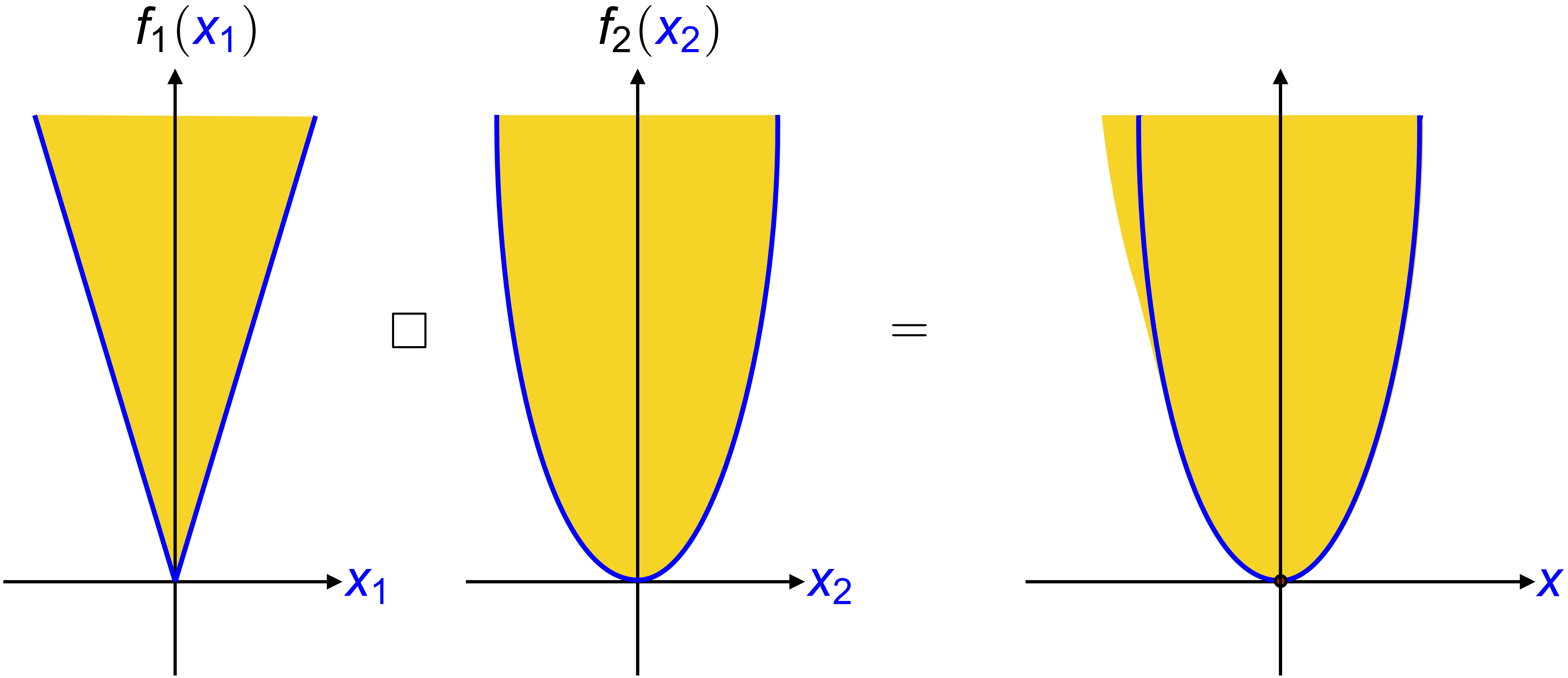

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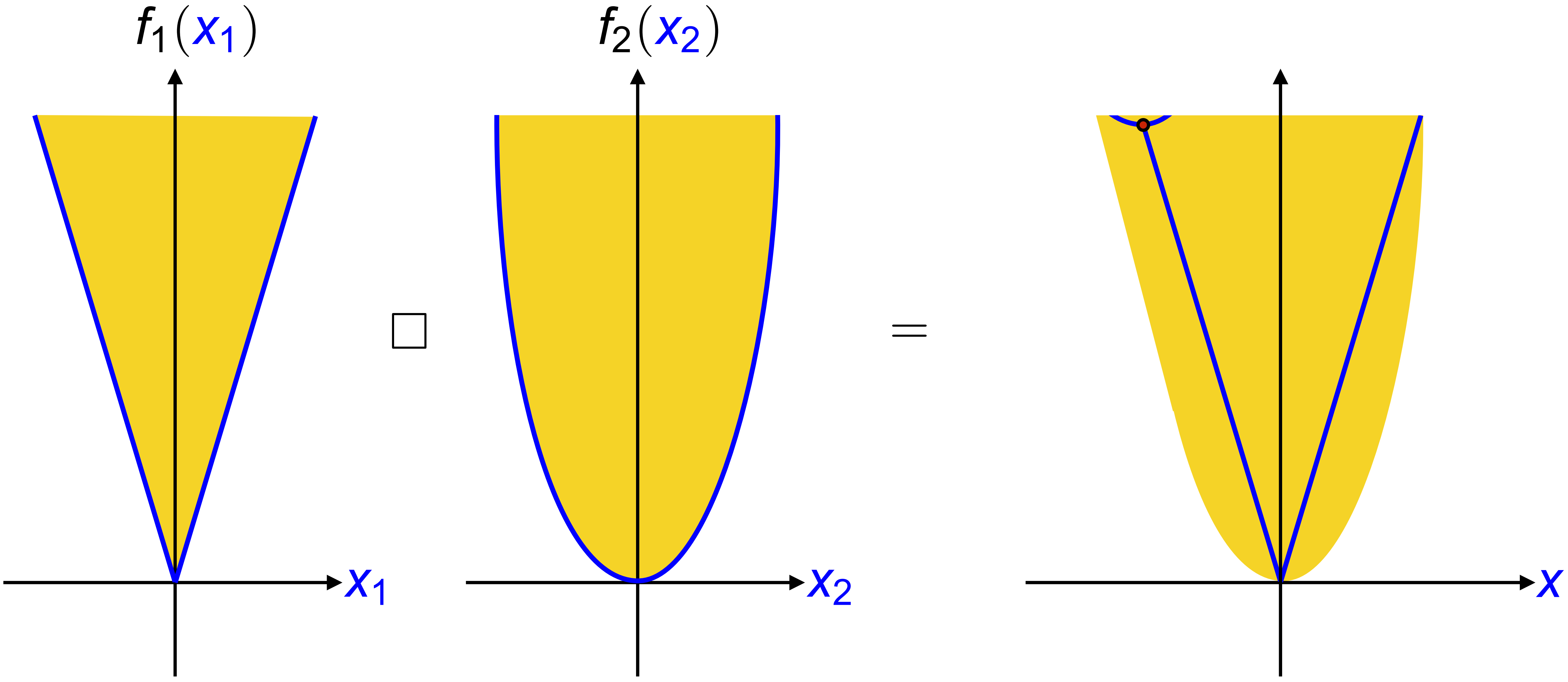

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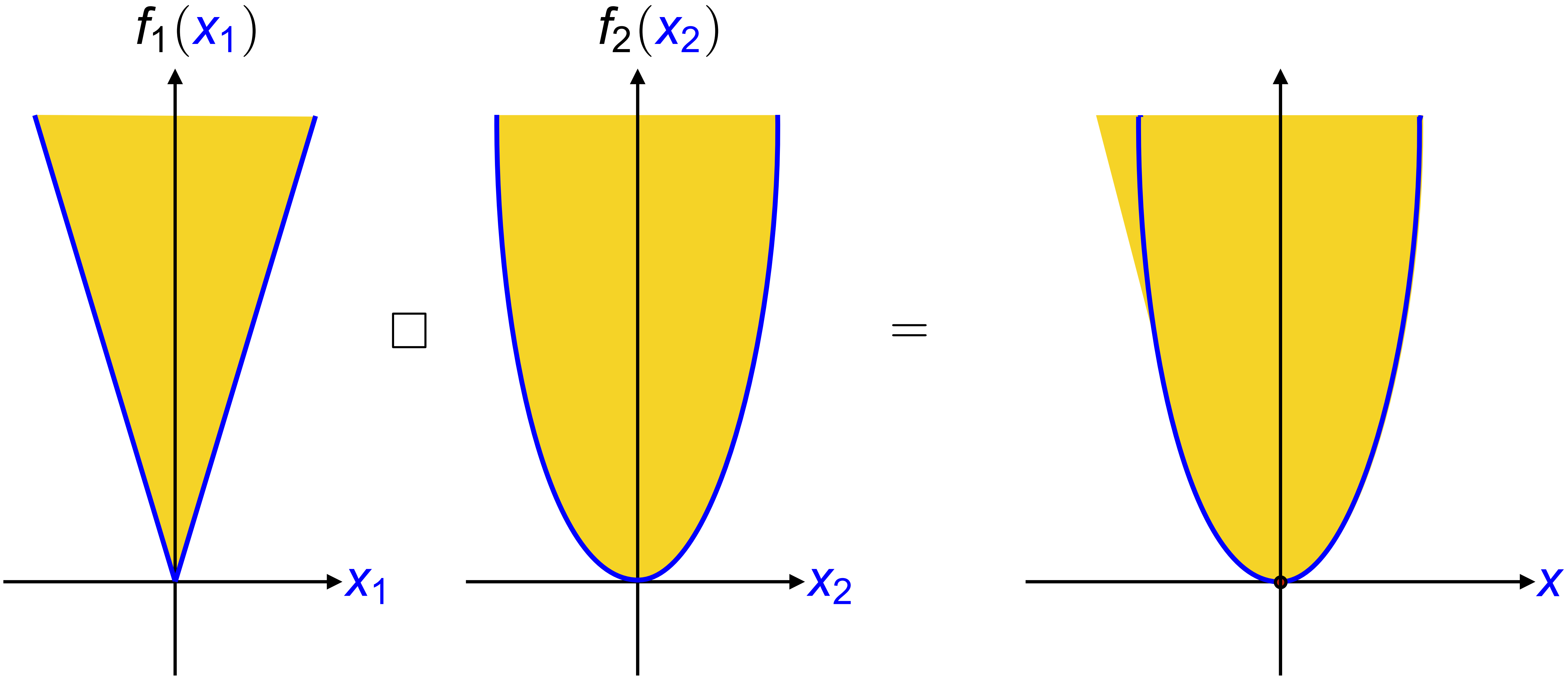

Infimal Convolution

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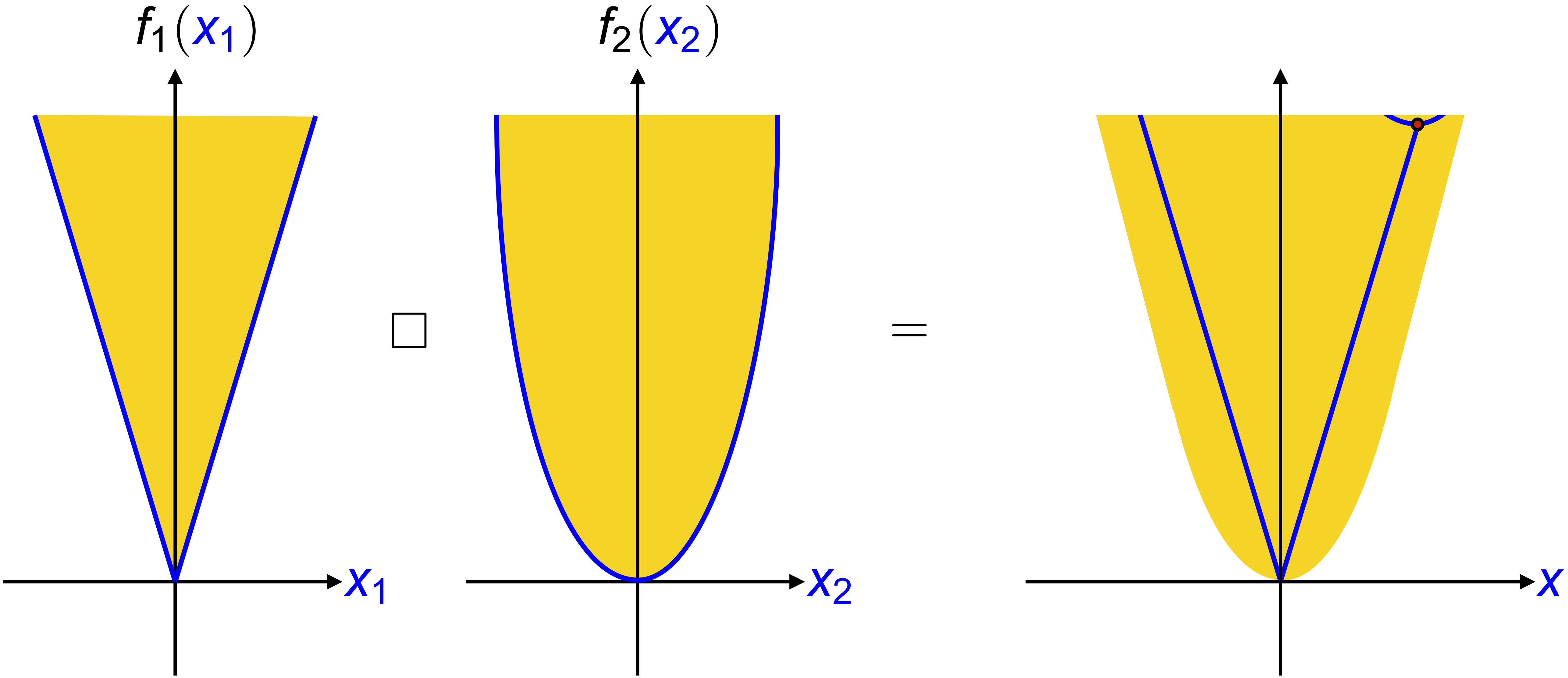
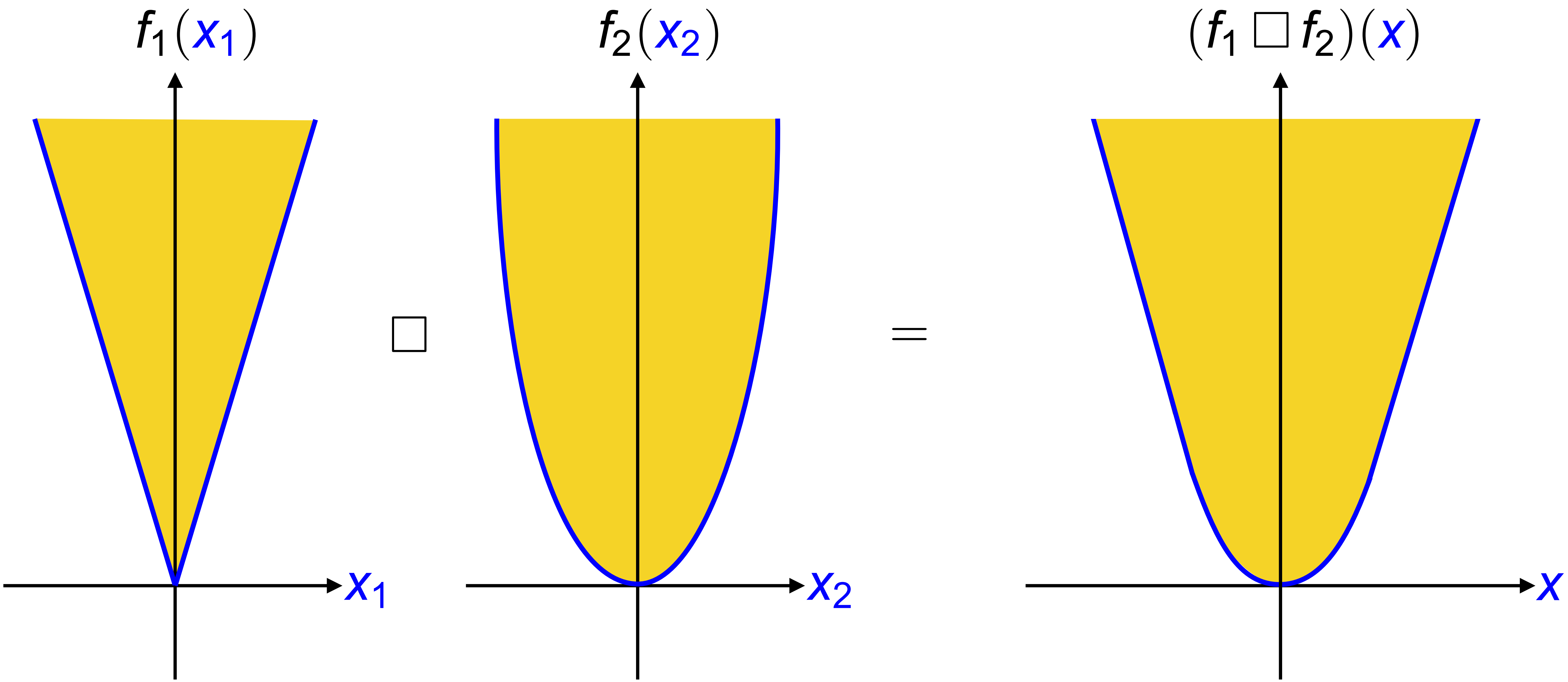


Infimal Convolution

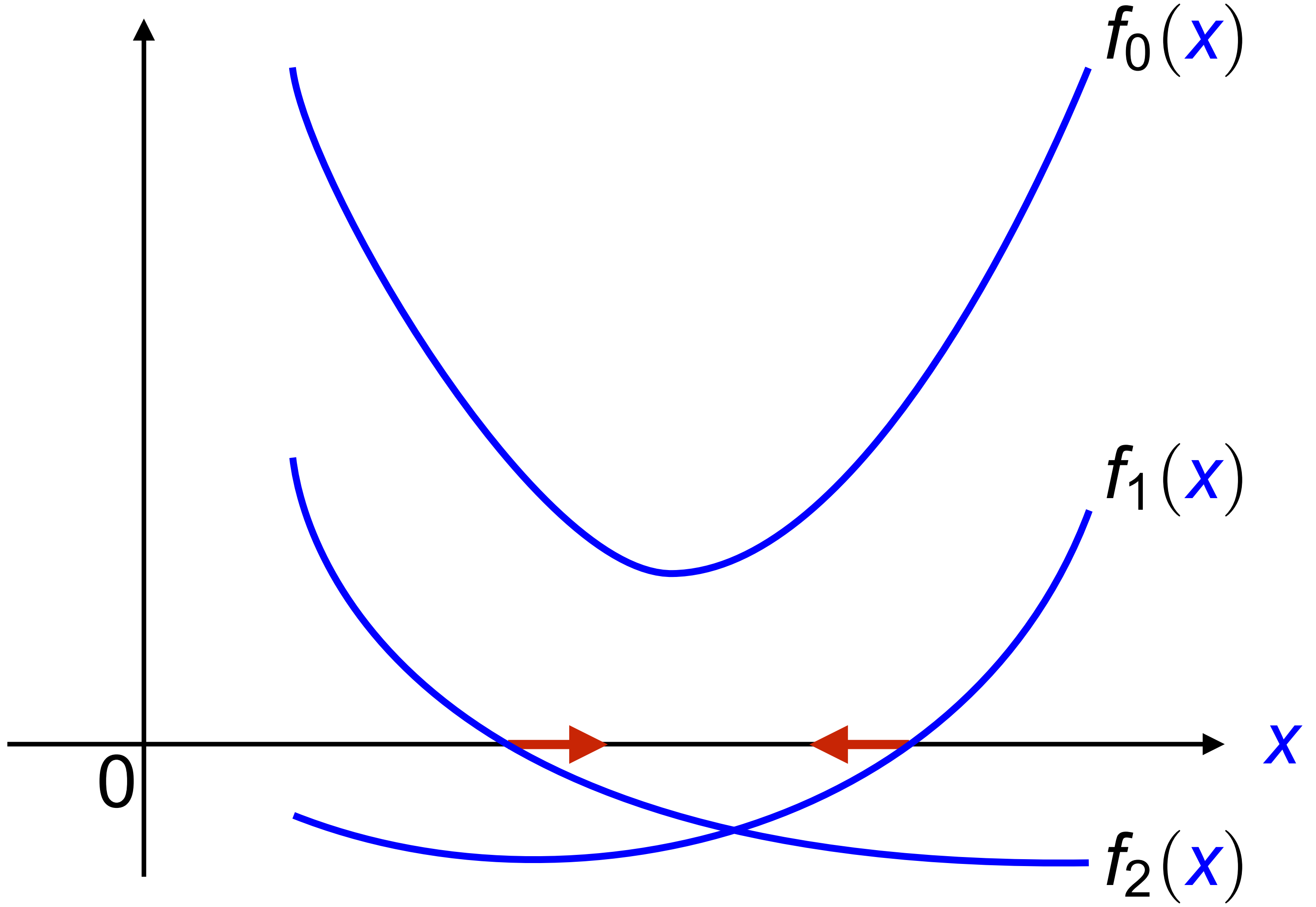
$$f_1(x_1),\, f_2(x_2) \quad \rightsquigarrow \quad (f_1 \,\square\, f_2)(x) = \inf_{x_1 + x_2 = x} f_1(x_1) + f_2(x_2)$$



