Distributionally Robust Optimization

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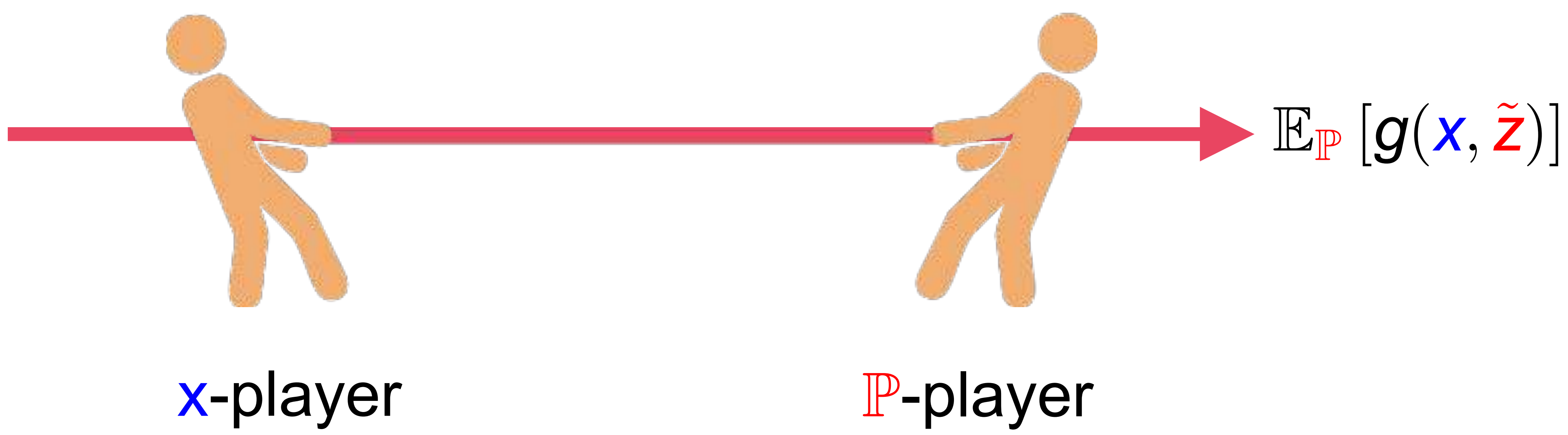

Uncertainty Quantification

$$\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}_{\mathbb{P}} \left[g(\tilde{z}) \right] \qquad \text{(P-UQ)}$$

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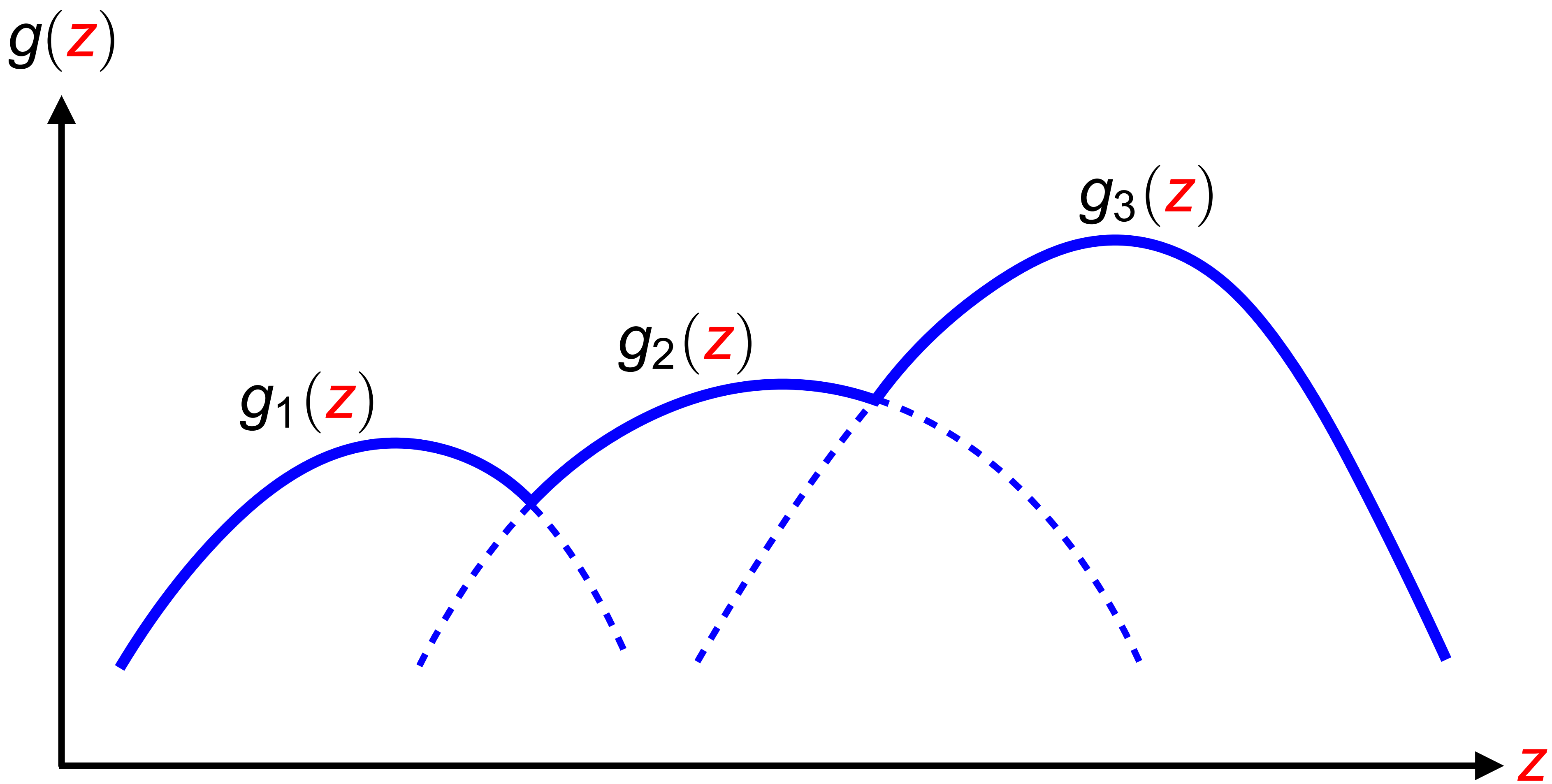
$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}_{\mathbb{P}}\left[g(\tilde{z})\right] \qquad \text{(P-UQ)}$$

Disutility function: $g(z) = \max_{i=1,\ldots,I} g_i(z)$, and $-g_i(z)$ is pcc $\forall i$



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Ambiguity set: $\mathcal{P} = \{ \mathbb{P} \text{ distribution on } \mathcal{S} \mid \mathbb{E}_{\mathbb{P}} \left[h(\tilde{z}) \right] \leq \mu \}$

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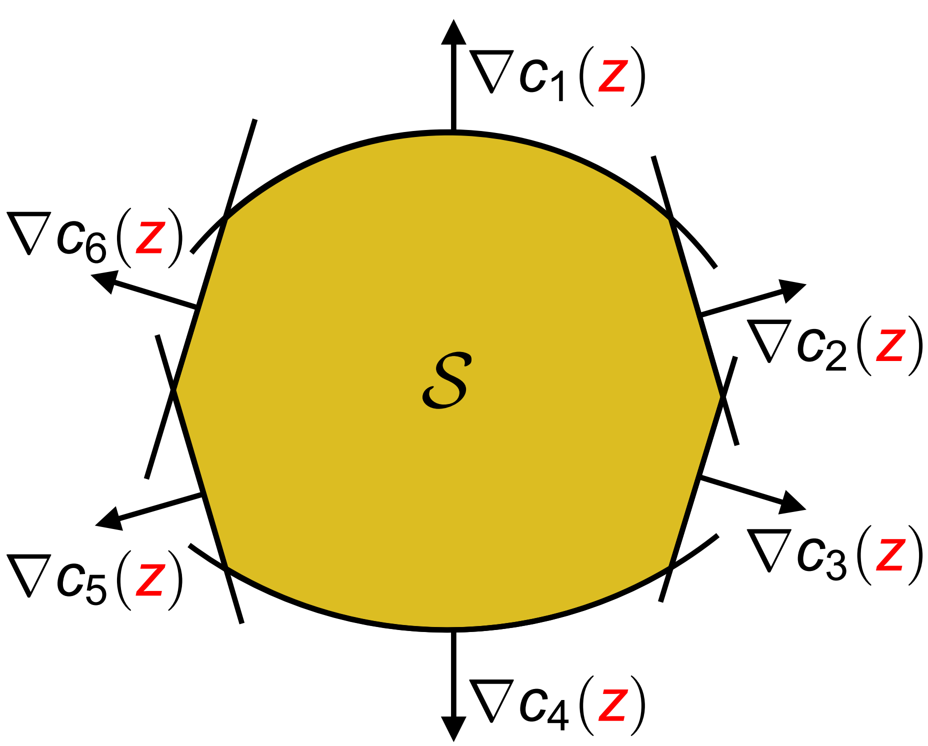
Support set: $\mathcal{S} = \{z : c_\ell(z) \leq 0 \ \forall \ell = 1, \ldots, L\}$