Convex Optimization

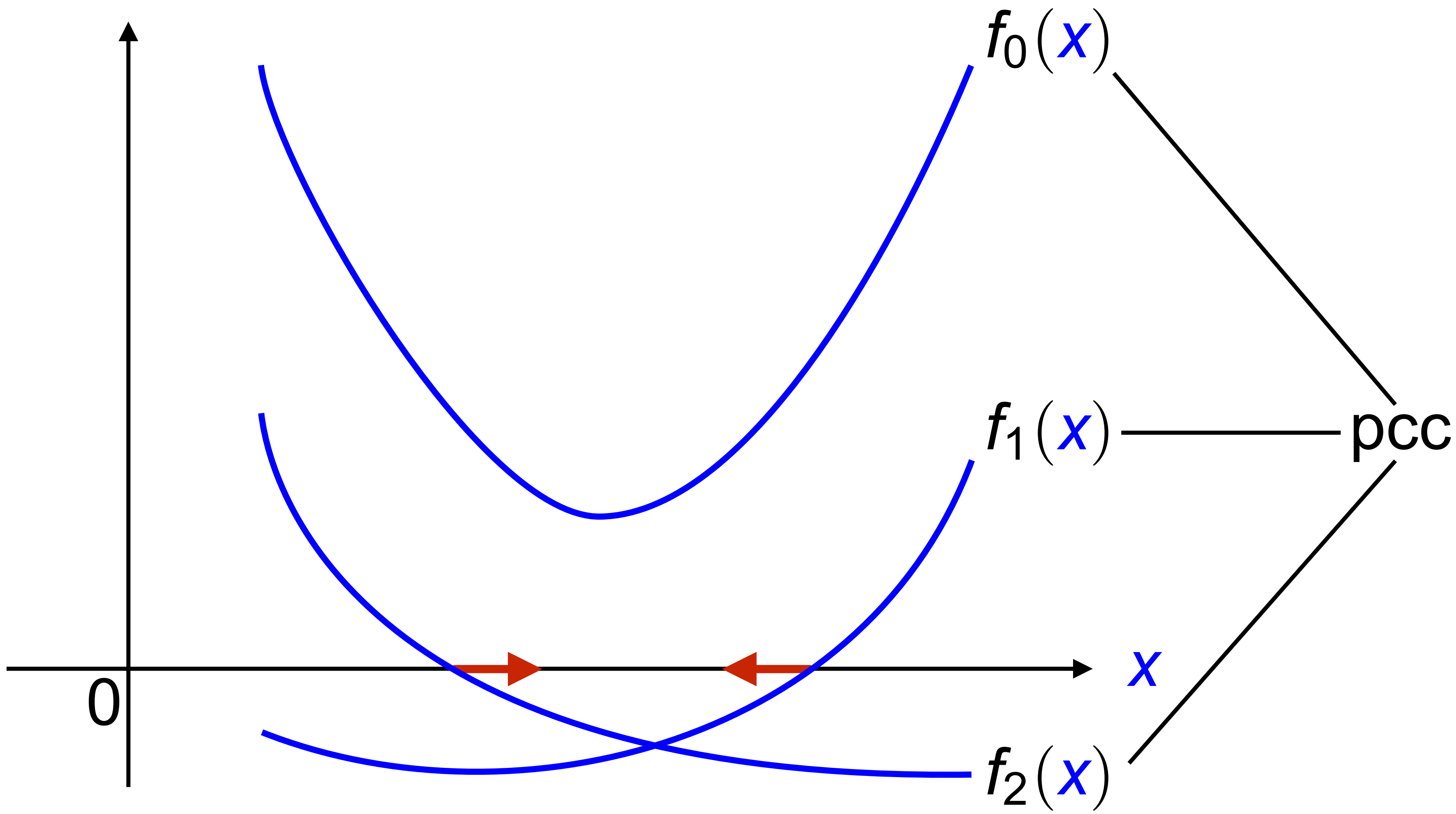
Primal Problem

$$\inf_{x} \quad f_0(x)$$
$$\text{s.t.} \quad f_i(x) \leq 0 \quad \forall i = 1, \ldots, l \qquad \text{(P)}$$



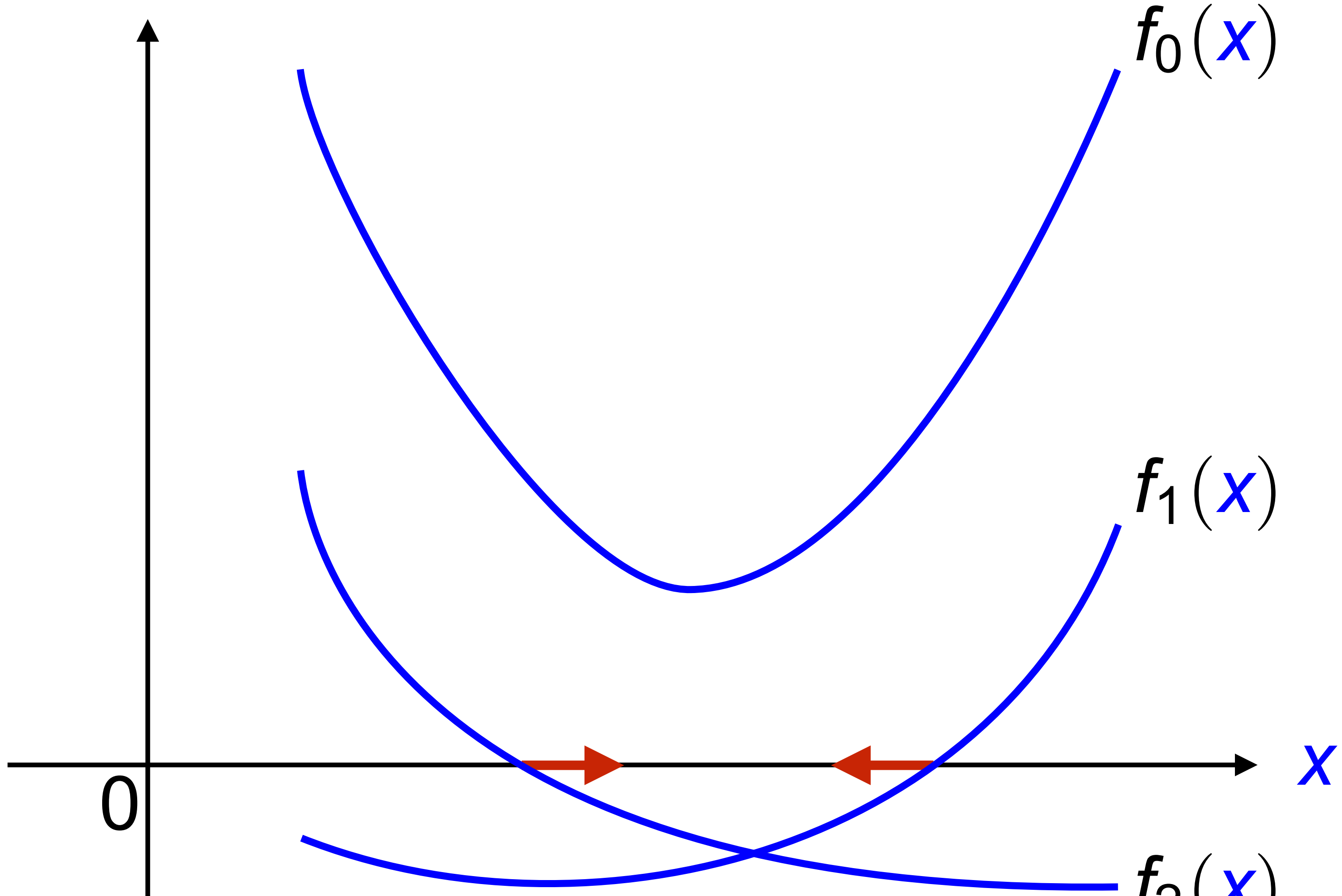
Primal Problem

$$\inf_{x} \quad f_0(x)$$
$$\text{s.t.} \quad f_i(x) \leq 0 \quad \forall i = 1, \ldots, l \qquad \text{(P)}$$



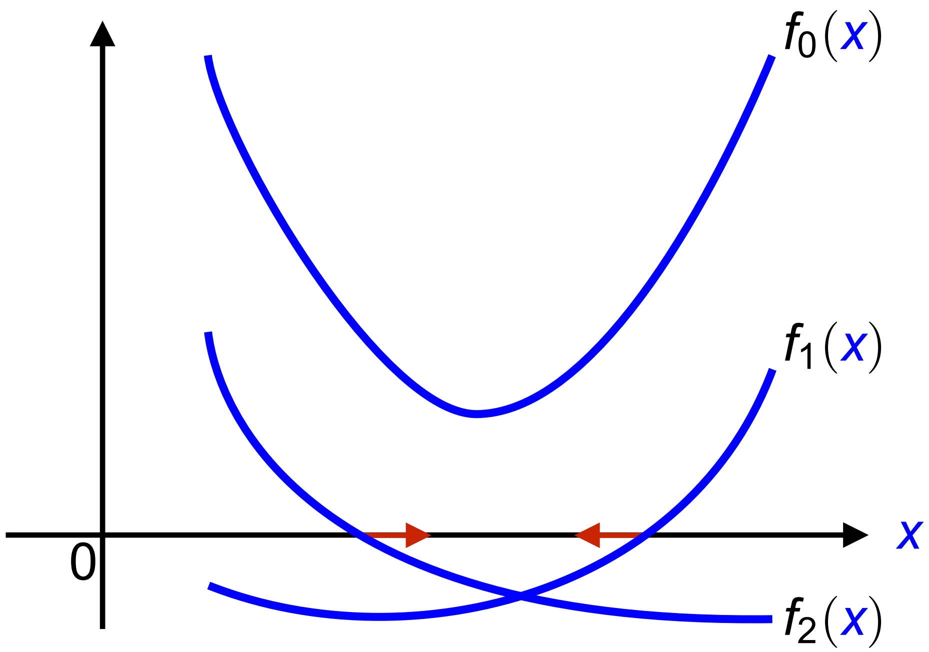
Primal Problem

$$\inf_{x} \quad f_0(x)$$
$$\text{s.t.} \quad f_i(x) \leq 0 \quad \forall i = 1, \ldots, l \qquad \text{(P)}$$



Primal Problem

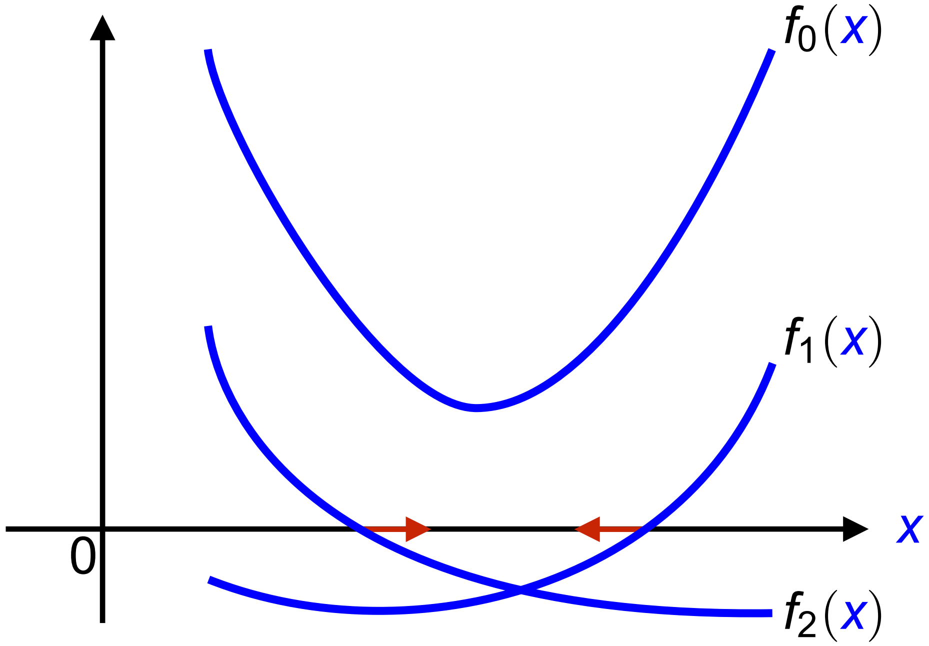
$$\inf_{x} \quad f_0(x)$$
$$\text{s.t.} \quad f_i(x) \leq 0 \quad \forall i = 1, \ldots, l \quad \longleftarrow \lambda_i$$



Joseph-Luis Lagrange

Primal Problem

$$\inf_{x} \sup_{\lambda \geq 0} f_0(x) + \sum_{i=1}^{l} \lambda_i f_i(x)$$

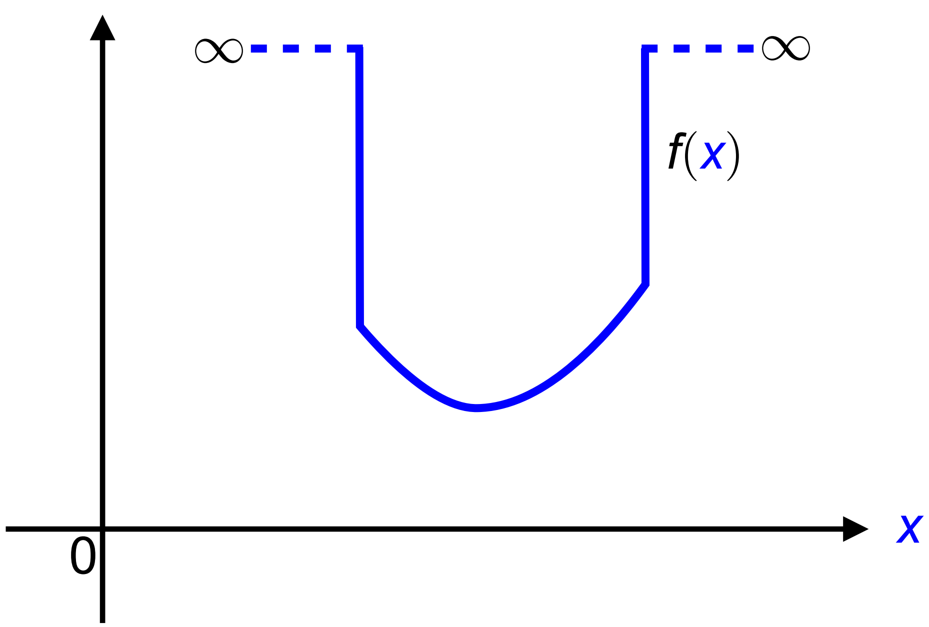


Joseph-Luis Lagrange

Primal Problem

$$\inf_{x} \sup_{\lambda \geq 0} f_0(x) + \sum_{i=1}^{l} \lambda_i f_i(x)$$

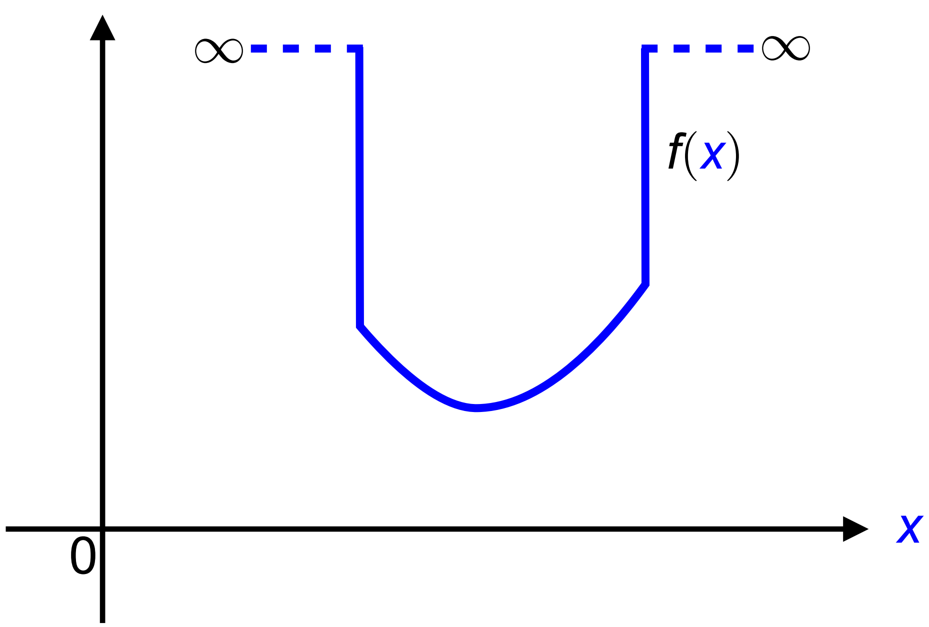
$$= f(x)$$



Joseph-Luis Lagrange

Dual Problem

$$\sup_{\lambda \geq 0} \ \inf_{x} \ f_0(x) + \sum_{i=1}^{l} \lambda_i f_i(x)$$

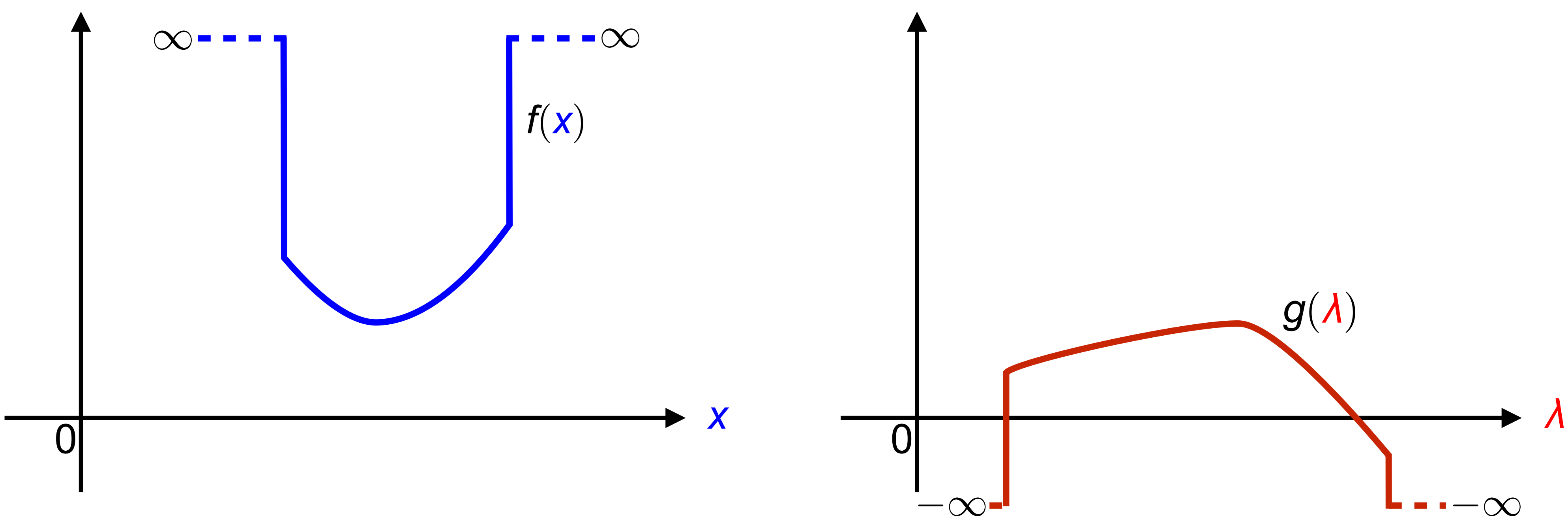


Joseph-Luis Lagrange

Dual Problem

$$\sup_{\lambda \geq 0} \; \boxed{\inf_{x} \; f_0(x) + \sum_{i=1}^{l} \lambda_i f_i(x)}$$

$$= g(\lambda)$$



Dual Problem

$$\sup_{\lambda \geq 0} \; \inf_{x} \; 0^\top x + f_0(x) + \sum_{i=1}^{l} \lambda_i f_i(x)$$

Werner Fenchel

Dual Problem

$$\sup_{\lambda \geq 0} \; -\sup_{x} \; 0^\top x - f_0(x) - \sum_{i=1}^{l} \lambda_i f_i(x)$$

Werner Fenchel

Dual Problem

$$\sup_{\lambda \geq 0} \; - \sup_{x} \; 0^\top x - f_0(x) - \sum_{i=1}^{l} \lambda_i f_i(x)$$

conjugate of Lagrangian

Werner Fenchel

Dual Problem

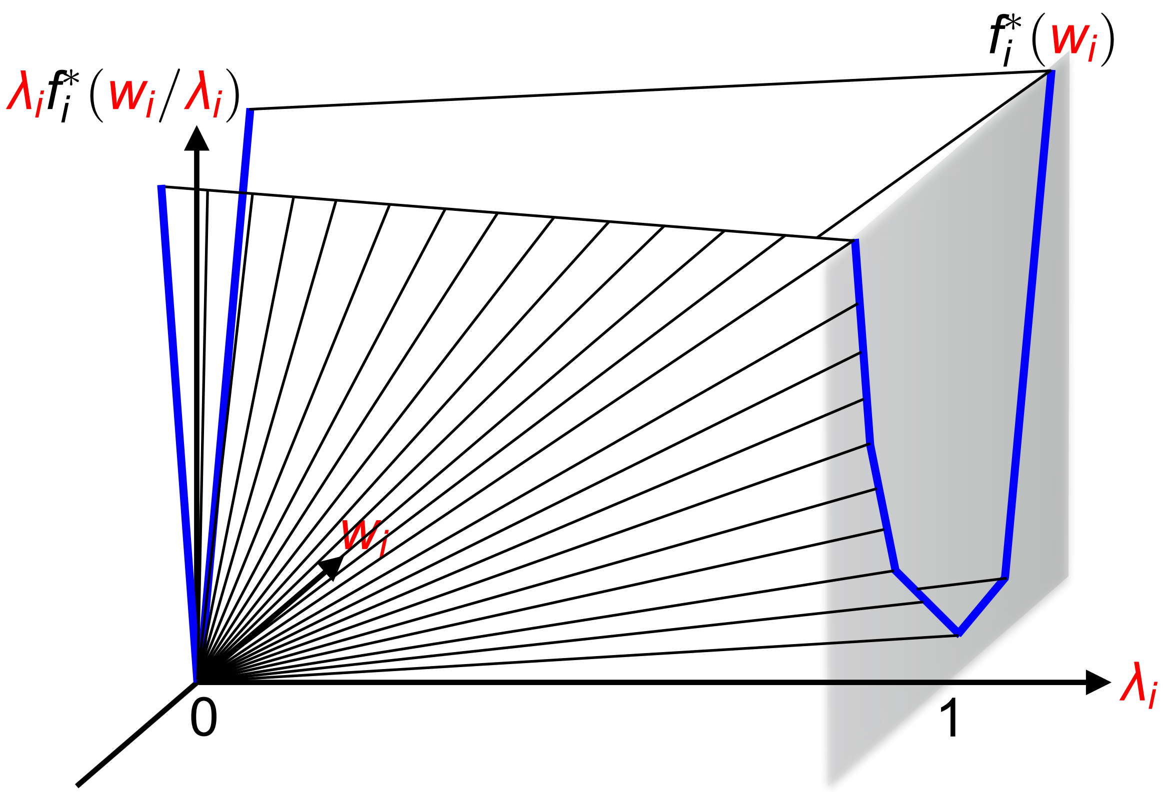
$$\sup_{\lambda \geq 0} \; - \sup_{x} \; 0^\top x - f_0(x) - \sum_{i=1}^{l} \lambda_i f_i(x)$$

$$= \text{ inf-convolution of } f_0^* \text{ and } (\lambda_i f_i)^*, \; i = 1, \ldots, l$$

Werner Fenchel

Dual Problem

$$\sup_{w,\lambda \geq 0} \quad -f_0^*(w_0) - \sum_{i=1}^{l} \lambda_i f_i^*(w_i/\lambda_i)$$
$$\text{s.t.} \quad \sum_{i=0}^{l} w_i = 0$$



Werner Fenchel

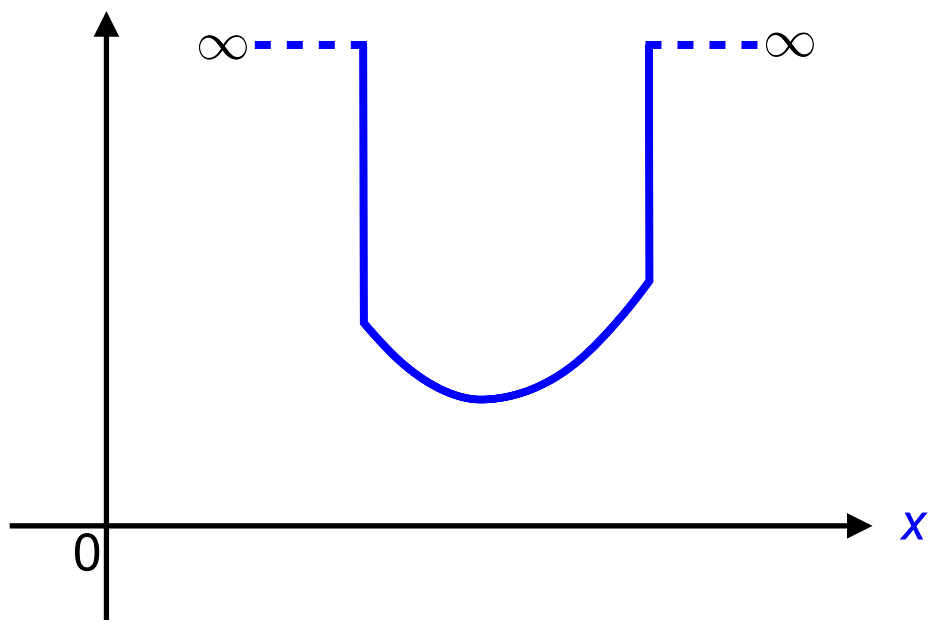
Dual Problem

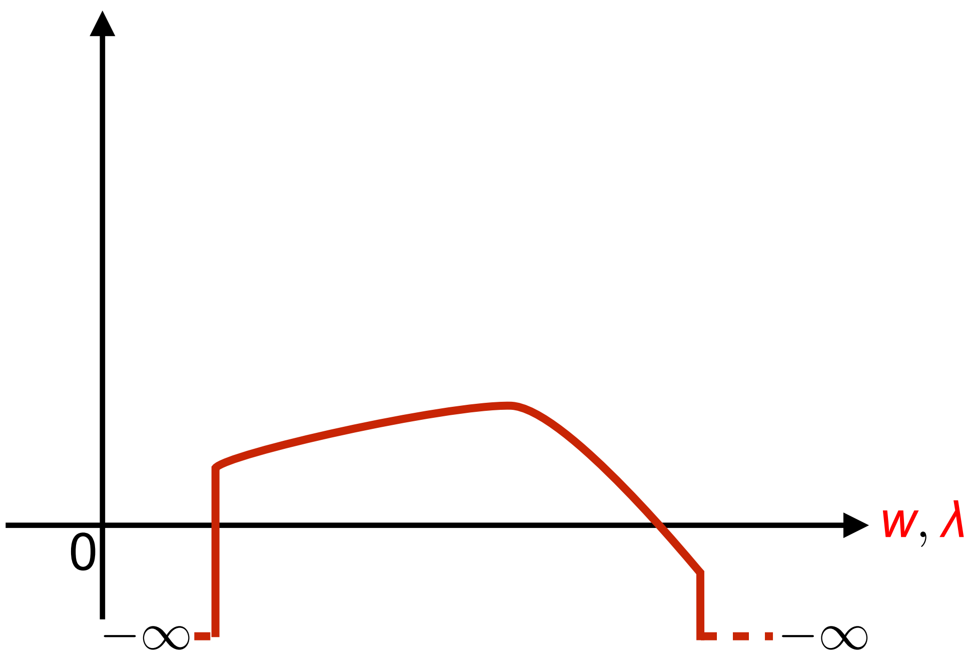
$$\sup_{w,\lambda \geq 0} \quad -f_0^*(w_0) - \sum_{i=1}^{l} \lambda_i f_i^*(w_i/\lambda_i) \qquad \text{(D)}$$

$$\text{s.t.} \quad \sum_{i=0}^{l} w_i = 0$$

Strong Duality

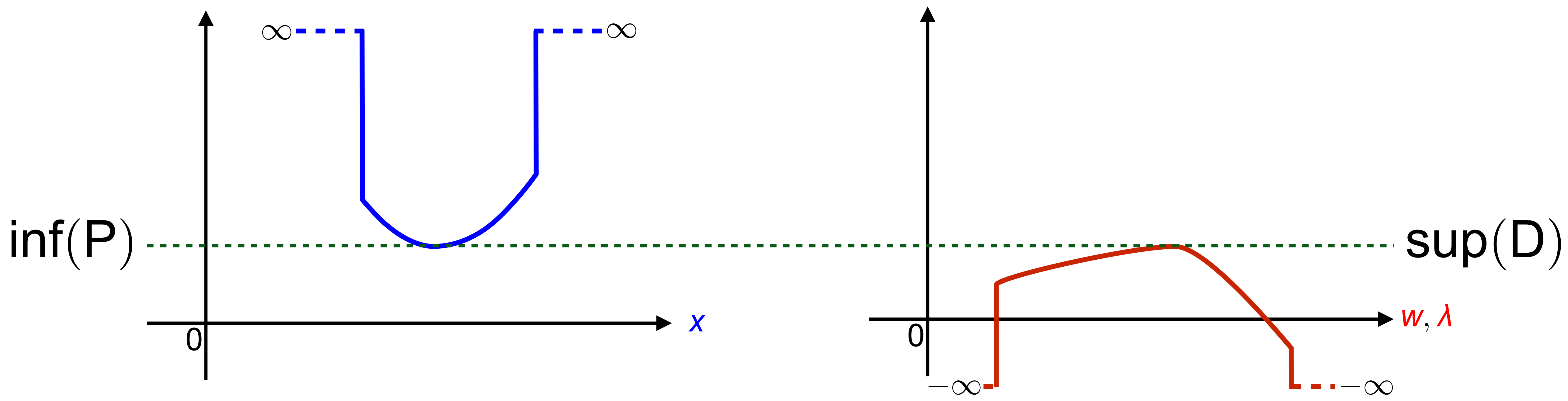
$$\sup_{w,\lambda \geq 0} \quad -f_0^*(w_0) - \sum_{i=1}^{l} \lambda_i f_i^*(w_i/\lambda_i) \qquad \text{(D)}$$
$$\text{s.t.} \quad \sum_{i=0}^{l} w_i = 0$$





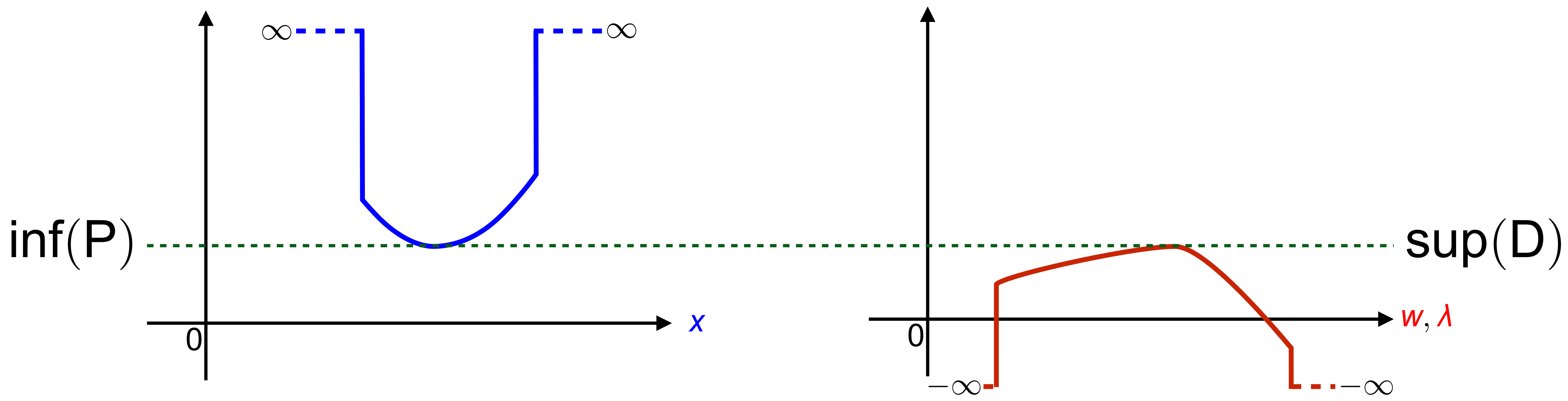
Strong Duality

$$\begin{aligned} \sup_{w,\lambda\geq 0} \quad & -f_0^*\left(w_0\right) - \sum_{i=1}^{l} \lambda_i f_i^*\left(w_i/\lambda_i\right) \\ \text{s.t.} \quad & \sum_{i=0}^{l} w_i = 0 \end{aligned} \qquad \text{(D)}$$



Strong Duality

$$\begin{array}{lll} \sup\limits_{w,\lambda\geq 0} & -f_0^*(w_0) - \sum_{i=1}^{l} \lambda_i f_i^*(w_i/\lambda_i) & \text{(D)} \\ \text{s.t.} & \sum_{i=0}^{l} w_i = 0 & \end{array}$$



if feasible set of (P) *or* (D) is bounded *or* has Slater point

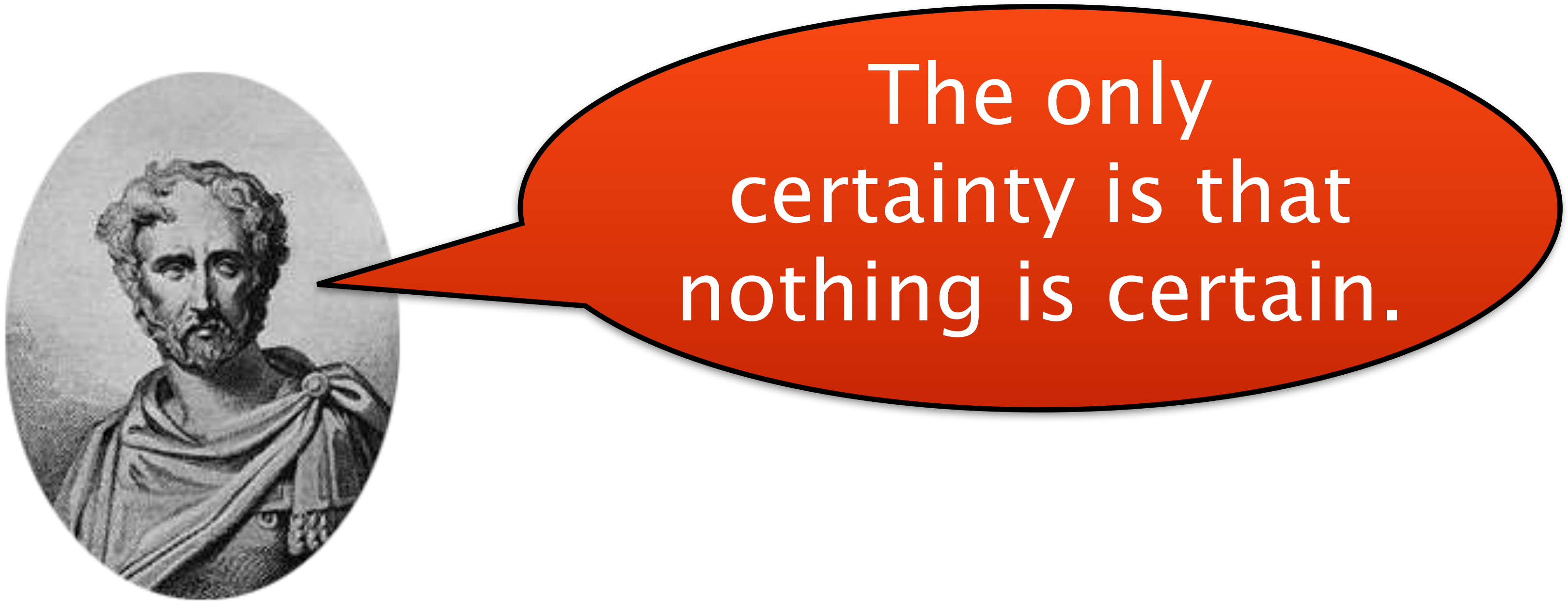
Robust Convex Optimization

Optimization under Uncertainty

Pliny the Elder

The only certainty is that nothing is certain.

Optimization under Uncertainty



Pliny the Elder

Real decision problems depend on uncertain parameters z.

Scenario Problems

Fix any uncertainty realization $z = (z_0, \ldots, z_l)$.