$c_\ell(z)$ pcc for every ℓ



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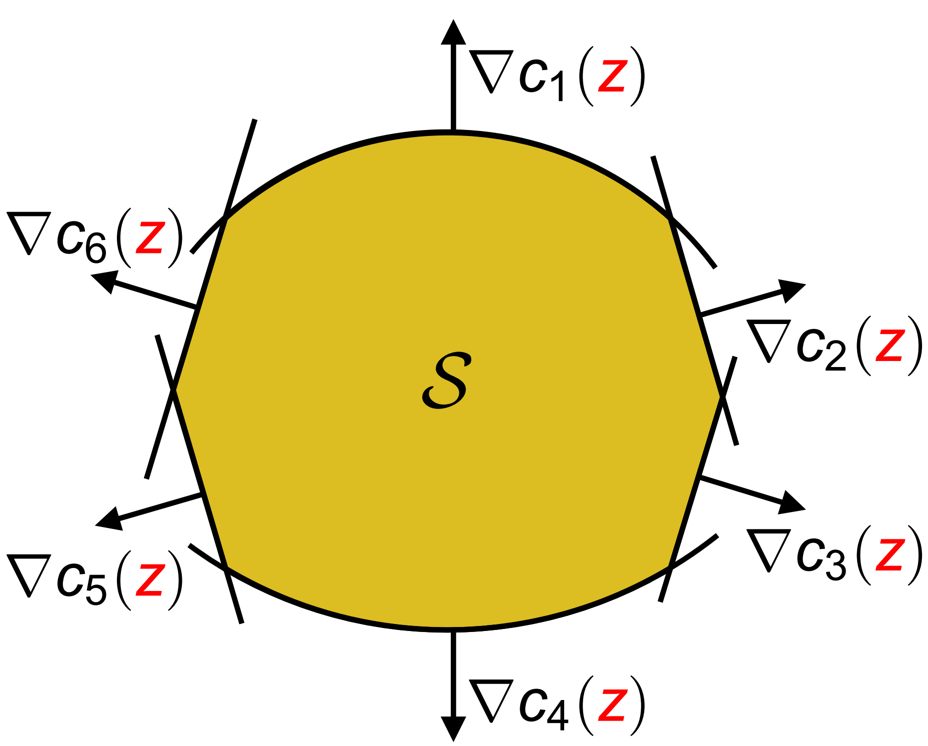
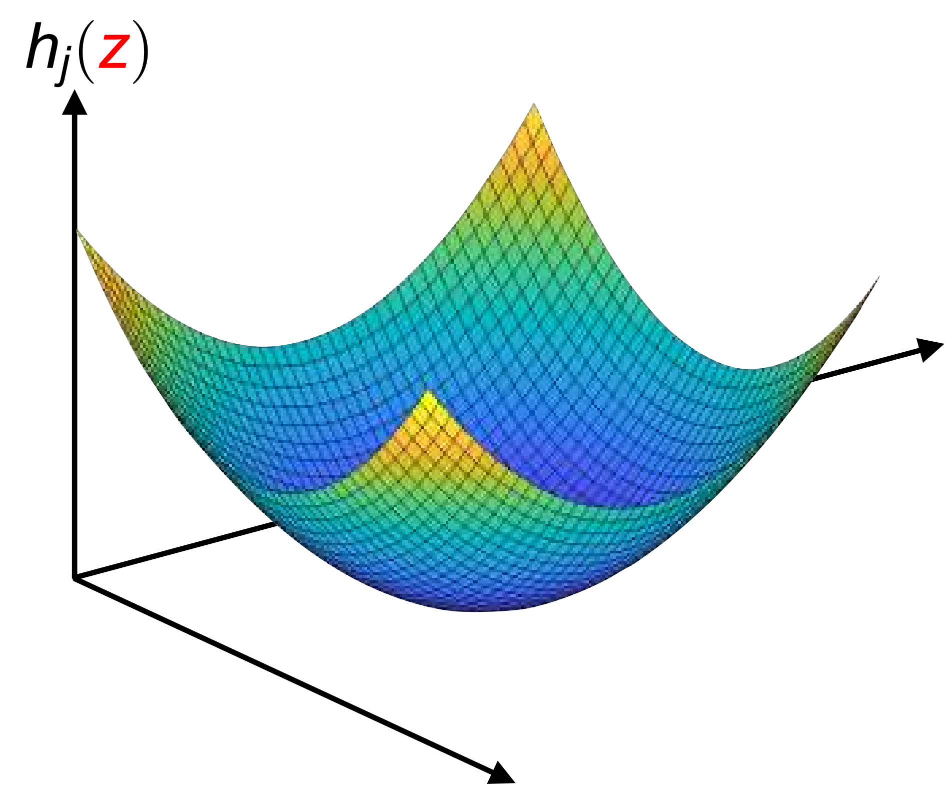


$h_j(z)$ pcc for every j



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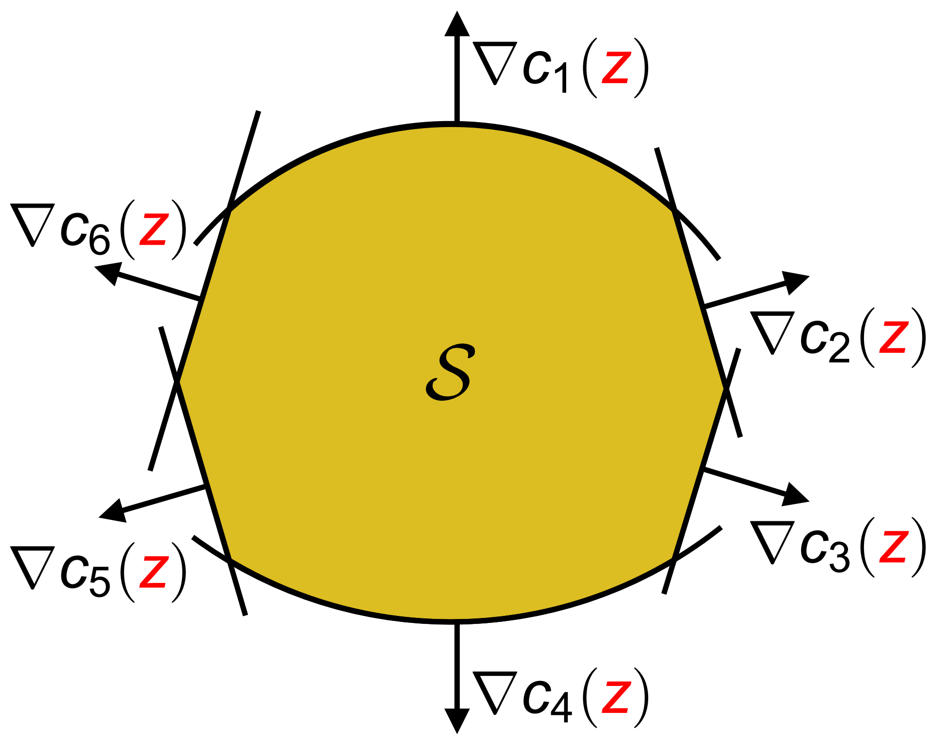

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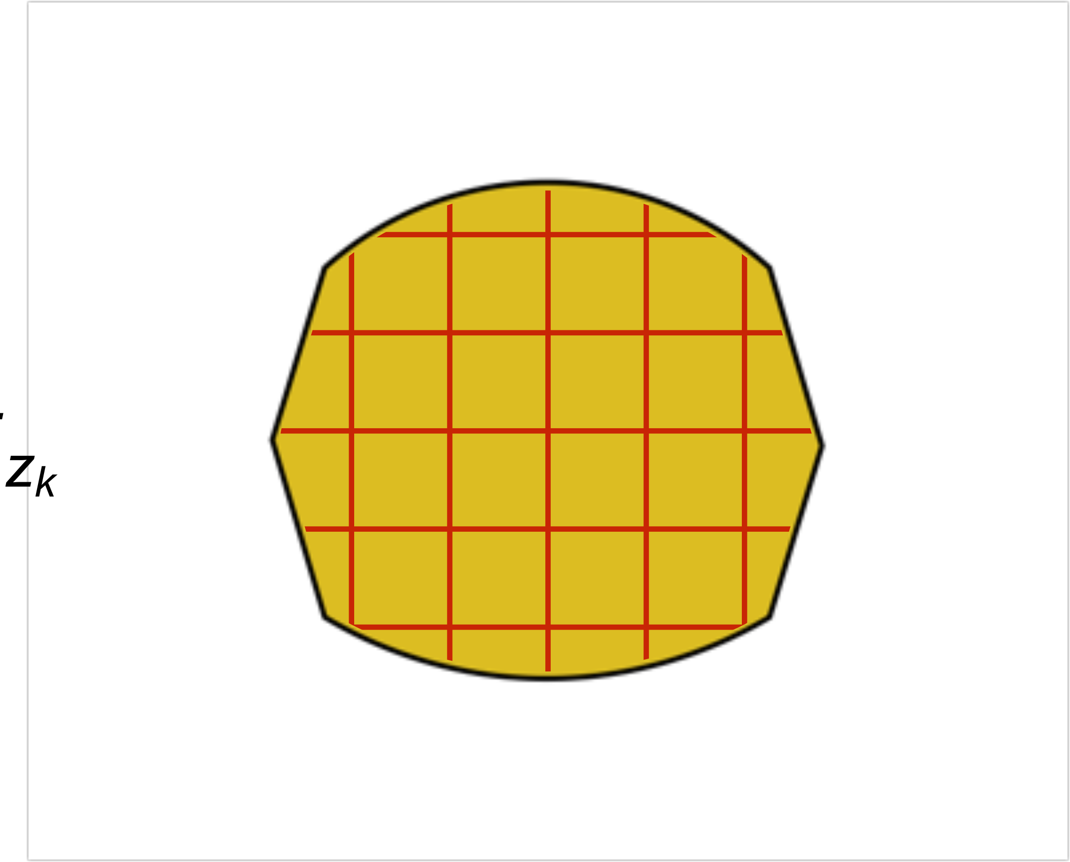
$$\sup_{\mathbb{P} \in \mathcal{P}} \ \mathbb{E}_{\mathbb{P}} \left[g(\tilde{z}) \right] \qquad \text{(P-UQ)}$$

Discretization:

$\mathbb{P} = \sum_{k=1}^{K} p_k \delta_{z_k}$



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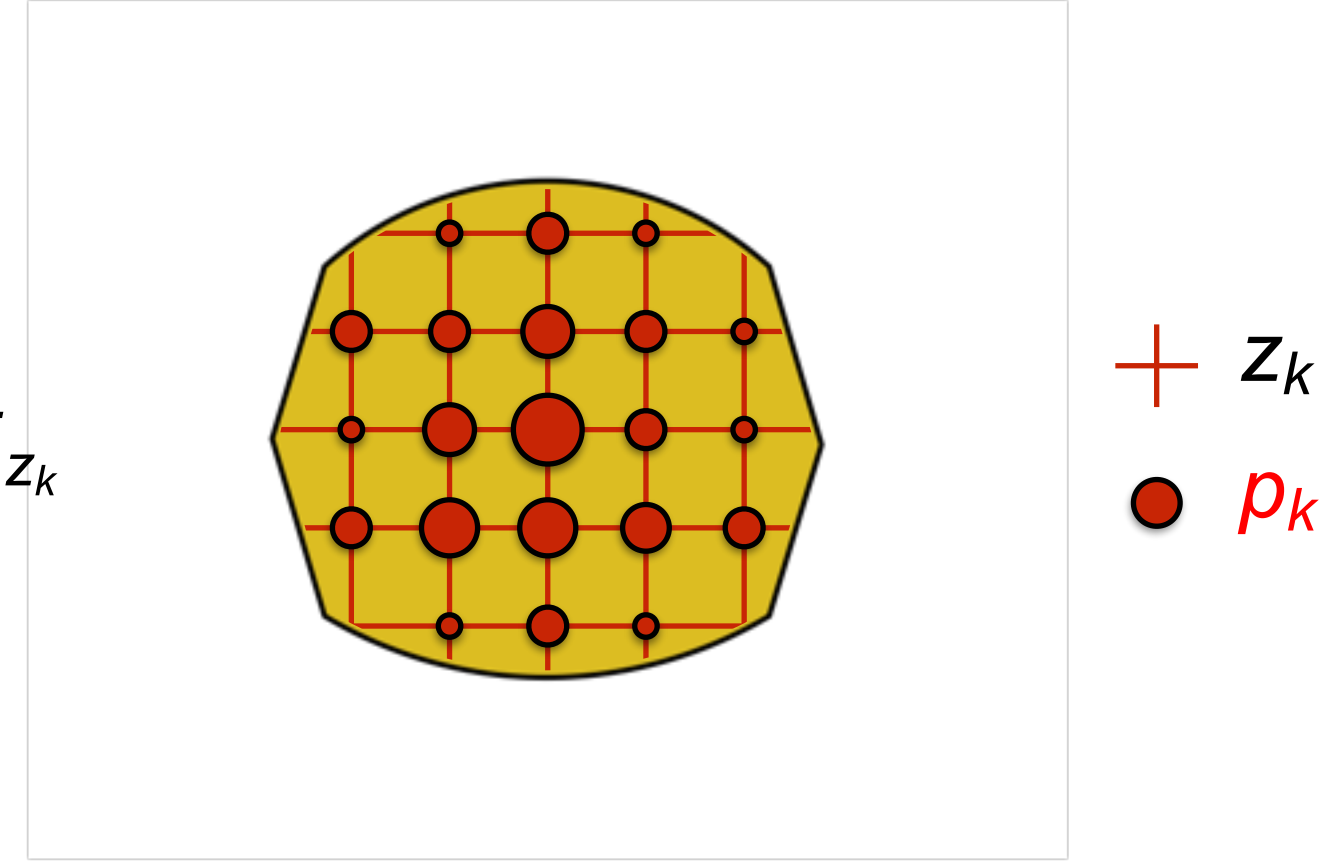