Scenario Problems

Fix any uncertainty realization $z = (z_0, \ldots, z_I)$.

$$\begin{aligned} \inf_{x} \quad & f_0(x, z_0) \\ \text{s.t.} \quad & f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I \end{aligned} \qquad (\mathrm{P}(z))$$

Scenario Problems

Fix any uncertainty realization $z = (z_0, \ldots, z_I)$.

$$\begin{array}{lll} \inf\limits_{x} & f_0(x, z_0) & \\ \text{s.t.} & f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I \end{array} \qquad (\text{P}(z))$$

pcc in x

Scenario Problems

Fix any uncertainty realization $z = (z_0, \ldots, z_I)$.

$$\begin{aligned} \inf_{x} \quad & f_0(x, z_0) \\ \text{s.t.} \quad & f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I \end{aligned} \qquad (\mathrm{P}(z))$$

$$\begin{aligned} \sup_{w,\lambda \geq 0} \quad & -f_0^{*1}(w_0, z_0) - \sum_{i=1}^{I} \lambda_i f_i^{*1}(w_i/\lambda_i, z_i) \\ \text{s.t.} \quad & \sum_{i=0}^{I} w_i = 0 \end{aligned} \qquad (\mathrm{D}(z))$$

Scenario Problems

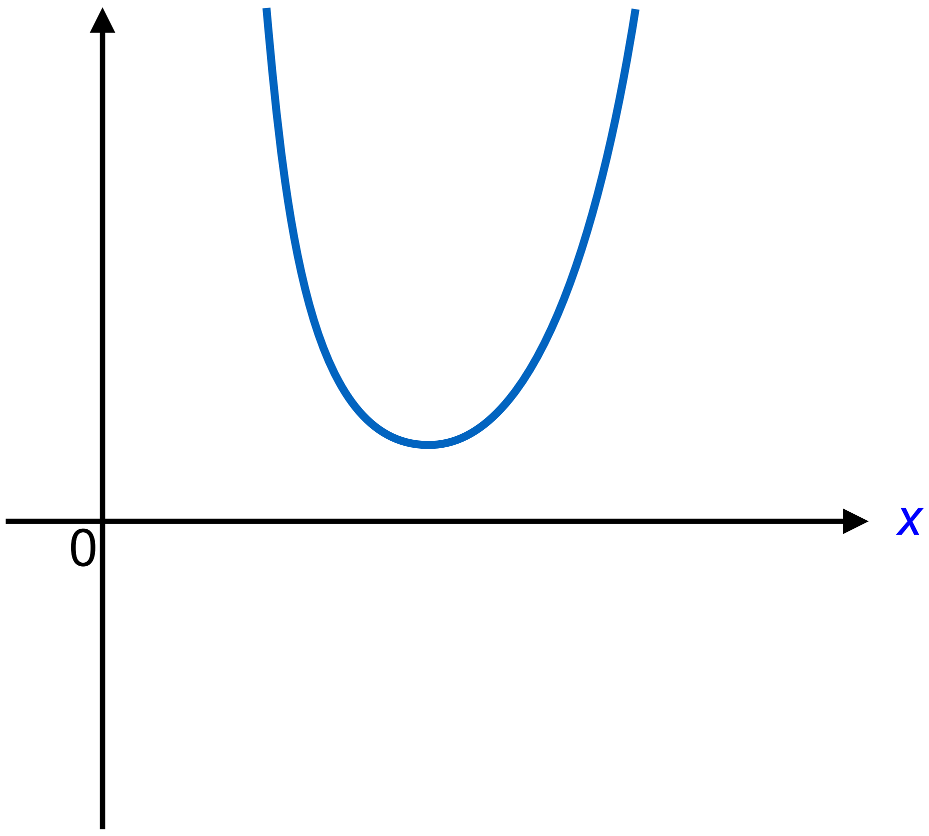
Fix any uncertainty realization $z = (z_0, \ldots, z_I)$.

$$\begin{array}{lll} \inf_{x} & f_0(x, z_0) & \\ \text{s.t.} & f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I & \end{array} \qquad (\mathrm{P}(z))$$

$$\begin{array}{ll} \sup_{w, \lambda \geq 0} & -f_0^{*1}(w_0, z_0) - \sum_{i=1}^{I} \lambda_i f_i^{*1}(w_i/\lambda_i, z_i) \\ \text{s.t.} & \sum_{i=0}^{I} w_i = 0 \end{array} \qquad (\mathrm{D}(z))$$

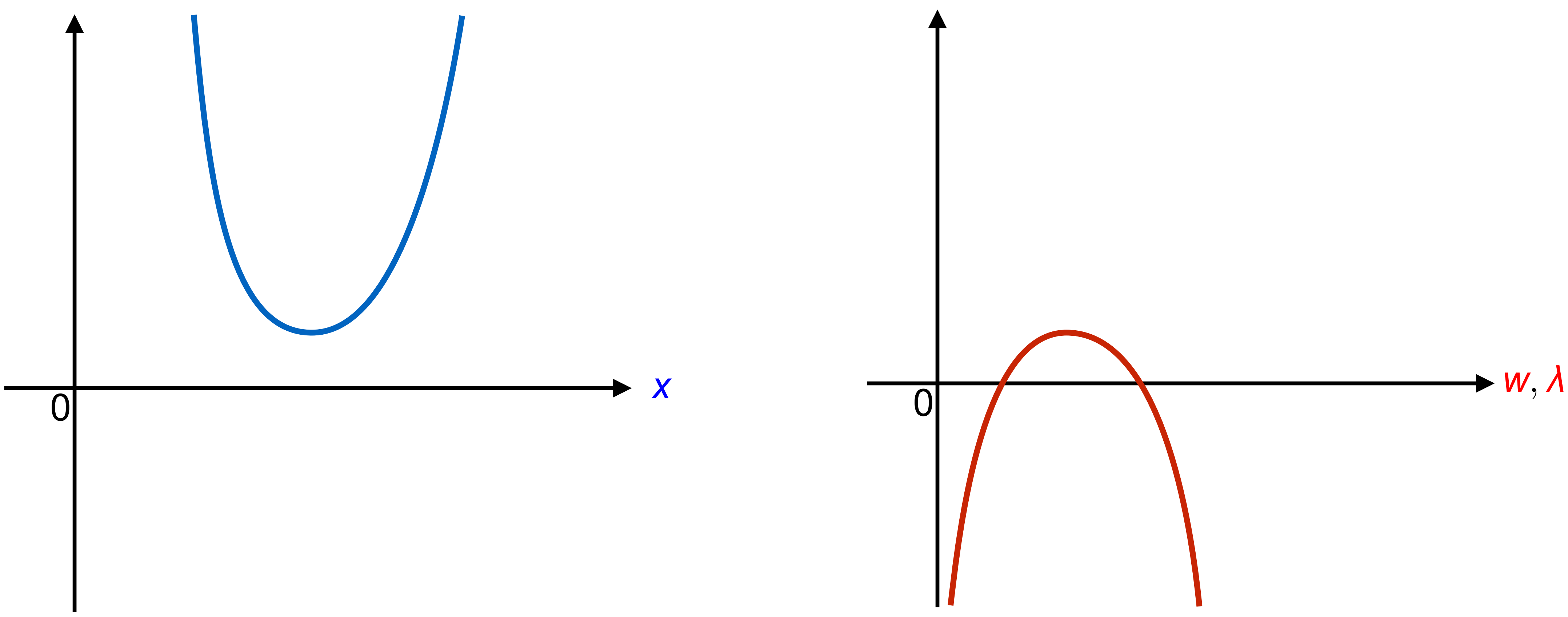
conjugates w.r.t. x

Scenario Problems



$P(z^{(1)})$

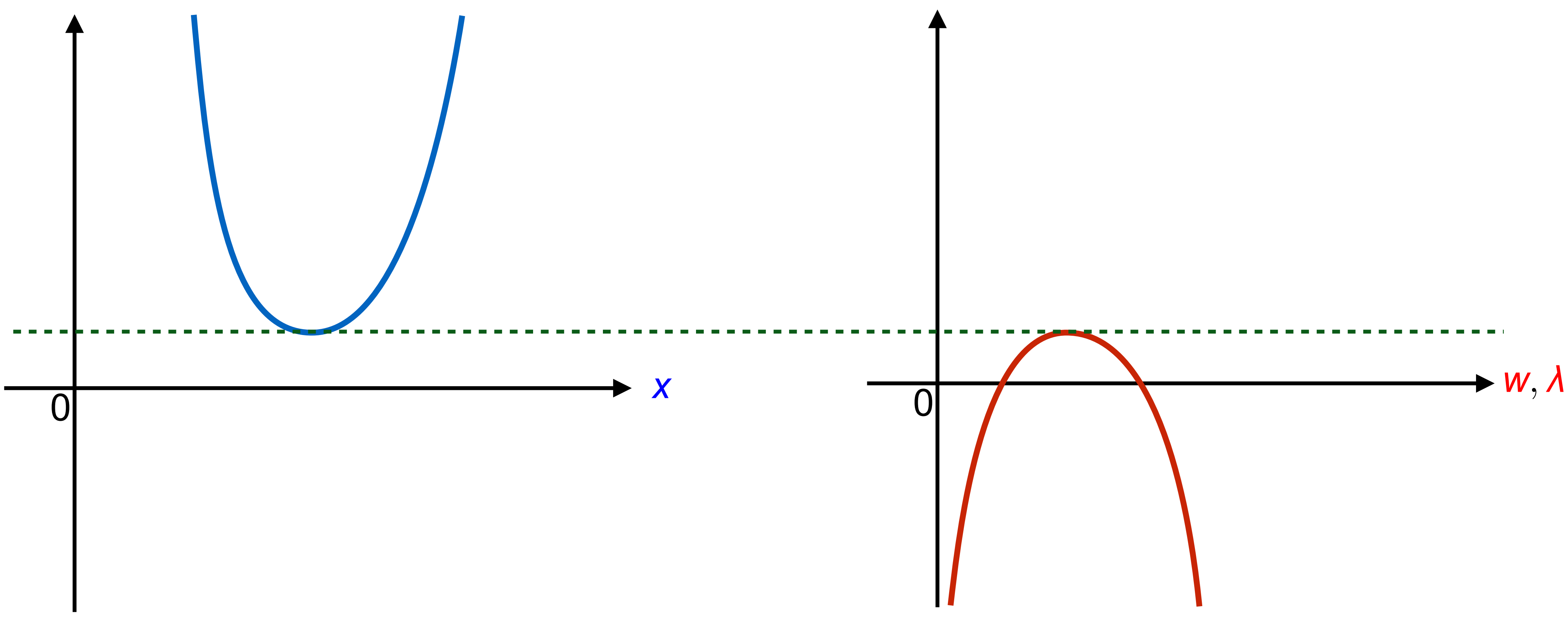
Scenario Problems



$P(z^{(1)})$ $D(z^{(1)})$

Scenario Problems

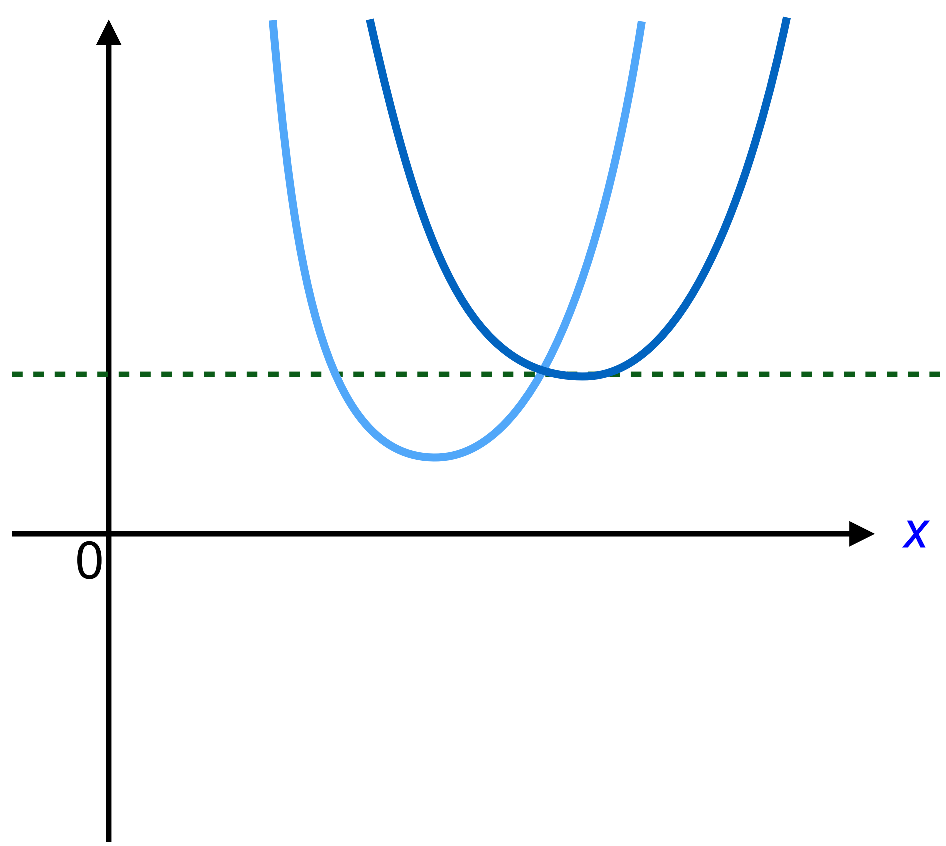
$$\inf \mathrm{P}(z^{(1)}) = \sup \mathrm{D}(z^{(1)})$$



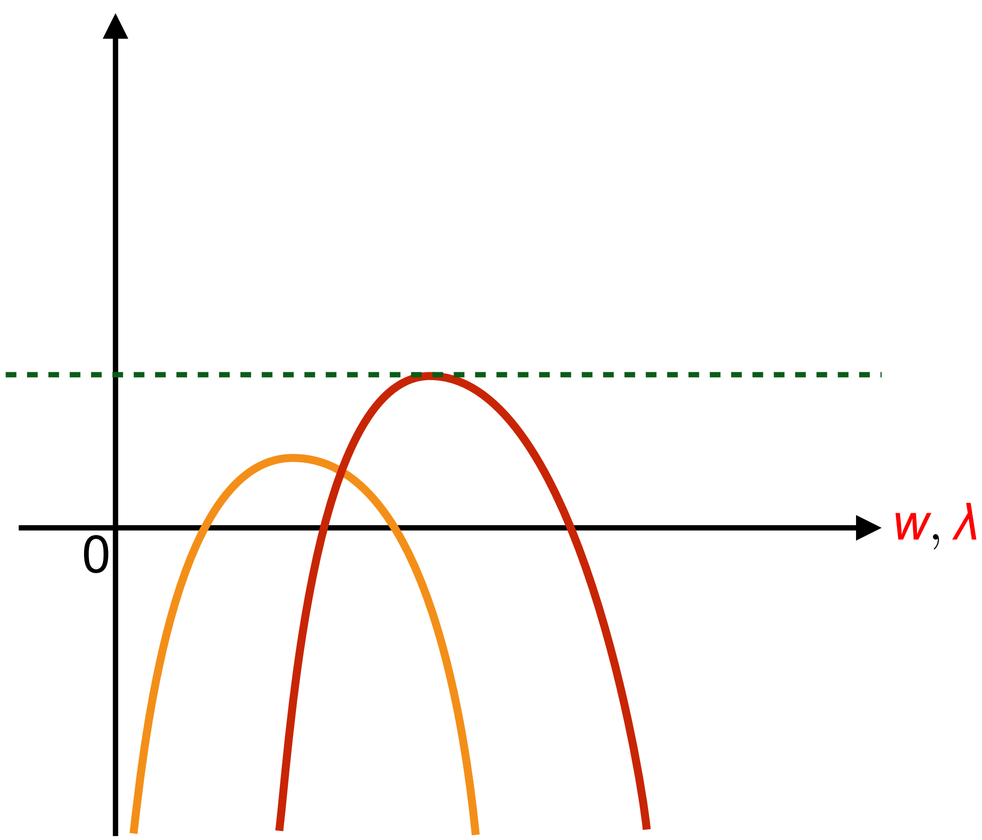
$\mathrm{P}(z^{(1)})$ $\mathrm{D}(z^{(1)})$