Uncertainty Quantification

$$\sup_{p \geq 0} \quad \sum_{k=1}^{K} p_k \, g(z_k)$$

$$\text{s.t.} \quad \sum_{k=1}^{K} p_k = 1$$

$$\sum_{k=1}^{K} p_k \, h(z_k) \leq \mu$$

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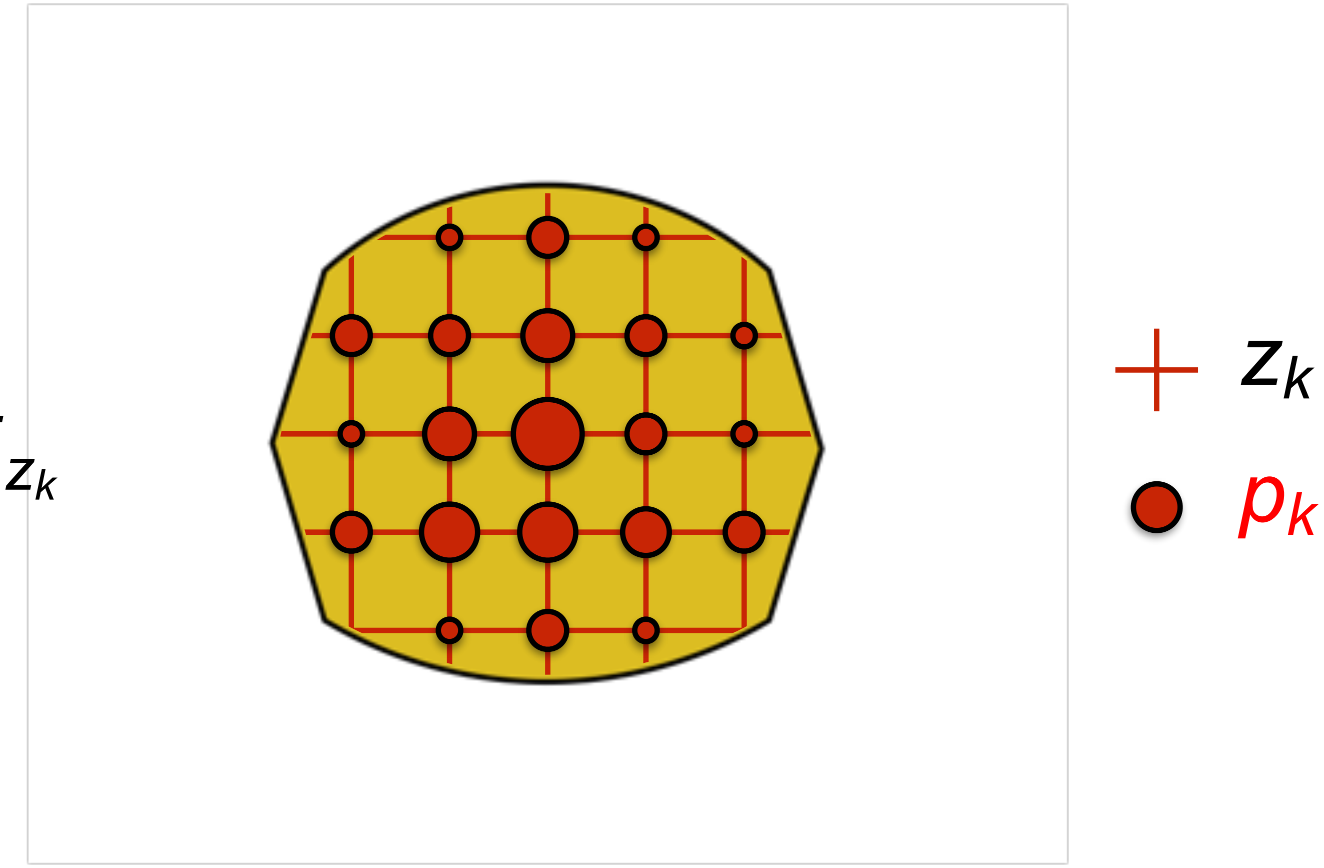


Uncertainty Quantification

$$\sup_{p \geq 0} \quad \sum_{k=1}^{K} p_k \, g(z_k)$$

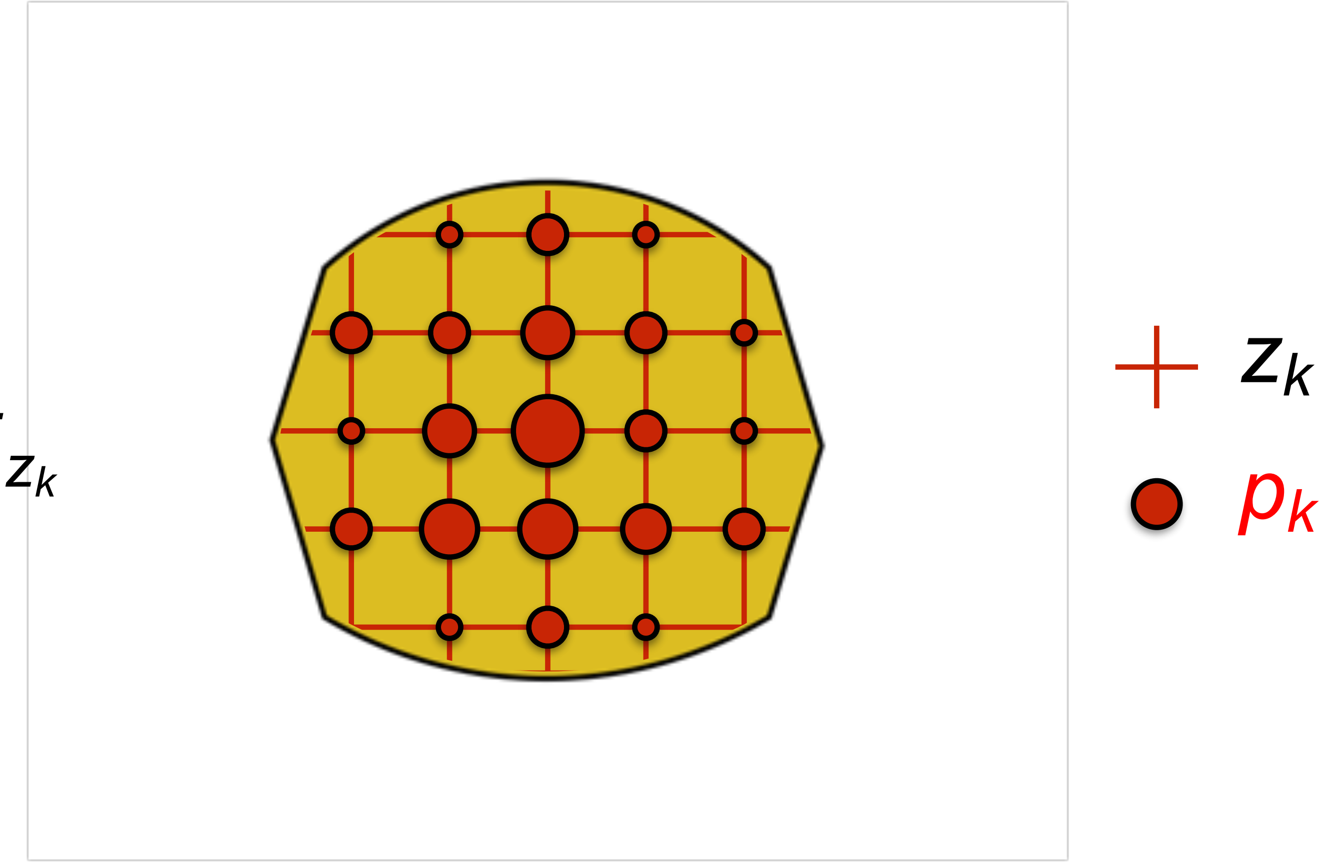
$$\text{s.t.} \quad \sum_{k=1}^{K} p_k = 1 \quad \longleftarrow \quad \alpha$$

$$\sum_{k=1}^{K} p_k \, h(z_k) \leq \mu \quad \longleftarrow \quad \beta$$

Lagrange multipliers

Discretization:

$$\mathbb{P} = \sum_{k=1}^{K} p_k \delta_{z_k}$$



Uncertainty Quantification

$$\inf_{\alpha,\beta\geq 0} \quad \alpha + \mu^\top \beta$$
$$\text{s.t.} \quad g(z_k) - \alpha - h(z_k)^\top \beta \leq 0 \quad \forall k = 1, \ldots, K$$