Scenario Problems

$$\inf \mathrm{P}(z^{(2)}) = \sup \mathrm{D}(z^{(2)})$$



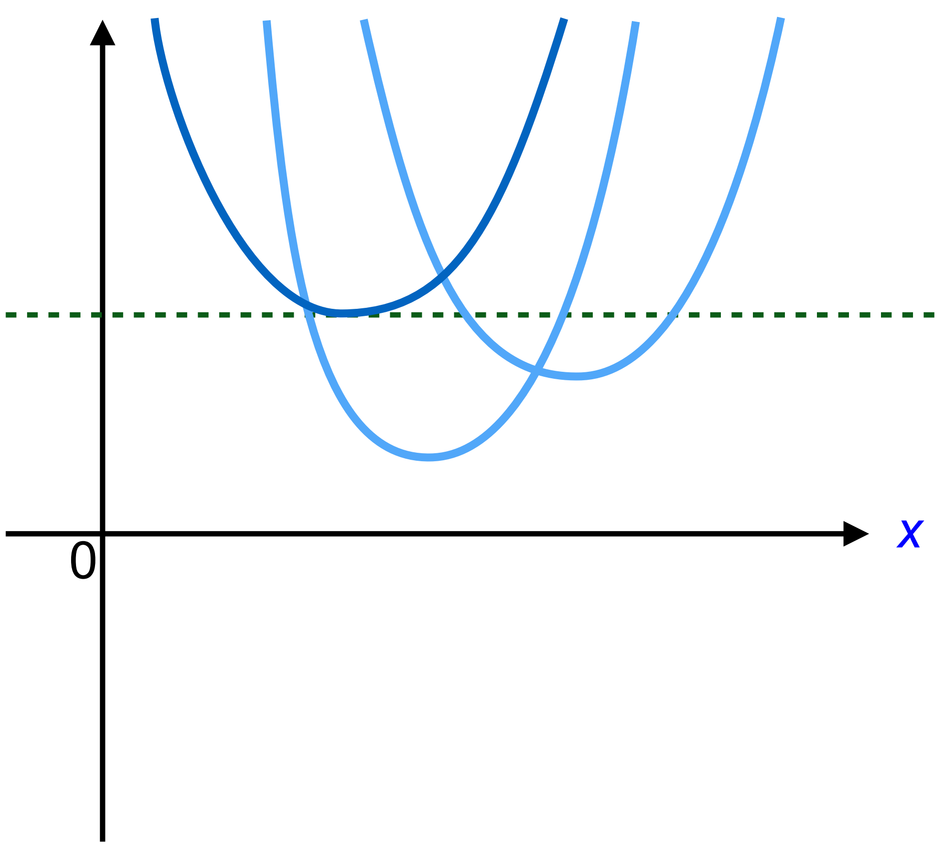
$\mathrm{P}(z^{(2)})$

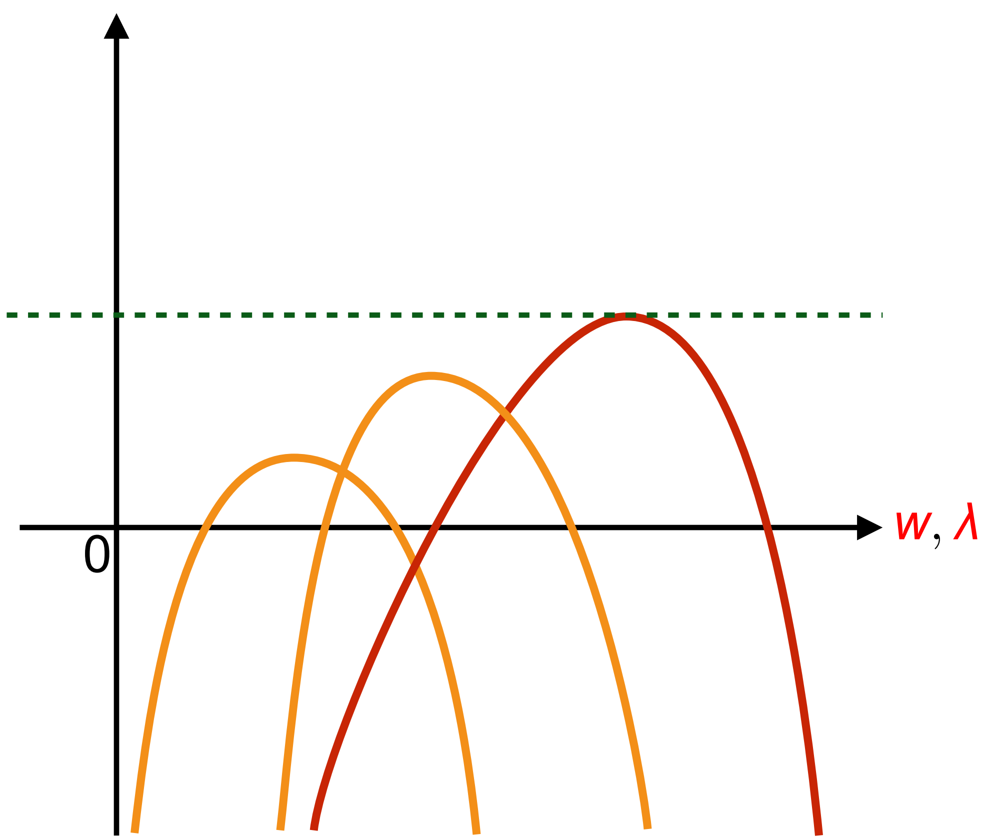


$\mathrm{D}(z^{(2)})$

Scenario Problems

$$\inf \mathrm{P}(z^{(3)}) = \sup \mathrm{D}(z^{(3)})$$



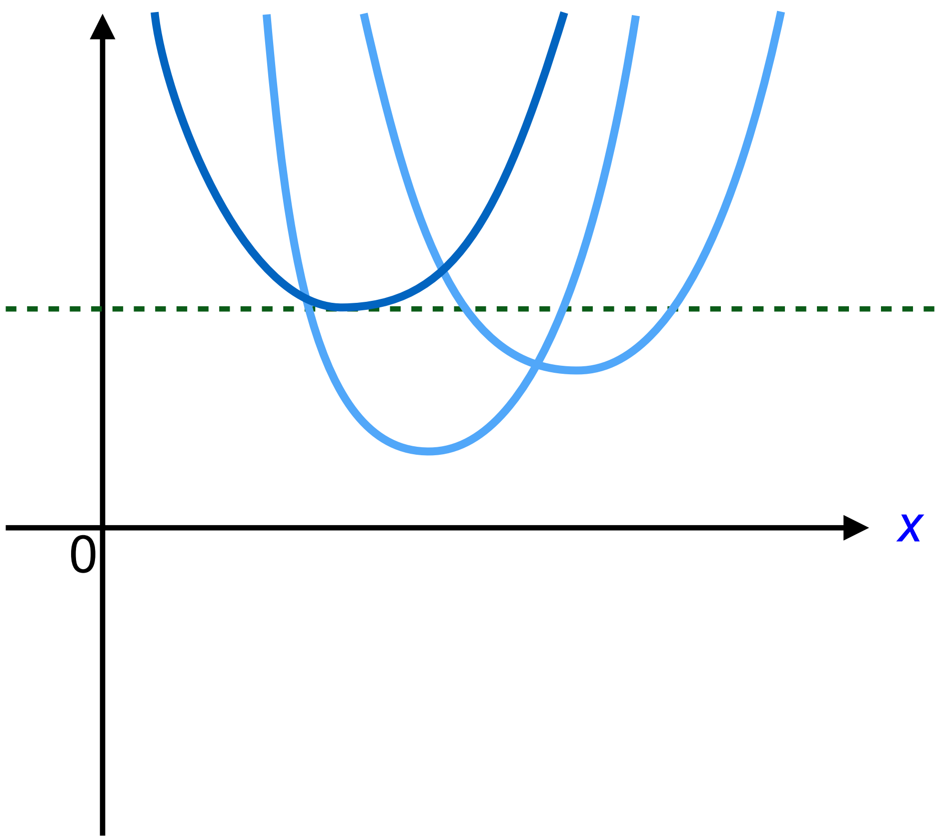


$\mathrm{P}(z^{(3)})$

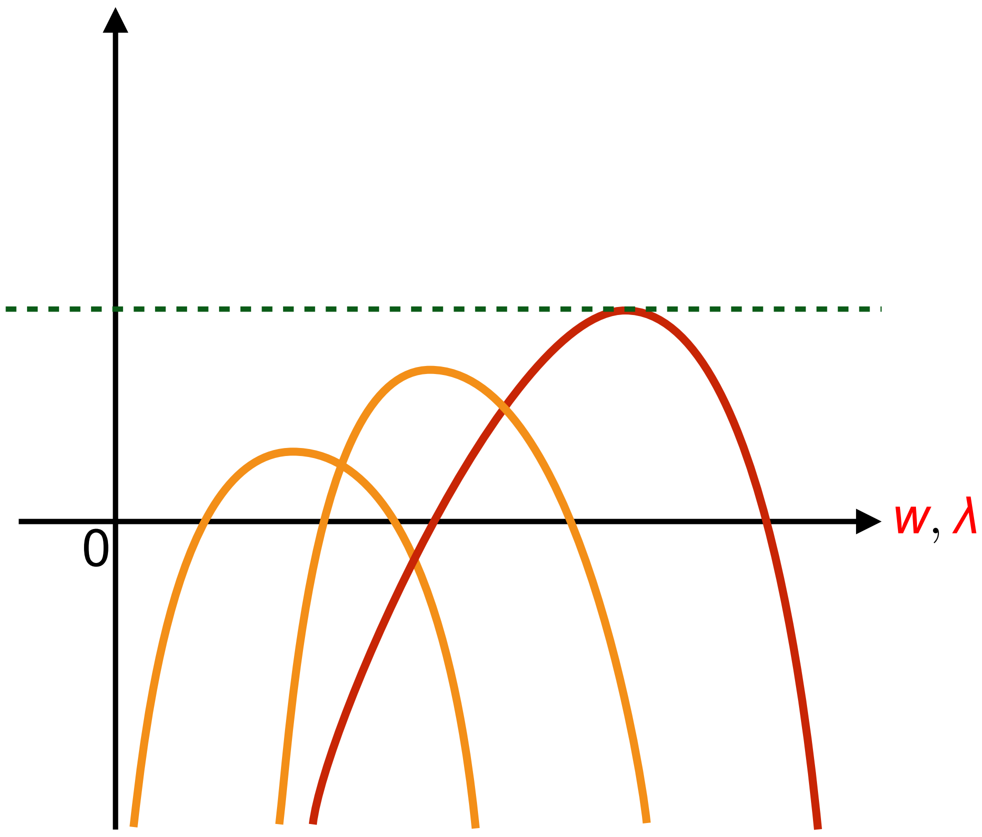
$\mathrm{D}(z^{(3)})$

Duality of Robust Optimization

$$\sup_{z \in \mathcal{Z}} \inf \mathrm{P}(z) = \sup_{z \in \mathcal{Z}} \sup \mathrm{D}(z)$$



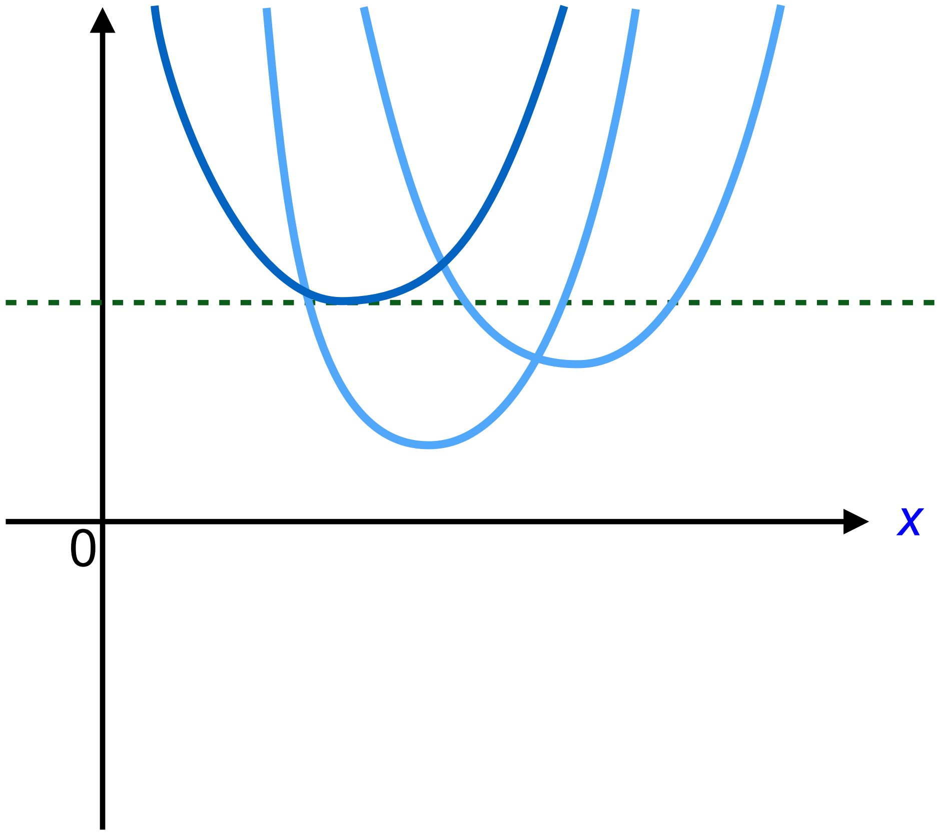
P(z)



D(z)

Primal Worst

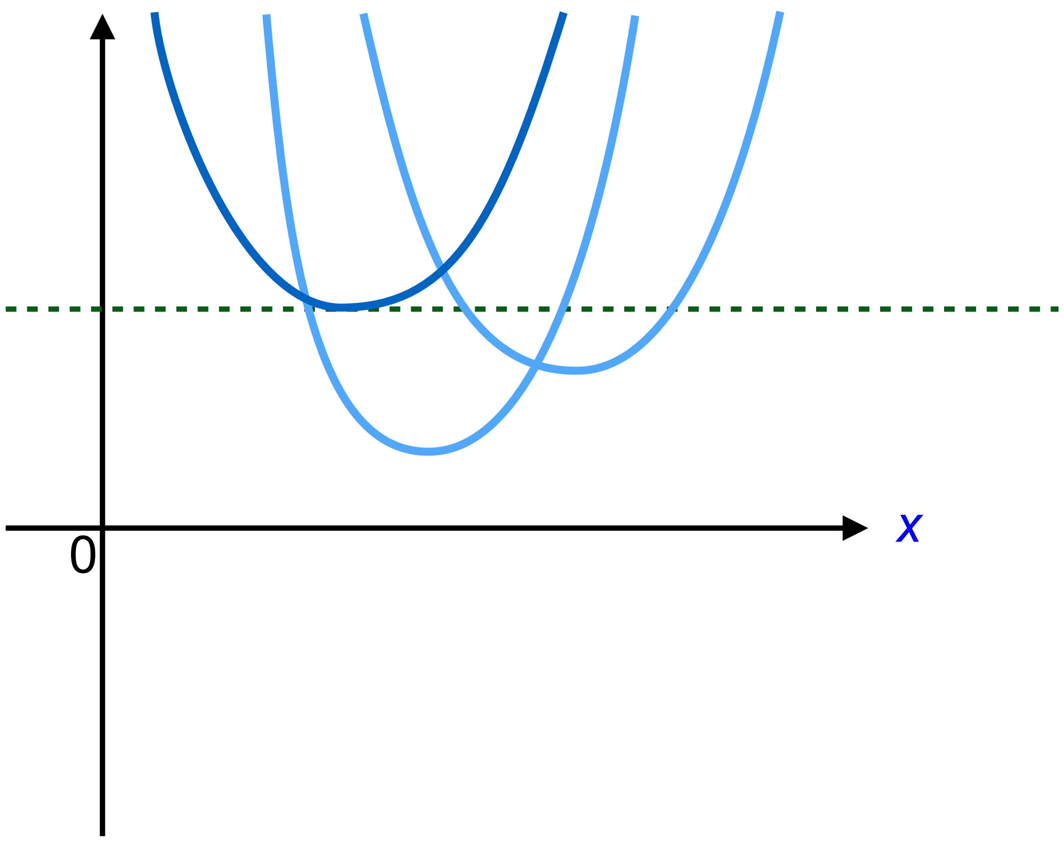
$$\sup_{z \in \mathcal{Z}} \inf \mathrm{P}(z) = \sup_{z \in \mathcal{Z}} \begin{cases} \inf_{x} & f_0(x, z_0) \\ \text{s.t.} & f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, l \end{cases}$$



$\mathrm{P}(z)$

Primal Worst

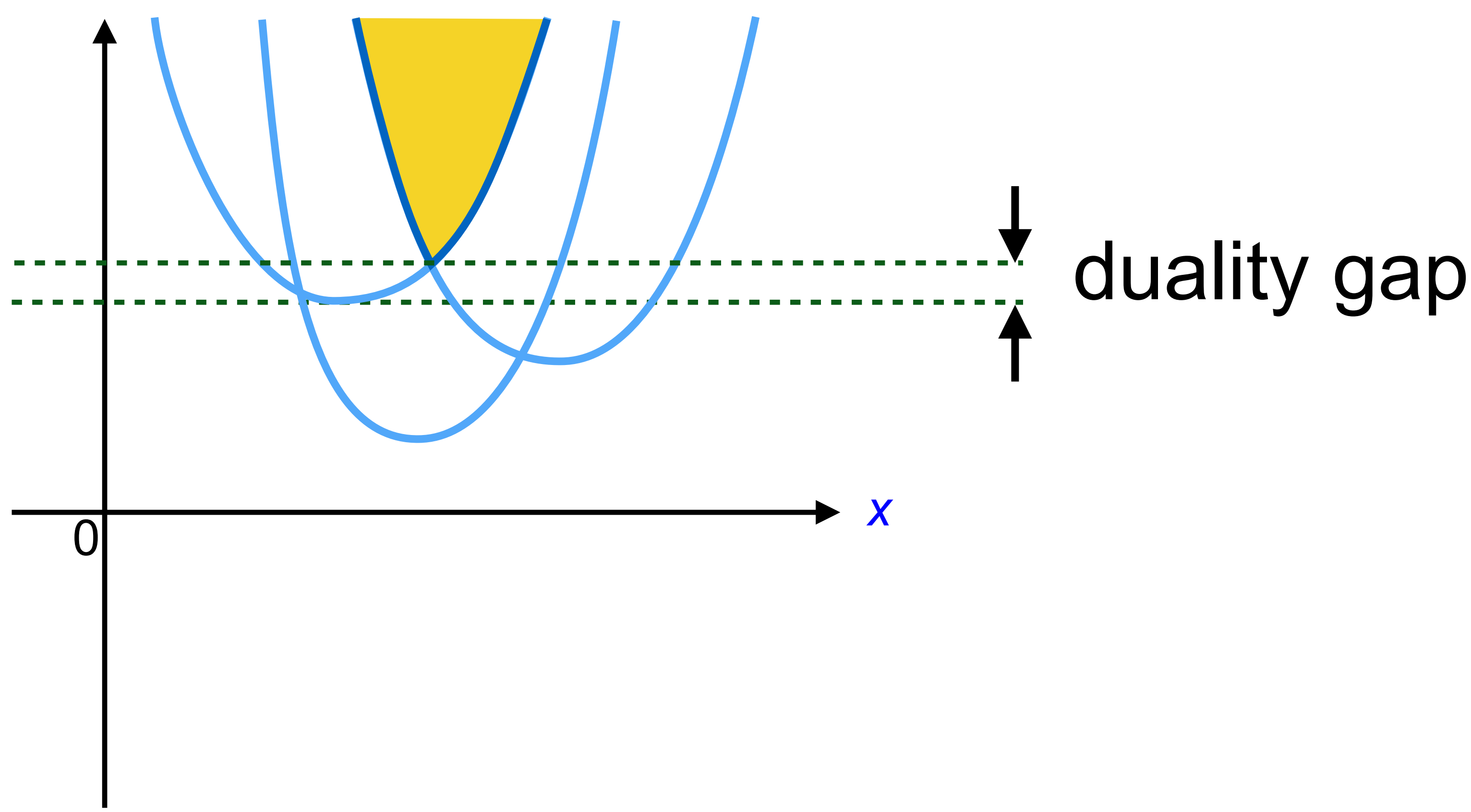
$$\sup_{z \in \mathcal{Z}} \inf \mathrm{P}(z) \leq \begin{cases} \inf\limits_{x} & \sup\limits_{z_0 \in \mathcal{Z}} f_0(x, z_0) \\ \text{s.t.} & \sup\limits_{z_i \in \mathcal{Z}} f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I \end{cases}$$



P(*z*)

Primal Worst

$$\sup_{z \in \mathcal{Z}} \inf P(z) \leq \begin{cases} \inf_{x} & \sup_{z_0 \in \mathcal{Z}} f_0(x, z_0) \\ \text{s.t.} & \sup_{z_i \in \mathcal{Z}} f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I \end{cases}$$

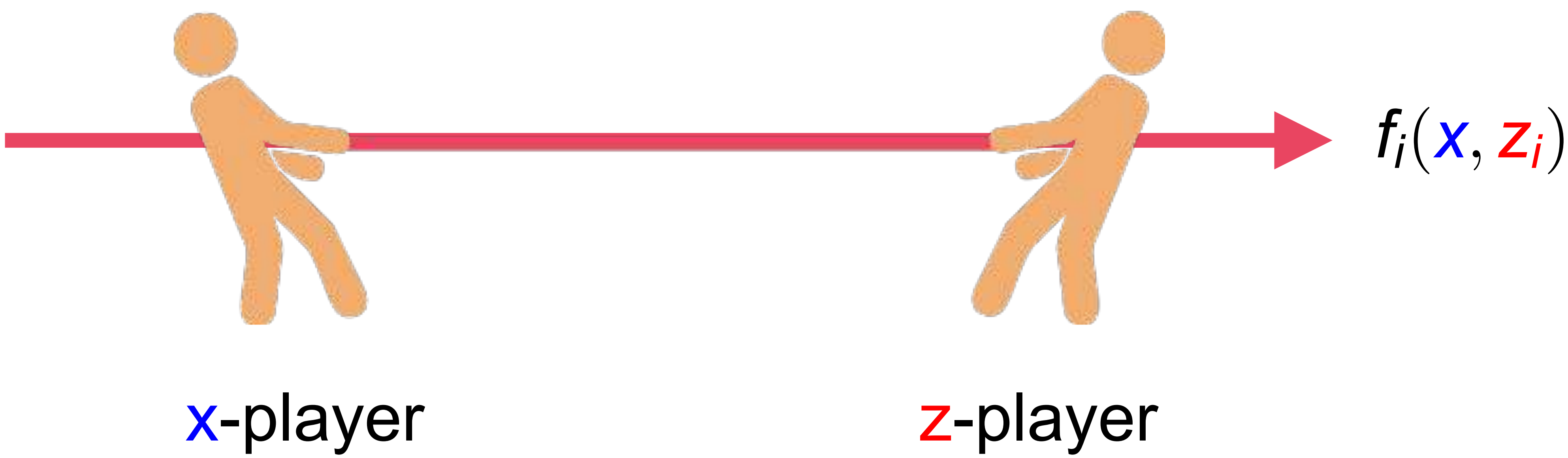


Primal Worst

$$\begin{aligned} \inf_{x} \quad & \sup_{z_0 \in \mathcal{Z}} f_0(x, z_0) \\ \text{s.t.} \quad & \sup_{z_i \in \mathcal{Z}} f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I \end{aligned} \qquad \text{(P-W)}$$