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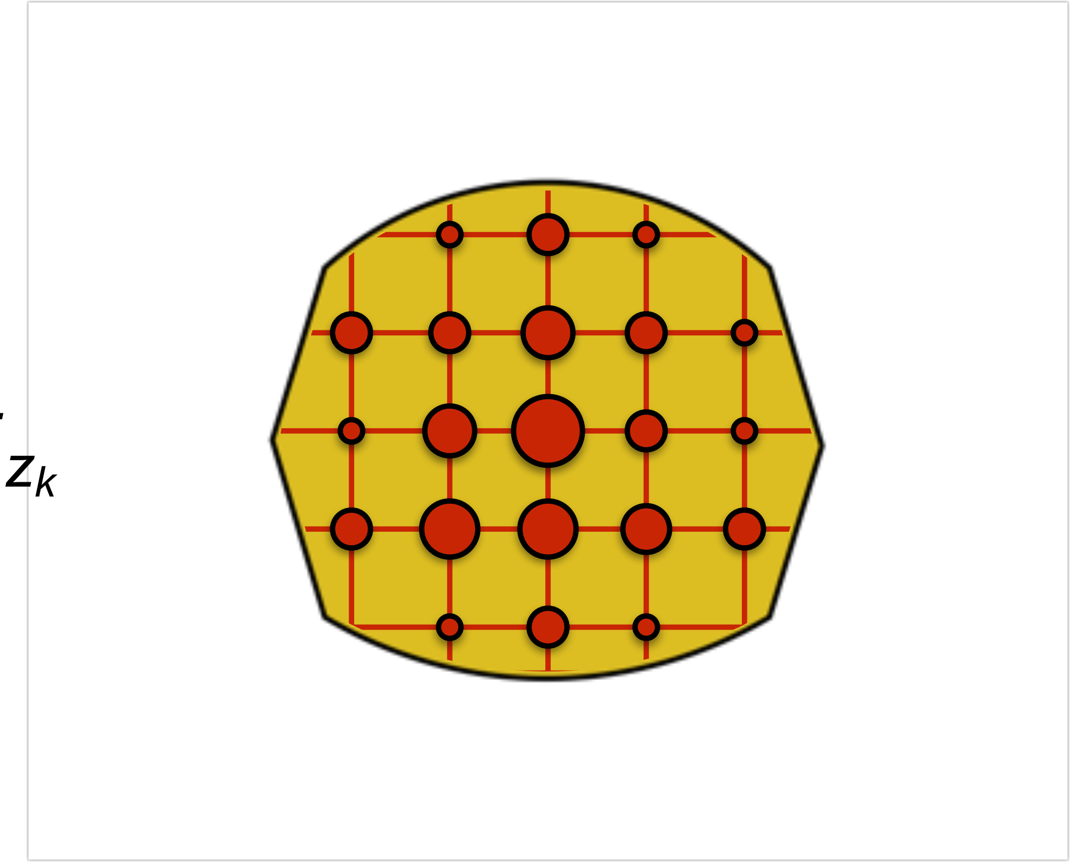

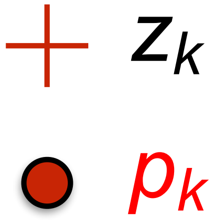

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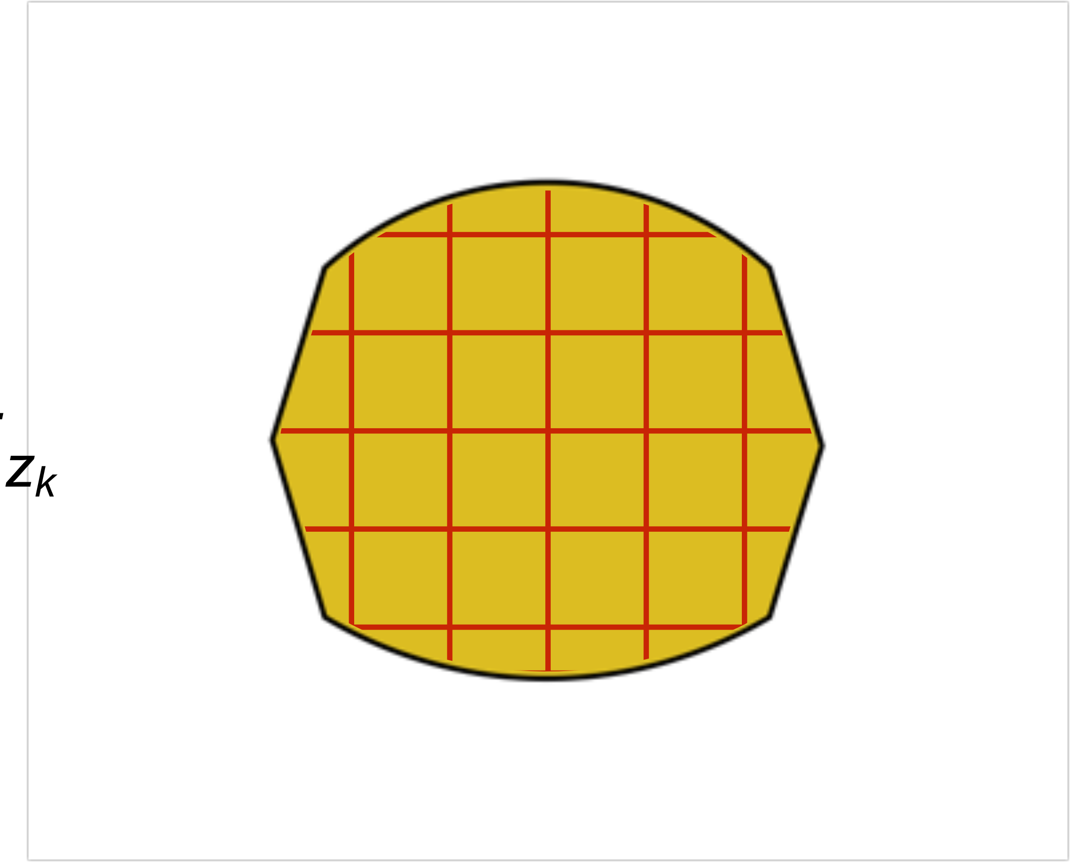

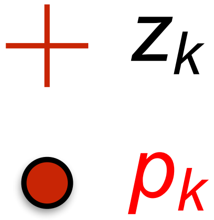

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$$\begin{array}{ll} \inf\limits_{\alpha,\beta\geq 0} & \alpha + \mu^\top \beta \\ \text{s.t.} & g(z_k) - \alpha - h(z_k)^\top \beta \leq 0 \quad \forall k = 1, \ldots, K \end{array}$$

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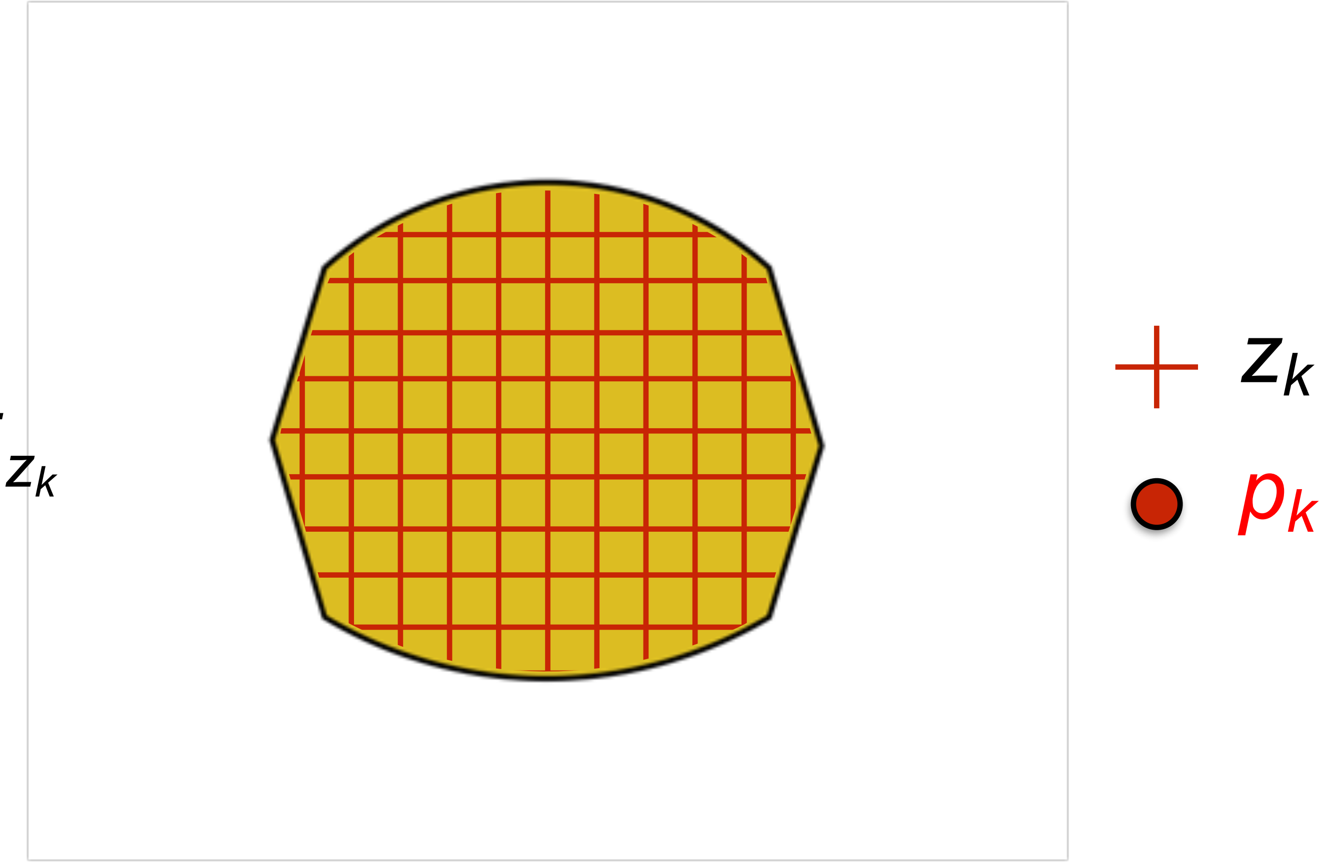