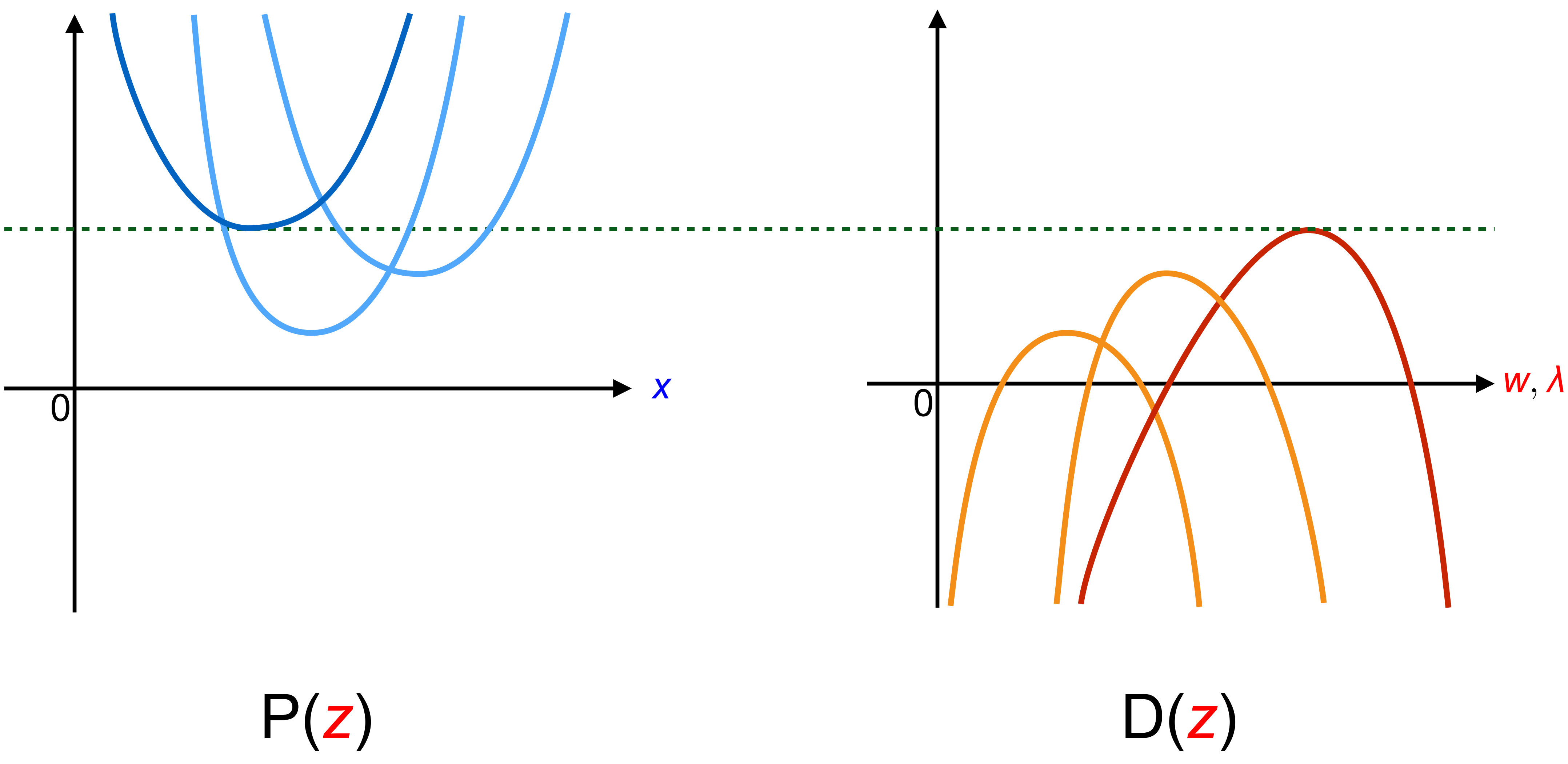
Primal Worst

$$\inf_{x} \quad \sup_{z_0 \in \mathcal{Z}} f_0(x, z_0)$$
$$\text{s.t.} \quad \sup_{z_i \in \mathcal{Z}} f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I \qquad \text{(P-W)}$$



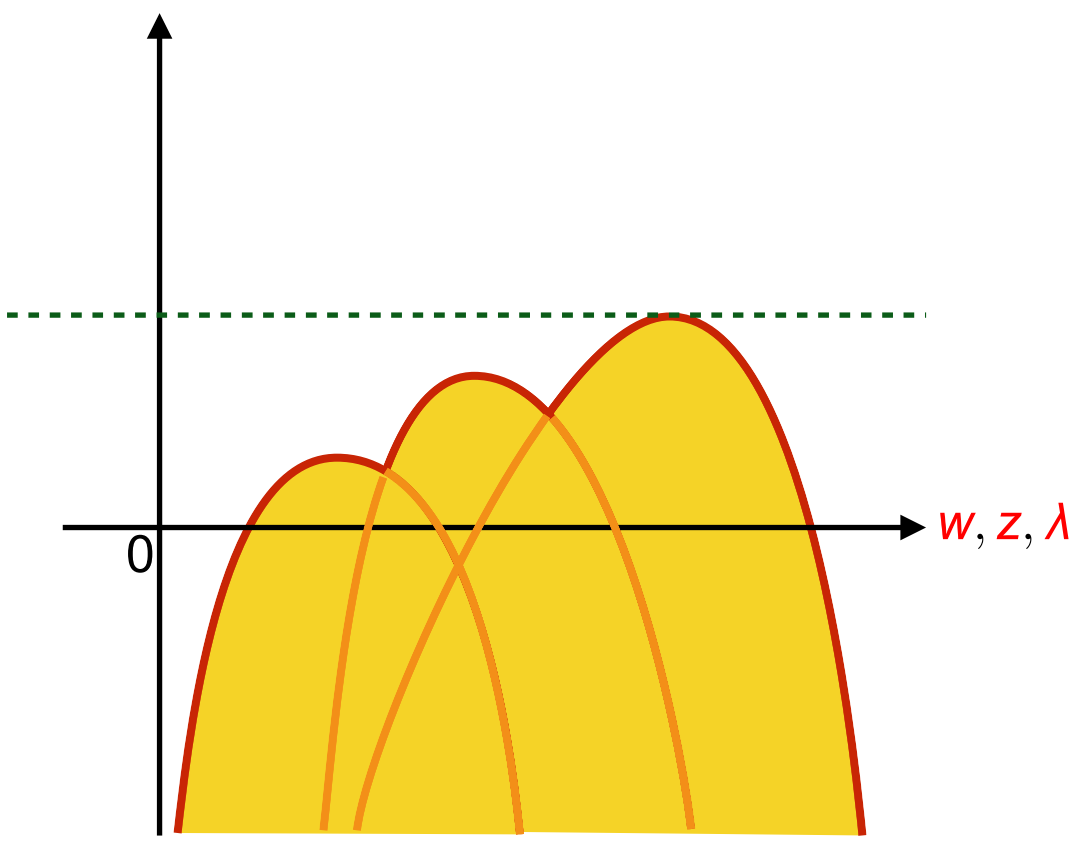
Duality of Robust Optimization

$$\sup_{z \in \mathcal{Z}} \inf \mathrm{P}(z) = \sup_{z \in \mathcal{Z}} \sup \mathrm{D}(z)$$



Dual Best

$$\sup_{z \in \mathcal{Z}} \sup D(z) = \begin{cases} \sup\limits_{w,z,\lambda \geq 0} & -f_0^{*1}(w_0, z_0) - \sum_{i=1}^{l} \lambda_i f_i^{*1}(w_i/\lambda_i, z_i) \\ \text{s.t.} & \sum_{i=0}^{l} w_i = 0, \;\; z_i \in \mathcal{Z} \; \forall i = 0, \ldots, l \end{cases}$$



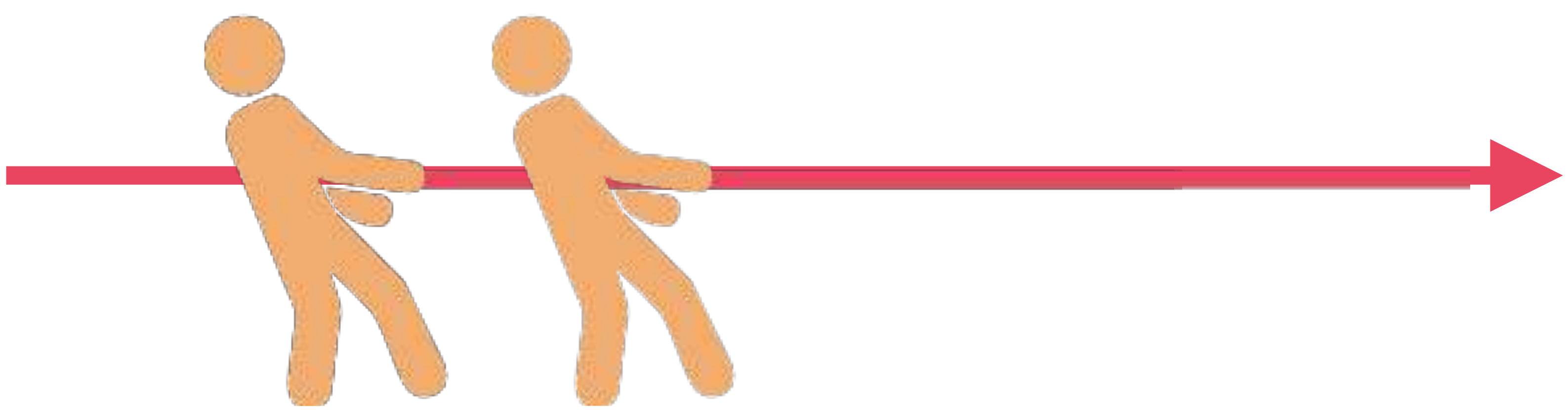
Dual Best

$$\sup_{w,z,\lambda\geq 0} \quad -f_0^{*1}(w_0, z_0) - \sum_{i=1}^{l} \lambda_i f_i^{*1}(w_i/\lambda_i, z_i) \qquad \text{(D-B)}$$

$$\text{s.t.} \quad \sum_{i=0}^{l} w_i = 0, \; z_i \in \mathcal{Z} \; \forall i = 0, \ldots, l$$

Dual Best

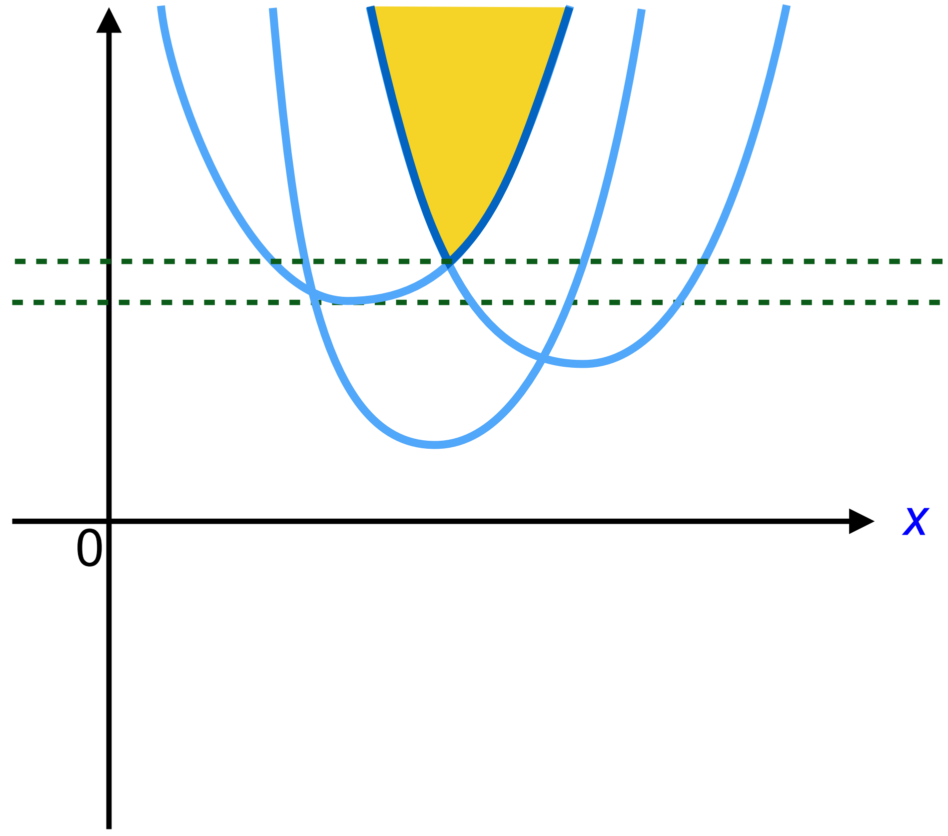
$$
\begin{aligned}
\sup_{w,z,\lambda\geq 0} \quad & -f_0^{*1}(w_0, z_0) - \sum_{i=1}^{I} \lambda_i f_i^{*1}(w_i/\lambda_i, z_i) \\
\text{s.t.} \quad & \sum_{i=0}^{I} w_i = 0,\; z_i \in \mathcal{Z}\ \forall i = 0, \ldots, I
\end{aligned}
\qquad \text{(D-B)}
$$



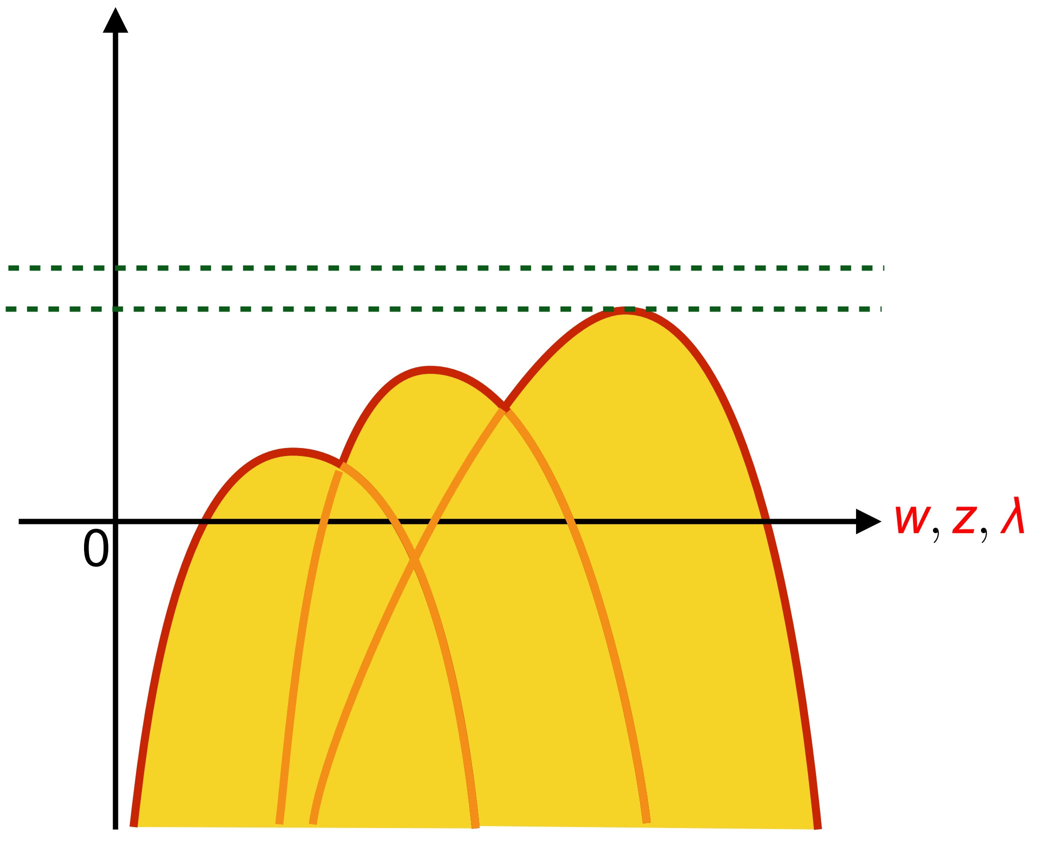
$\lambda_i f_i^{*1}(w_i/\lambda_i, z_i)$

(w, λ)-player z-player

Duality of Robust Optimization



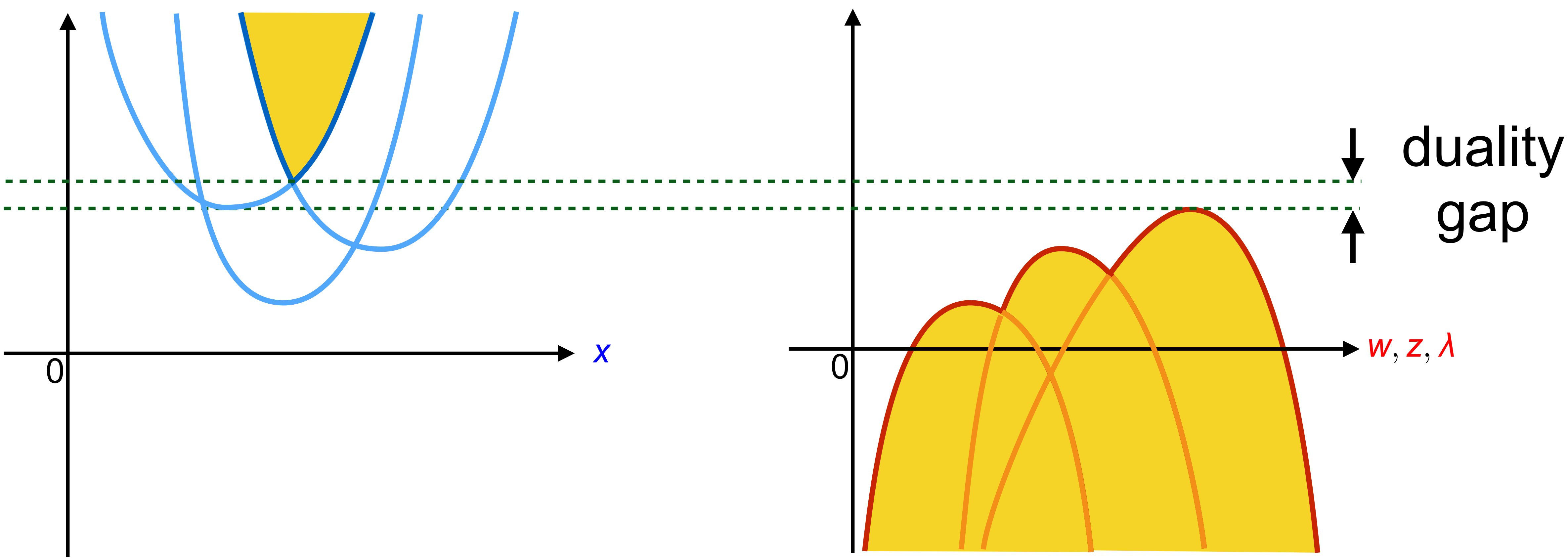
(P-W)



(D-B)

Duality of Robust Optimization

Conditions for gap-free duality?