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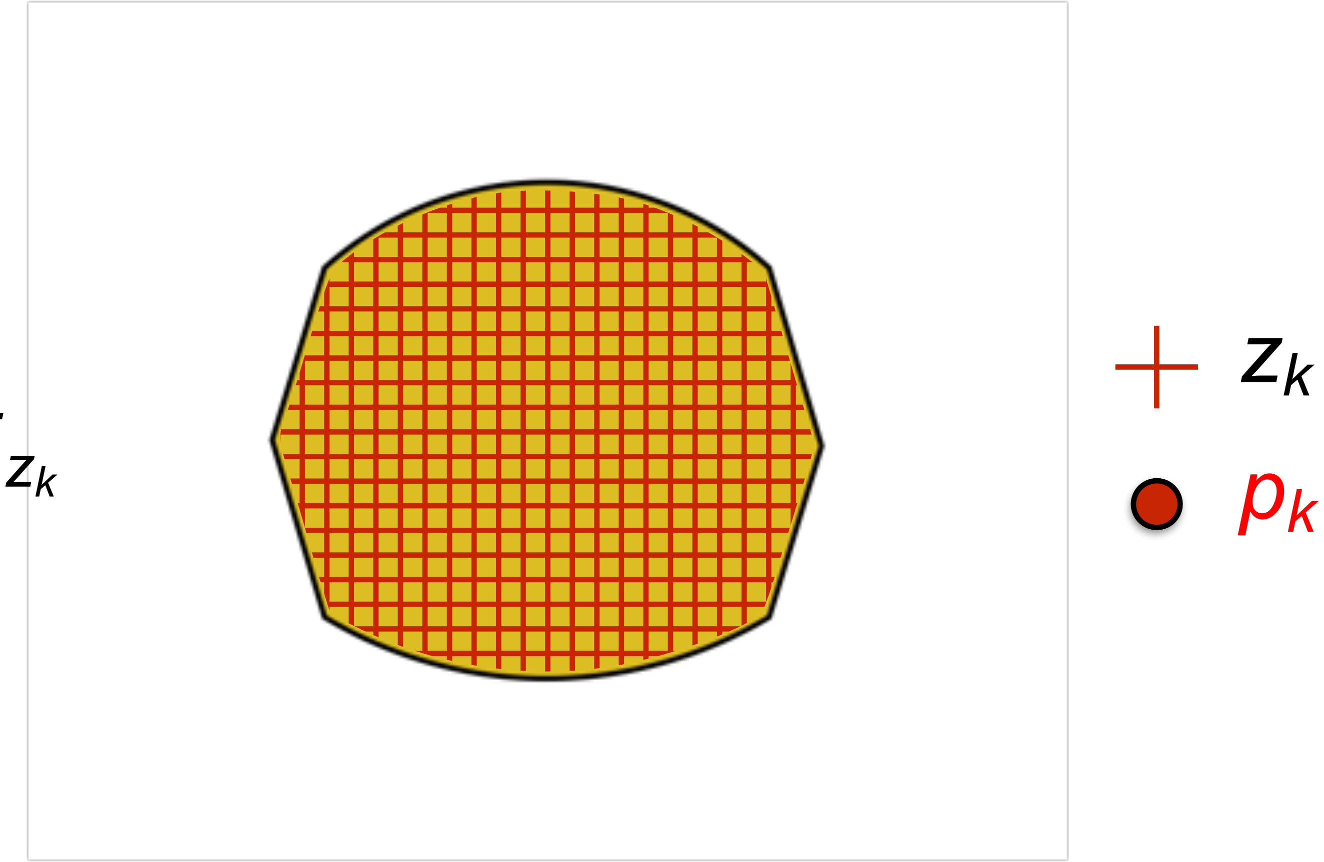

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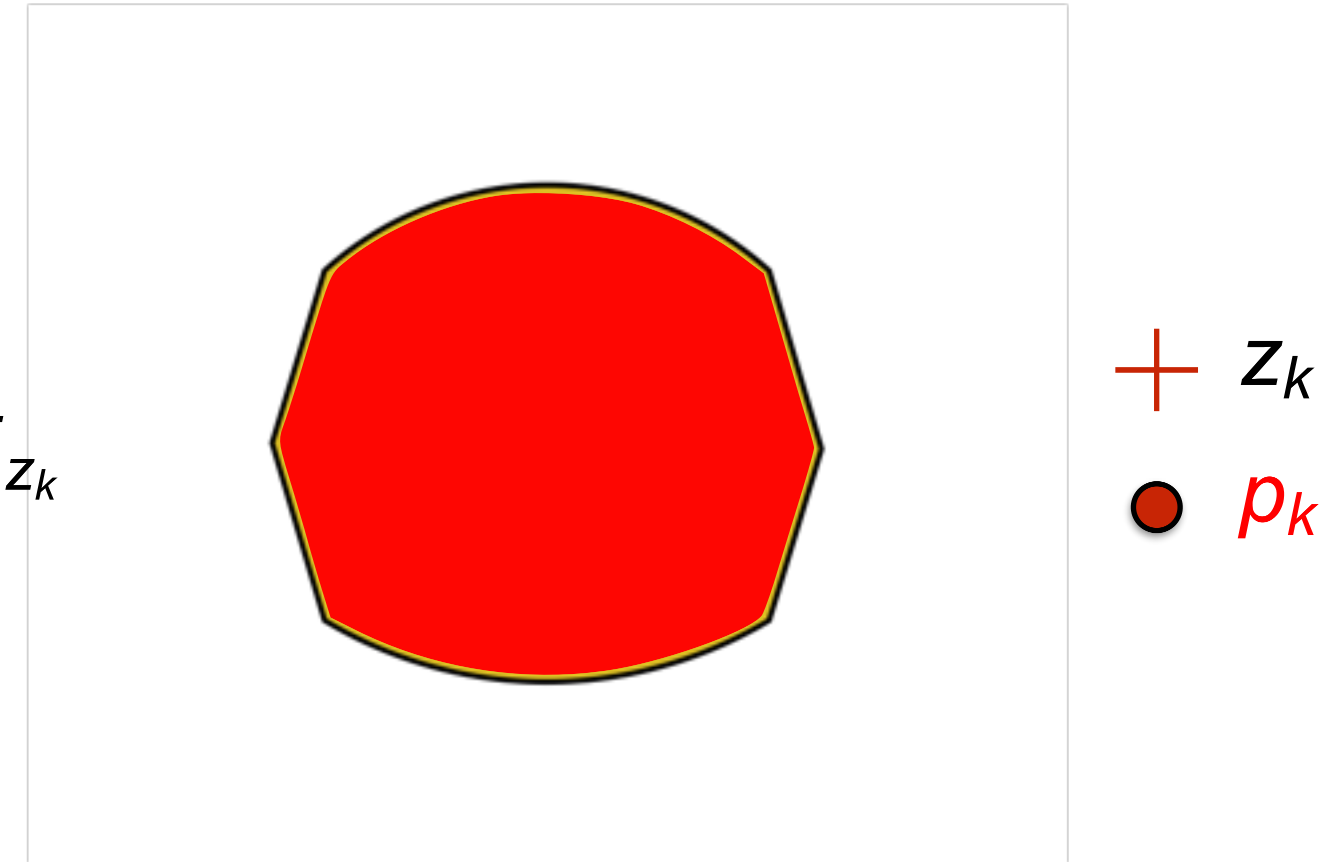


Uncertainty Quantification

$$\begin{aligned} \inf_{\alpha,\beta\geq 0} \quad & \alpha + \mu^\top \beta \\ \text{s.t.} \quad & g(z) - \alpha - h(z)^\top \beta \leq 0 \quad \forall z \in \mathcal{S} \end{aligned} \qquad \text{(D-UQ)}$$

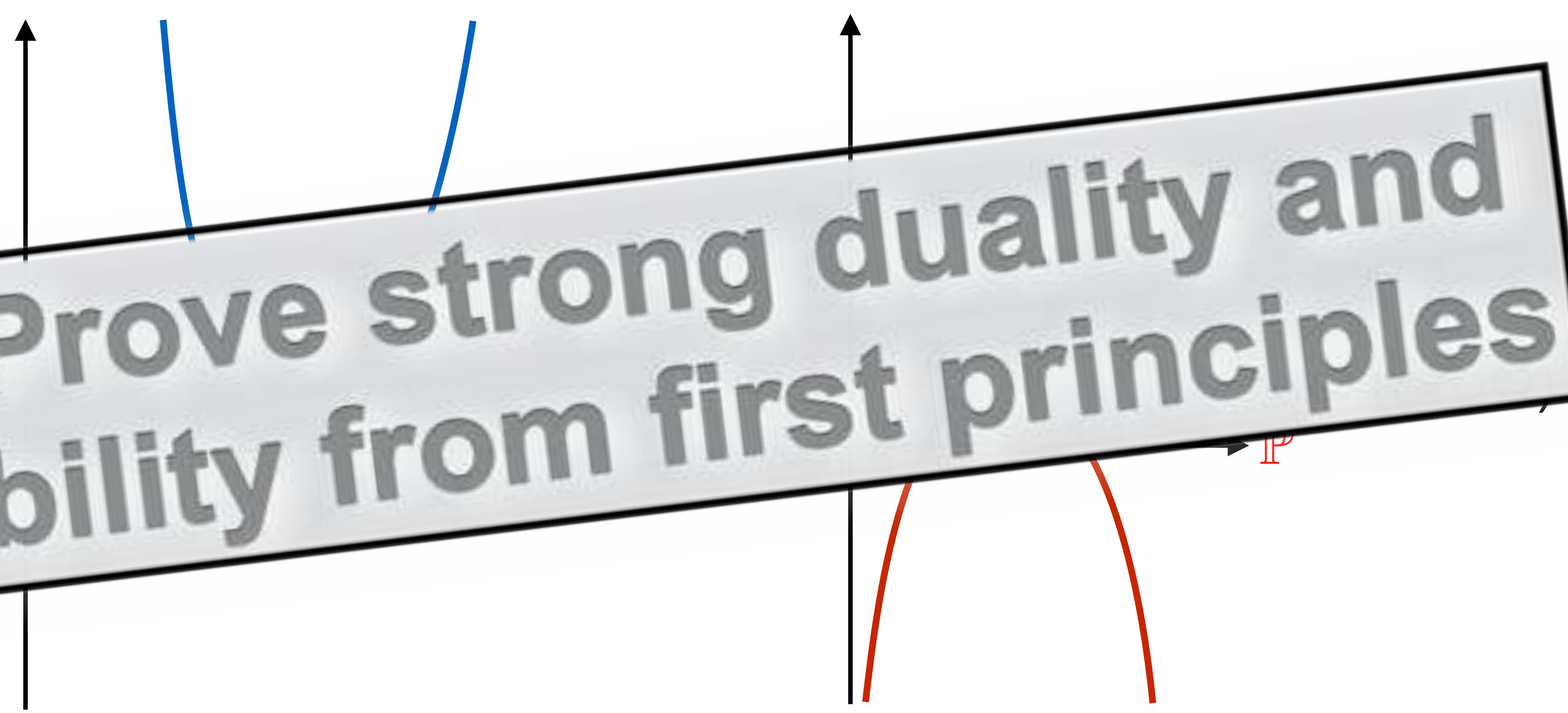
(D-UQ) is a RO problem akin to (P-W) with

- decisions α and β
- uncertain parameter z
- uncertainty set $\mathcal{S}$

Semi-Infinite Duality Theory

Want to prove strong duality between (P-UQ) and (D-UQ), but…

- conditions of semi-infinite duality theory are difficult to check
- need extra conditions for tractability of (P-UQ) and (D-UQ)



Aim: Prove strong duality and tractability from first principles

Finite Reduction

$$\begin{array}{lll} \sup\limits_{z,\lambda\geq 0} & \sum_{i=1}^{I} \lambda_i \, g(z_i) & \\ \text{s.t.} & \sum_{i=1}^{I} \lambda_i = 1 & \text{(FR)} \\ & \sum_{i=1}^{I} \lambda_i \, h(z_i) \leq \mu & \\ & z_i \in \mathcal{S} \quad \forall i = 1, \ldots, I & \end{array}$$

(FR) restricts $\mathcal{P}$ to distributions of the form $\mathbb{P} = \sum_{i=1}^{I} \lambda_i \delta_{z_i}$

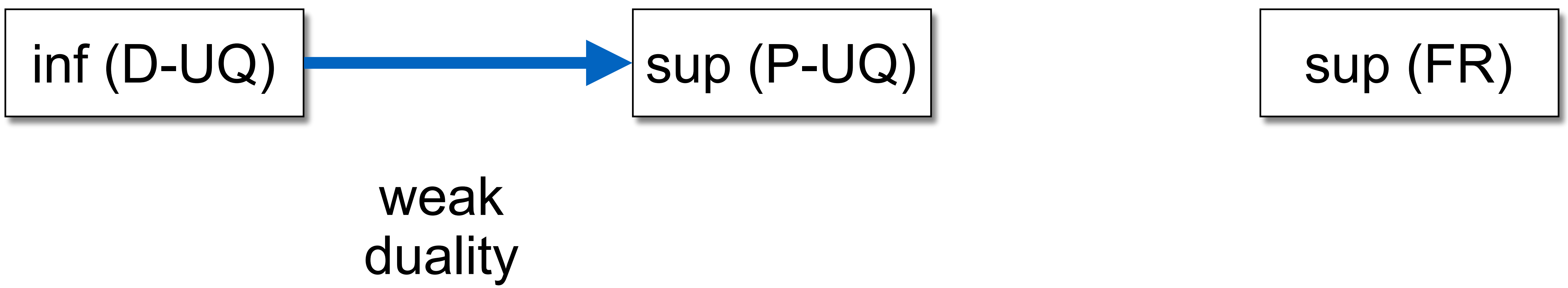
Universal Inequalities

inf (D-UQ)

sup (P-UQ)

sup (FR)

Universal Inequalities



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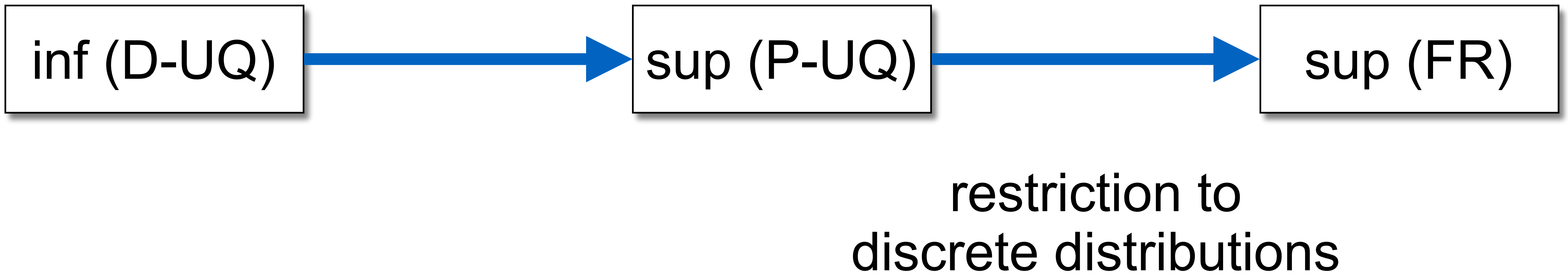