(P-W)

(D-B)

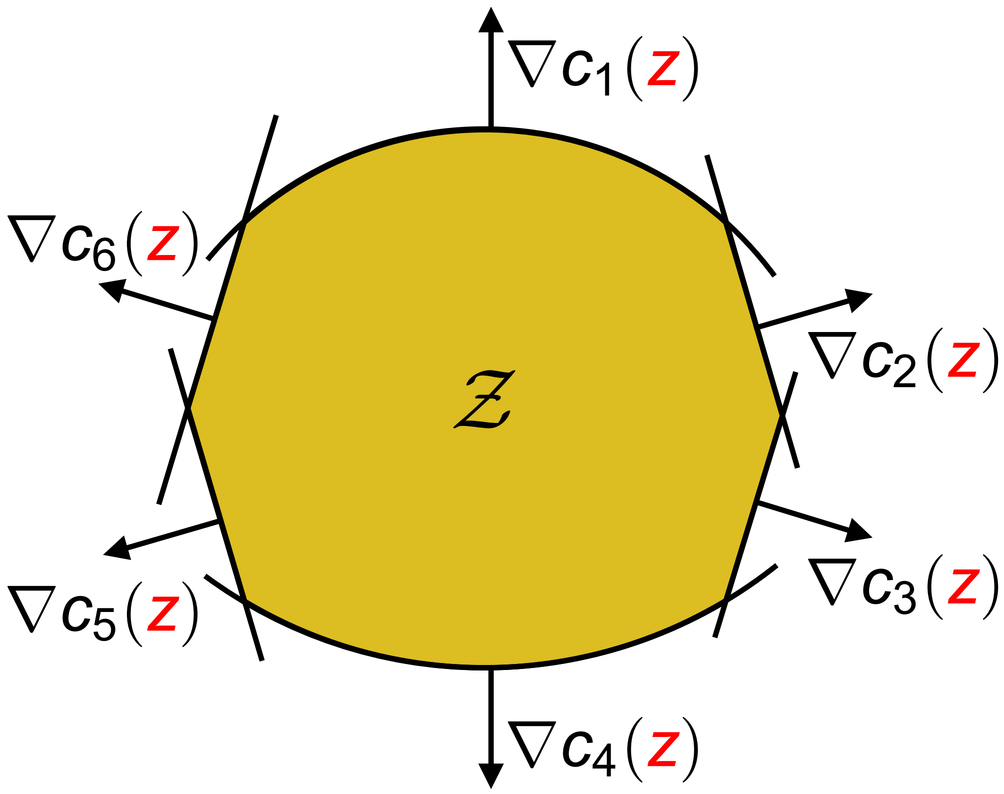
Convexity Assumptions

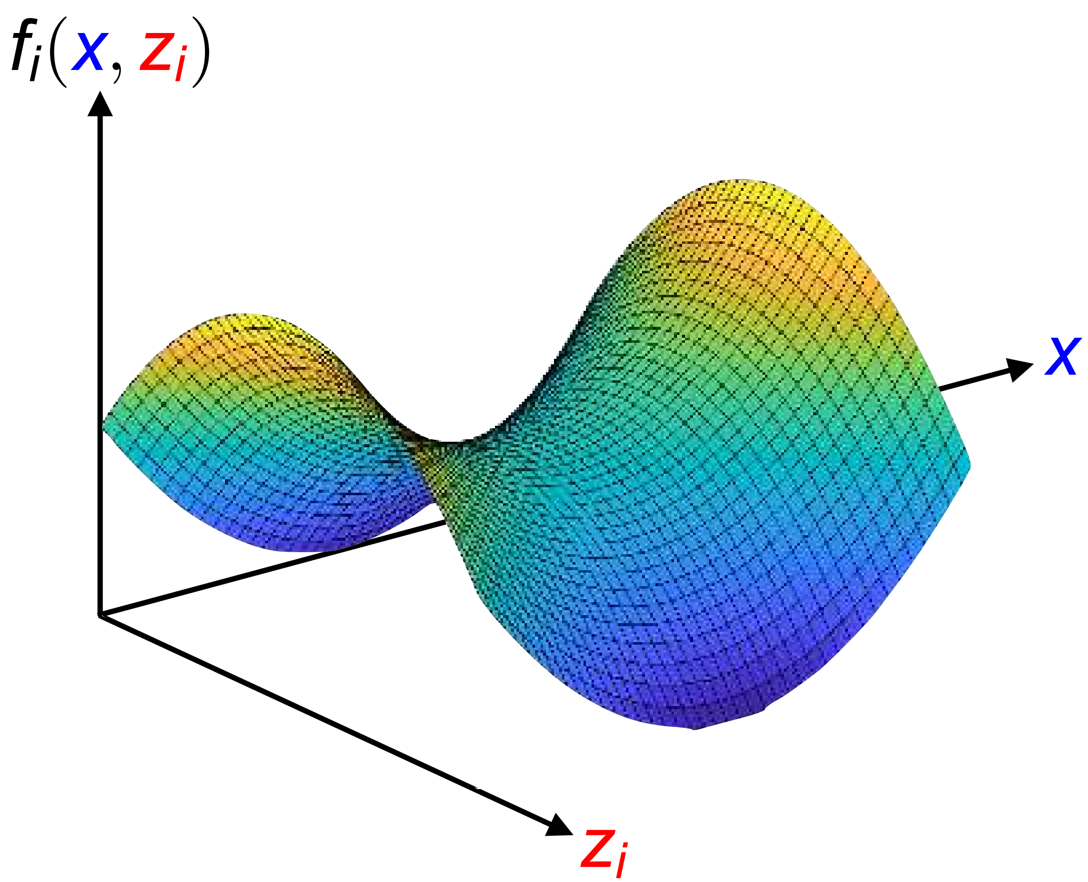
Uncertainty set: $\mathcal{Z} = \{z : c_\ell(z) \leq 0 \ \forall \ell = 1, \ldots, L\}$

Constraint functions:

$c_\ell(z)$ pcc in z for every ℓ

$-f_i(x, z_i)$ pcc in z_i for every i





Convex Reformulation of (P-W)

$$\begin{aligned} \inf_{x} \quad & \sup_{z_0 \in \mathcal{Z}} f_0(x, z_0) \\ \text{s.t.} \quad & \sup_{z_i \in \mathcal{Z}} f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I \end{aligned}$$

Convex Reformulation of (P-W)

$$\sup_{z_i \in \mathcal{Z}} f_i(x, z_i) = -\inf_{z_i \in \mathcal{Z}} -f_i(x, z_i)$$

Convex Reformulation of (P-W)

$$\sup_{z_i \in \mathcal{Z}} f_i(x, z_i) = -\left\{ \begin{array}{ll} \inf\limits_{z_i} & -f_i(x, z_i) \\ \text{s.t.} & c_\ell(z_i) \leq 0 \quad \forall \ell = 1, \ldots, L \end{array} \right.$$

Convex Reformulation of (P-W)

$$\sup_{z_i \in \mathcal{Z}} f_i(x, z_i) = -\left\{ \begin{array}{ll} \sup\limits_{y,v \geq 0} & -(-f_i)^{*2}(x, y_{i,\ell}) - \sum_{\ell=1}^{L} v_{i\ell} c_\ell^*(y_{i\ell}/v_{i\ell}) \\ \text{s.t.} & \sum_{\ell=0}^{L} y_{i\ell} = 0 \end{array} \right.$$

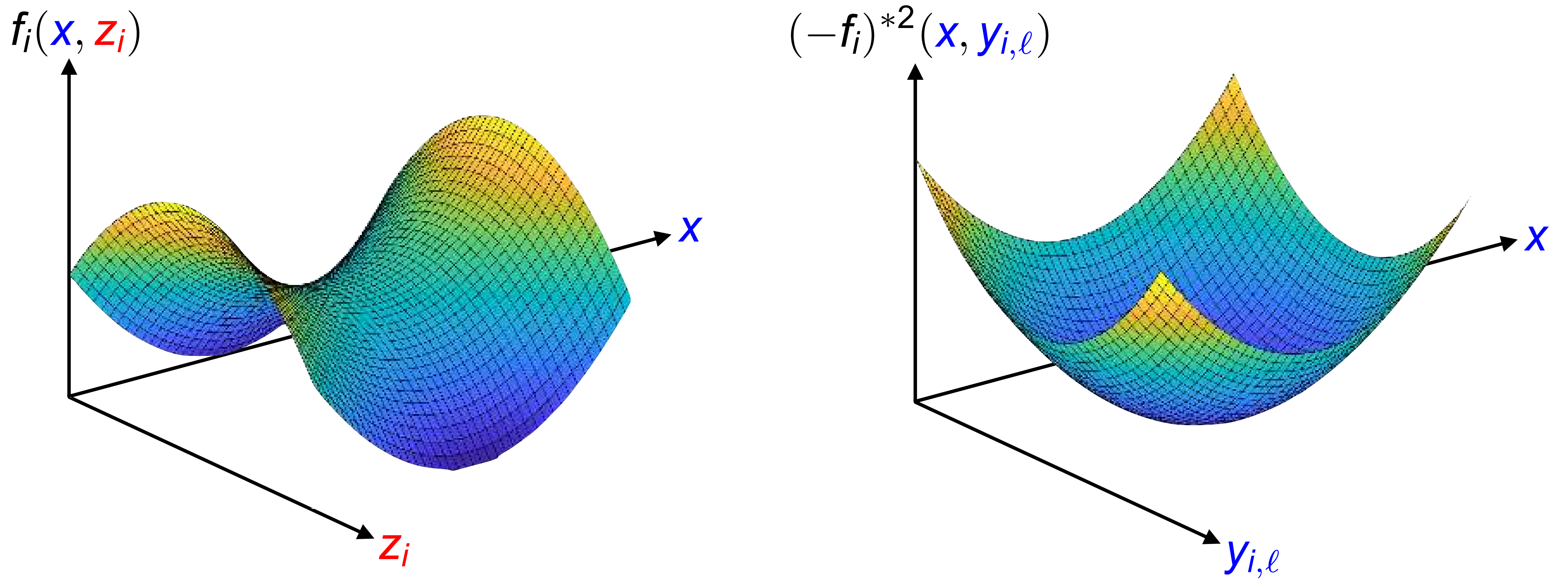
Convex Reformulation of (P-W)

$$\sup_{z_i \in \mathcal{Z}} f_i(x, z_i) = -\left\{ \begin{array}{ll} \sup\limits_{y,v \geq 0} & -(-f_i)^{*2}(x, y_{i,\ell}) - \sum_{\ell=1}^{L} v_{i\ell} c_\ell^*(y_{i\ell}/v_{i\ell}) \\ \text{s.t.} & \sum_{\ell=0}^{L} y_{i\ell} = 0 \end{array} \right.$$

conjugates w.r.t. z_i

Convex Reformulation of (P-W)

$$\sup_{z_i \in \mathcal{Z}} f_i(x, z_i) = -\left\{ \begin{array}{ll} \sup\limits_{y,v \geq 0} & -(-f_i)^{*2}(x, y_{i,\ell}) - \sum_{\ell=1}^{L} v_{i\ell} c_\ell^*(y_{i\ell}/v_{i\ell}) \\ \text{s.t.} & \sum_{\ell=0}^{L} y_{i\ell} = 0 \end{array} \right.$$



Convex Reformulation of (P-W)

$$\sup_{z_i \in \mathcal{Z}} f_i(x, z_i) = \begin{cases} \inf_{y,v \geq 0} & (-f_i)^{*2}(x, y_{i,\ell}) + \sum_{\ell=1}^{L} v_{i\ell} c_\ell^*(y_{i\ell}/v_{i\ell}) \\ \text{s.t.} & \sum_{\ell=0}^{L} y_{i\ell} = 0 \end{cases}$$

Convex Reformulation of (P-W)

$$\begin{aligned} \inf_{x} \quad & \sup_{z_0 \in \mathcal{Z}} f_0(x, z_0) \\ \text{s.t.} \quad & \sup_{z_i \in \mathcal{Z}} f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I \end{aligned} \qquad \text{(P-W)}$$

Convex Reformulation of (P-W)

$$
\begin{aligned}
\inf_{x} \quad & \sup_{z_0 \in \mathcal{Z}} f_0(x, z_0) & \\
\text{s.t.} \quad & \sup_{z_i \in \mathcal{Z}} f_i(x, z_i) \le 0 \quad \forall i = 1, \ldots, I
\end{aligned}
\qquad \text{(P-W)}
$$

$$
\begin{aligned}
\inf_{x,y,v \ge 0} \quad & (-f_0)^{*2}(x, y_{0,\ell}) + \textstyle\sum_{\ell=1}^{L} v_{0\ell} c_\ell^*(y_{0\ell}/v_{0\ell}) & \\
\text{s.t.} \quad & (-f_i)^{*2}(x, y_{i,\ell}) + \textstyle\sum_{\ell=1}^{L} v_{i\ell} c_\ell^*(y_{i\ell}/v_{i\ell}) \le 0 & \forall i = 1, \ldots, I \\
& \textstyle\sum_{\ell=0}^{L} y_{i\ell} = 0 & \forall i = 1, \ldots, I
\end{aligned}
\qquad \text{(P-W')}
$$

Convex Reformulation of (P-W)

$$\inf_{x} \sup_{z_0 \in \mathcal{Z}} f_0(x, z_0)$$
$$\text{s.t.} \sup_{z_i \in \mathcal{Z}} f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I \qquad \text{(P-W)}$$

$\Updownarrow$ (subject to regularity conditions)

$$\inf_{x,y,v \geq 0} (-f_0)^{*2}(x, y_{0,\ell}) + \sum_{\ell=1}^{L} v_{0\ell} c_\ell^*(y_{0\ell}/v_{0\ell})$$
$$\text{s.t.} \quad (-f_i)^{*2}(x, y_{i,\ell}) + \sum_{\ell=1}^{L} v_{i\ell} c_\ell^*(y_{i\ell}/v_{i\ell}) \leq 0 \quad \forall i = 1, \ldots, I \qquad \text{(P-W')}$$
$$\sum_{\ell=0}^{L} y_{i\ell} = 0 \quad \forall i = 1, \ldots, I$$

Convex Reformulation of (P-W)

$$\begin{aligned} \inf_{x} \quad & \sup_{z_0 \in \mathcal{Z}} f_0(x, z_0) \\ \text{s.t.} \quad & \sup_{z_i \in \mathcal{Z}} f_i(x, z_i) \leq 0 \quad \forall i = 1, \ldots, I \end{aligned} \qquad \text{(P-W)}$$

$\Updownarrow$ (subject to regularity conditions)

$$\begin{aligned} \inf_{x,y,v \geq 0} \quad & (-f_0)^{*2}(x, y_{0,\ell}) + \textstyle\sum_{\ell=1}^{L} v_{0\ell} c_\ell^*(y_{0\ell}/v_{0\ell}) \\ \text{s.t.} \quad & (-f_i)^{*2}(x, y_{i,\ell}) + \textstyle\sum_{\ell=1}^{L} v_{i\ell} c_\ell^*(y_{i\ell}/v_{i\ell}) \leq 0 \quad \forall i = 1, \ldots, I \\ & \textstyle\sum_{\ell=0}^{L} y_{i\ell} = 0 \quad \forall i = 1, \ldots, I \end{aligned} \qquad \text{(P-W')}$$

Ben-Tal, Nemirovski & El Ghaoui, *Princeton University Press*, 2009; Bertsimas & Sim, *Oper. Res.*, 2004; Ben-Tal, den Hertog & Vial, *Math. Program.*, 2015; etc.

Convex Reformulation of (D-B)

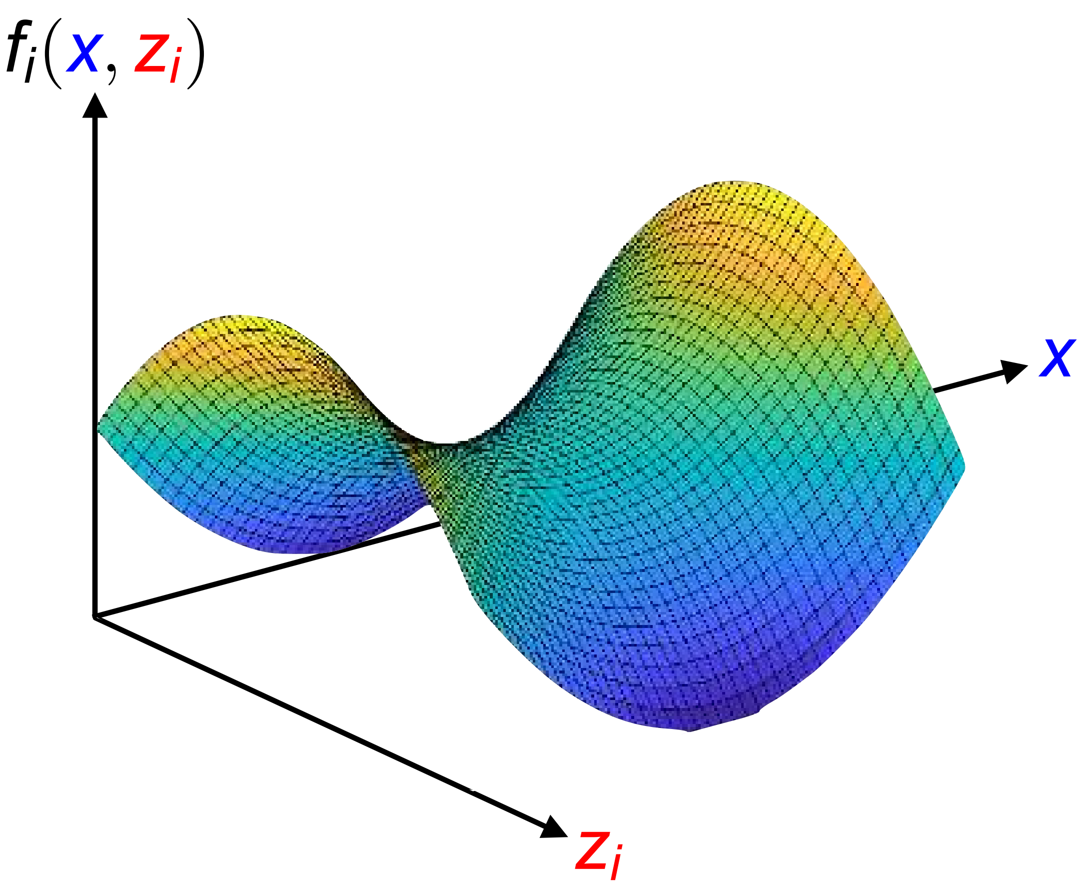
$$\begin{aligned} \sup_{w,z,\lambda \geq 0} \quad & -f_0^{*1}\left(w_0, z_0\right) - \sum_{i=1}^{I} \lambda_i f_i^{*1}\left(w_i/\lambda_i, z_i\right) \\ \text{s.t.} \quad & c_\ell(z) \leq 0 \quad \forall i = 0, \dots, I, \ \forall \ell = 1, \dots, L \\ & \sum_{i=0}^{I} w_i = 0 \end{aligned} \qquad \text{(D-B)}$$

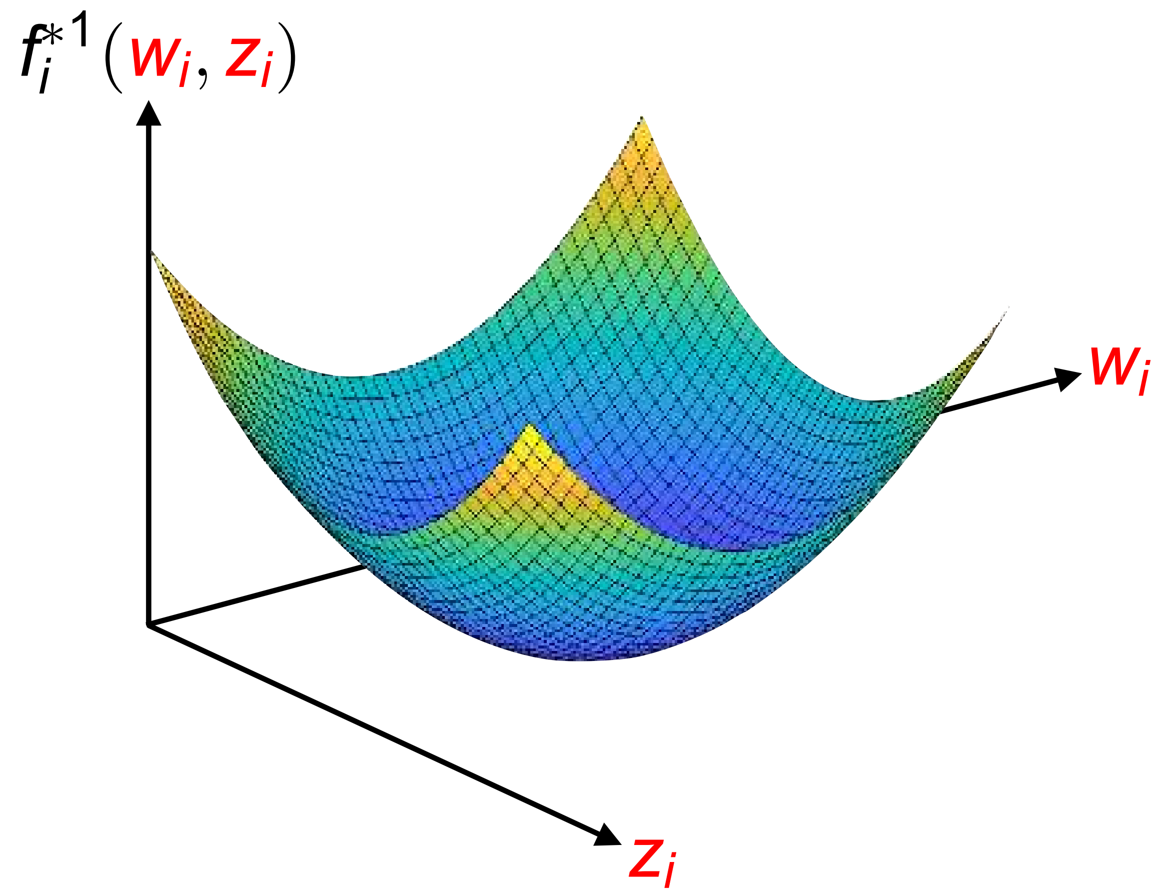
Convex Reformulation of (D-B)

$$\sup_{w,z,\lambda\geq 0} \quad -f_0^{*1}(w_0, z_0) - \sum_{i=1}^{I} \lambda_i f_i^{*1}(w_i/\lambda_i, z_i)$$

$$\text{s.t.} \quad c_\ell(z) \leq 0 \quad \forall i = 0, \ldots, I, \ \forall \ell = 1, \ldots, L \qquad \text{(D-B)}$$

$$\sum_{i=0}^{I} w_i = 0$$





Convex Reformulation of (D-B)

$$
\begin{aligned}
\sup_{w,z,\lambda \geq 0} \quad & -f_0^{*1}(w_0, z_0) - \sum_{i=1}^{I} \lambda_i f_i^{*1}(w_i/\lambda_i, z_i) \\
\text{s.t.} \quad & c_\ell(z) \leq 0 \quad \forall i = 0, \ldots, I, \ \forall \ell = 1, \ldots, L \\
& \sum_{i=0}^{I} w_i = 0
\end{aligned}
\qquad \text{(D-B)}
$$

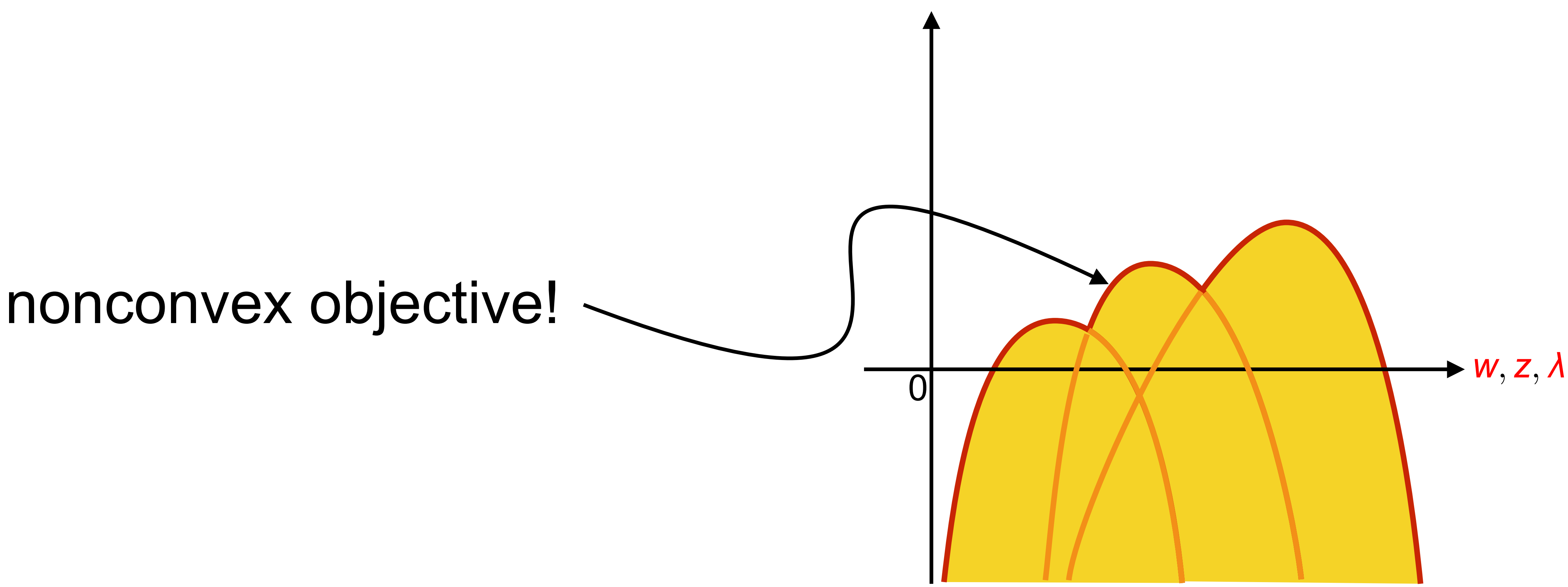
nonconvex objective!

Convex Reformulation of (D-B)

$$\sup_{w,z,\lambda\geq 0} \quad -f_0^{*1}(w_0, z_0) - \sum_{i=1}^{I} \lambda_i f_i^{*1}(w_i/\lambda_i, z_i)$$

$$\text{s.t.} \quad c_\ell(z) \leq 0 \quad \forall i = 0,\dots,I,\ \forall \ell = 1,\dots,L \qquad \text{(D-B)}$$

$$\sum_{i=0}^{I} w_i = 0$$



Convex Reformulation of (D-B)

$$\sup_{w,z,\lambda\geq 0} \quad -f_0^{*1}(w_0, z_0) - \sum_{i=1}^{I} \lambda_i f_i^{*1}(w_i/\lambda_i, z_i)$$

$$\text{s.t.} \quad c_\ell(z) \leq 0 \quad \forall i = 0,\ldots,I,\ \forall \ell = 1,\ldots,L \qquad \text{(D-B)}$$

$$\sum_{i=0}^{I} w_i = 0$$

variable substitution: $v_i \leftarrow \lambda_i z_i$

Gorissen et al., *Oper. Res.*, 2014.

Convex Reformulation of (D-B)

$$\begin{aligned} \sup_{w,z,\lambda\geq 0} \quad & -f_0^{*1}(w_0, z_0) - \textstyle\sum_{i=1}^{I} \lambda_i f_i^{*1}(w_i/\lambda_i, z_i) \\ \text{s.t.} \quad & c_\ell(z) \leq 0 \quad \forall i = 0, \ldots, I,\ \forall \ell = 1, \ldots, L \\ & \textstyle\sum_{i=0}^{I} w_i = 0 \end{aligned} \qquad \text{(D-B)}$$

$$\begin{aligned} \sup_{w,v,\lambda\geq 0} \quad & -f_0^{*1}(w_0, v_0) - \textstyle\sum_{i=1}^{I} \lambda_i f_i^{*1}(w_i/\lambda_i, v_i/\lambda_i) \\ \text{s.t.} \quad & \lambda_i c_\ell(v/\lambda_i) \leq 0 \quad \forall i = 0, \ldots, I,\ \forall \ell = 1, \ldots, L \\ & \textstyle\sum_{i=0}^{I} w_i = 0 \end{aligned} \qquad \text{(D-B')}$$

Gorissen et al., *Oper. Res.*, 2014.

Convex Reformulation of (D-B)

$$\begin{aligned} \sup_{w,z,\lambda\geq 0} \quad & -f_0^{*1}(w_0, z_0) - \textstyle\sum_{i=1}^{I} \lambda_i f_i^{*1}(w_i/\lambda_i, z_i) \\ \text{s.t.} \quad & c_\ell(z) \leq 0 \quad \forall i = 0, \ldots, I, \ \forall \ell = 1, \ldots, L \\ & \textstyle\sum_{i=0}^{I} w_i = 0 \end{aligned} \qquad \text{(D-B)}$$

$\Updownarrow$ (subject to regularity conditions)

$$\begin{aligned} \sup_{w,v,\lambda\geq 0} \quad & -f_0^{*1}(w_0, v_0) - \textstyle\sum_{i=1}^{I} \lambda_i f_i^{*1}(w_i/\lambda_i, v_i/\lambda_i) \\ \text{s.t.} \quad & \lambda_i c_\ell(v/\lambda_i) \leq 0 \quad \forall i = 0, \ldots, I, \ \forall \ell = 1, \ldots, L \\ & \textstyle\sum_{i=0}^{I} w_i = 0 \end{aligned} \qquad \text{(D-B')}$$

Gorissen et al., *Oper. Res.*, 2014.

Primal Worst = Dual Best

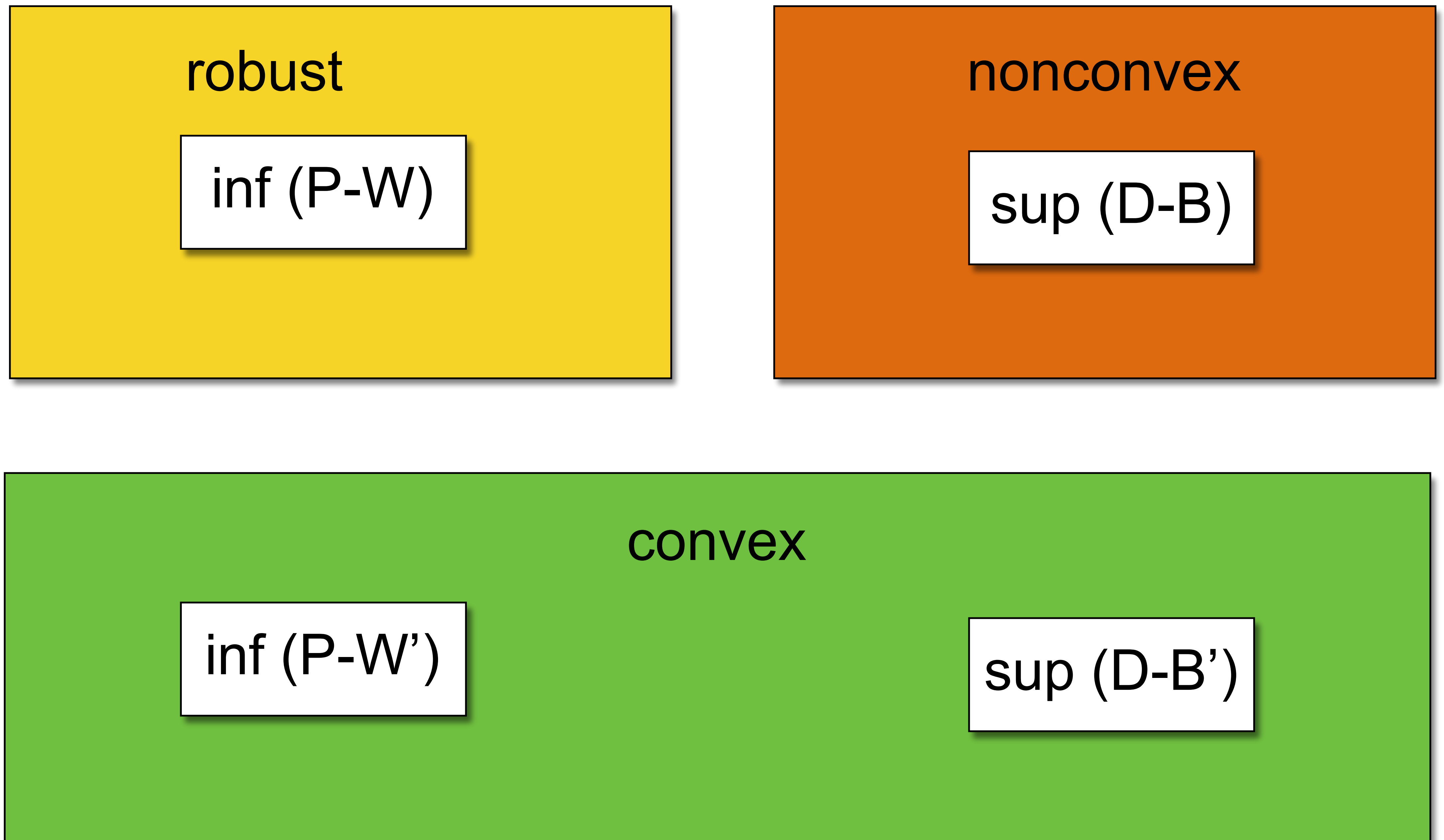
inf (P-W)

sup (D-B)

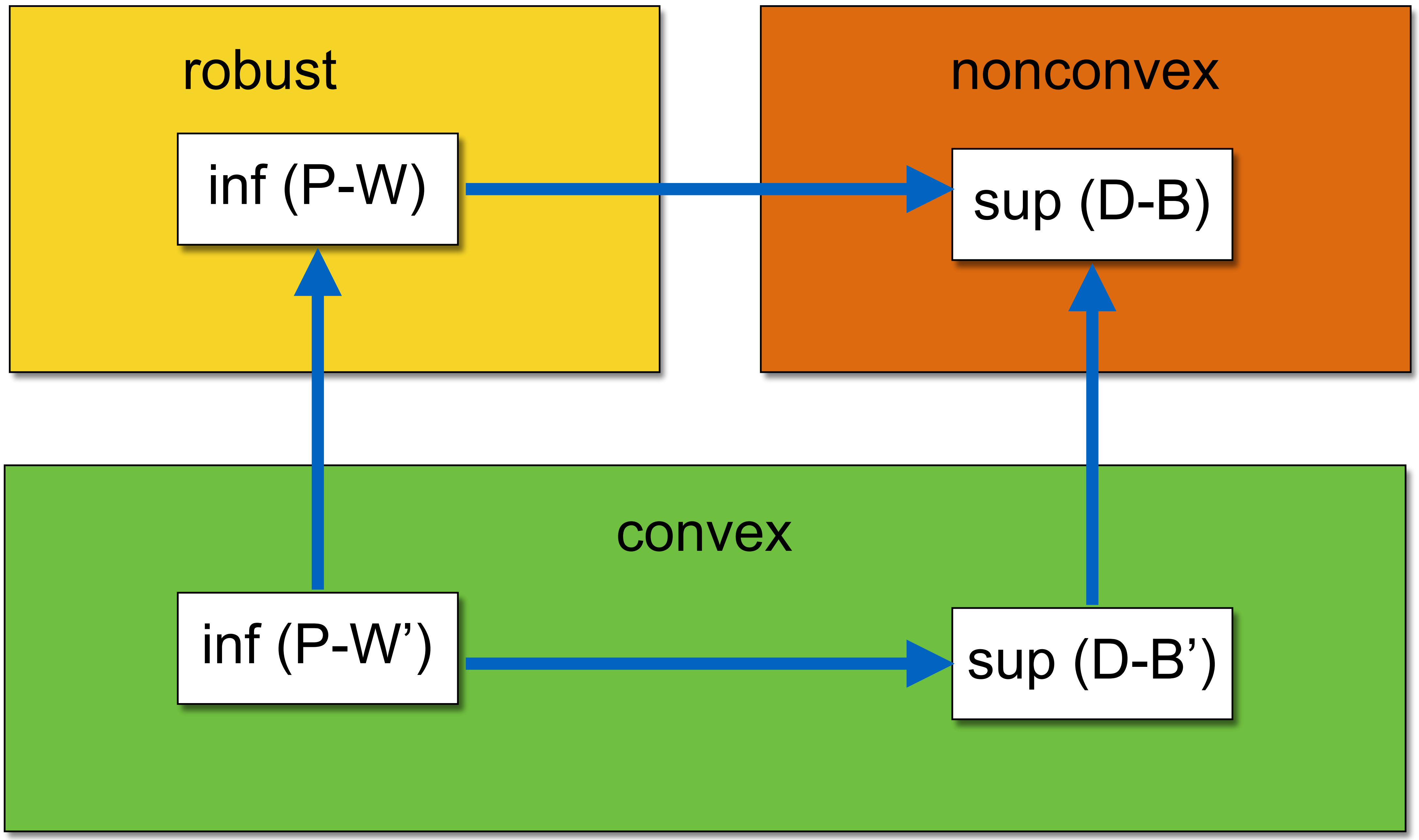
inf (P-W')

sup (D-B')

Primal Worst = Dual Best

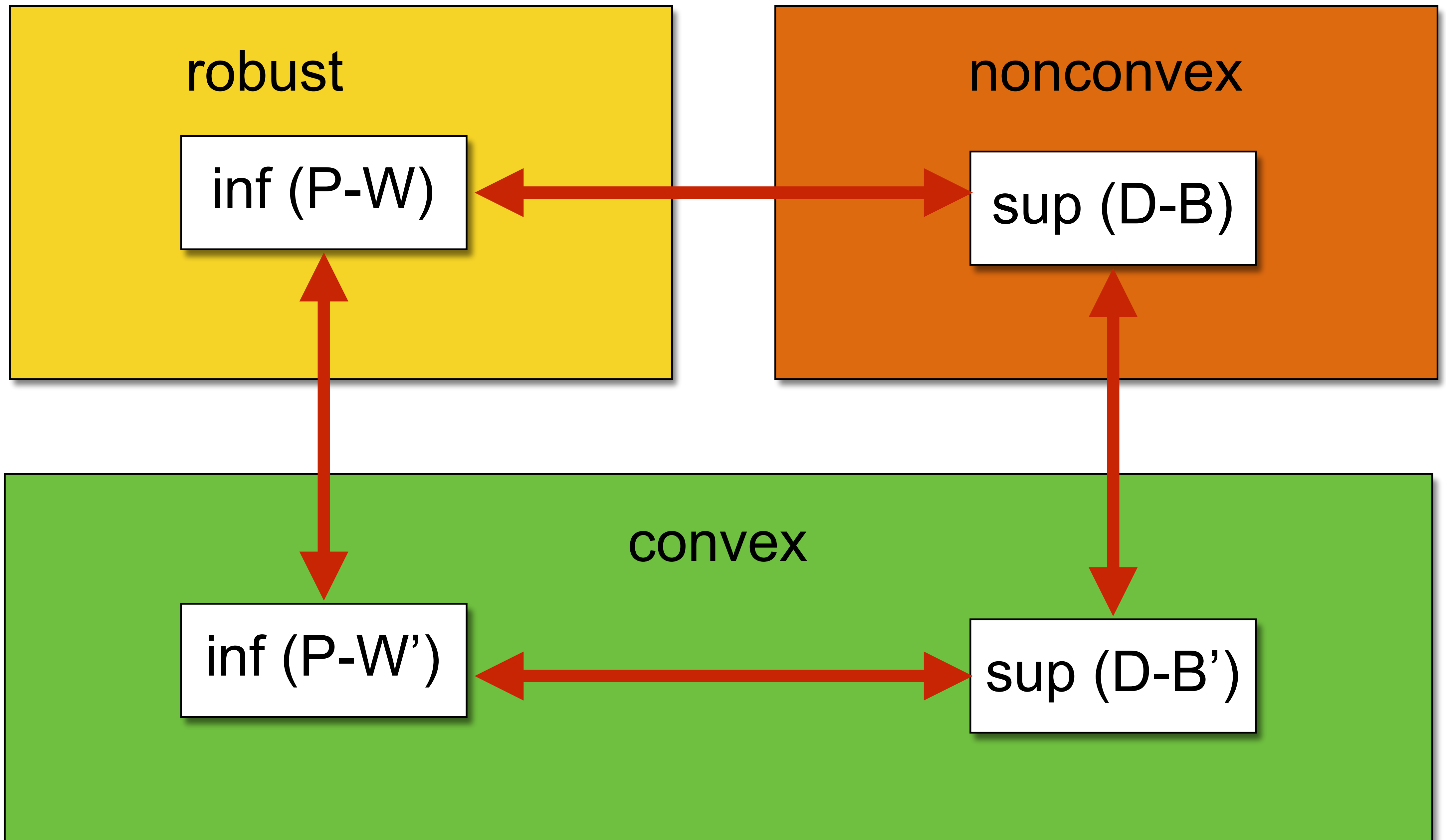


Primal Worst = Dual Best



Primal Worst = Dual Best

If (D-B) has a Slater point…



Beck & Ben-Tal, *Oper. Res. Lett.*, 2009.