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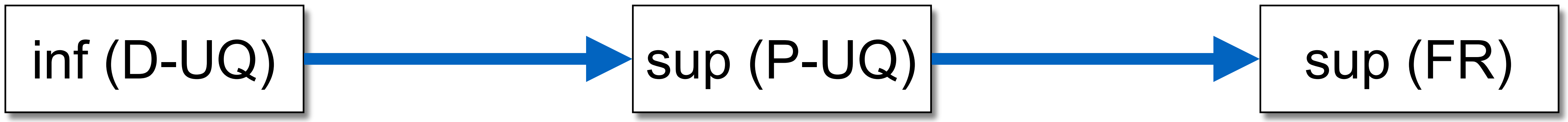

Idea:

- Interpret (D-UQ) as (P-W)
- interpret (FR) as (D-B)

Interpreting (D-UQ) as (P-W)

$$\begin{aligned}\inf_{\alpha,\beta\geq 0} \quad & \alpha + \mu^\top \beta \\ \text{s.t.} \quad & g(z) - \alpha - h(z)^\top \beta \leq 0 \quad \forall z \in \mathcal{S}\end{aligned}$$

Interpreting (D-UQ) as (P-W)

$$\begin{aligned}
\inf_{\alpha,\beta \geq 0} \quad & \alpha + \mu^\top \beta \\
\text{s.t.} \quad & \max_{i=1,\ldots,I} g_i(z) - \alpha - h(z)^\top \beta \leq 0 \quad \forall z \in \mathcal{S}
\end{aligned}$$

Interpreting (D-UQ) as (P-W)

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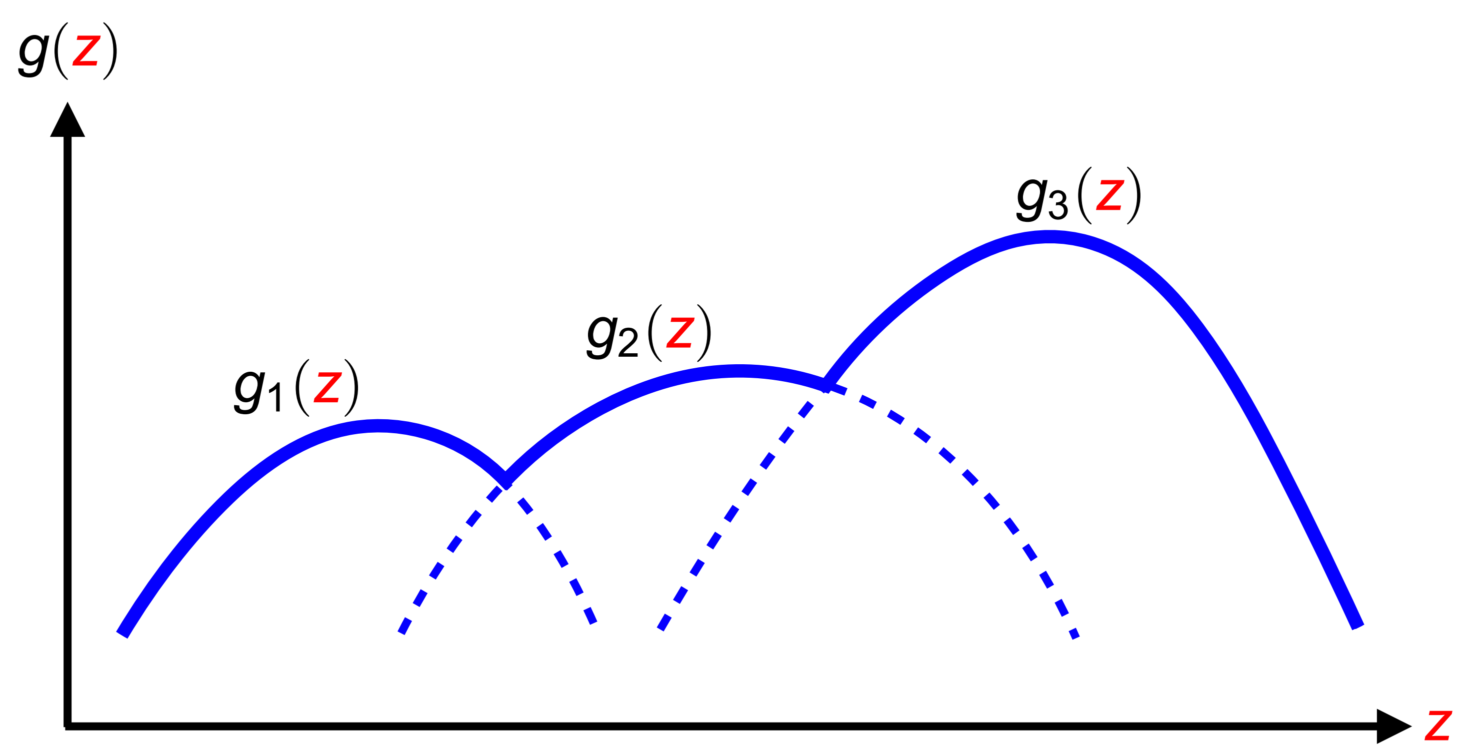
not concave in z!



Interpreting (D-UQ) as (P-W)

$$\begin{aligned} \inf_{\alpha,\beta\geq 0} \quad & \alpha + \mu^\top \beta \\ \text{s.t.} \quad & g_i(z_i) - \alpha - h(z_i)^\top \beta \leq 0 \quad \forall z_i \in \mathcal{S}, \ \forall i = 1, \ldots, I \end{aligned}$$

Interpreting (D-UQ) as (P-W)

$$\begin{aligned} \inf_{\alpha,\beta\geq 0} \quad & \alpha + \mu^\top \beta \\ \text{s.t.} \quad & \underbrace{g_i(z_i) - \alpha - h(z_i)^\top \beta}_{= f_i((\alpha,\beta),z_i)} \leq 0 \quad \forall z_i \in \mathcal{S},\ \forall i = 1,\ldots,I \end{aligned}$$

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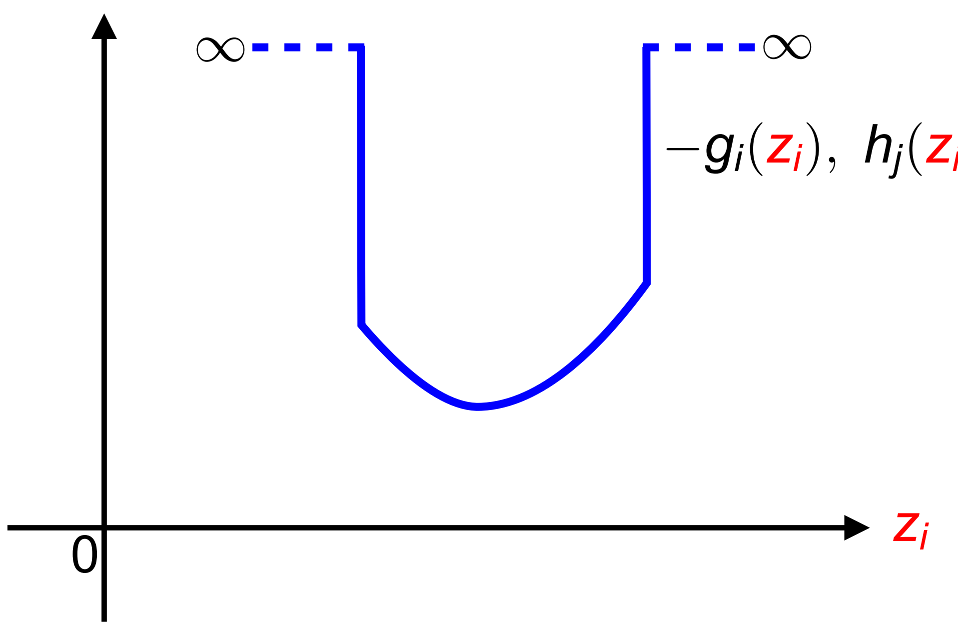

$\Longrightarrow \quad f_i((\alpha,\beta), z_i) = -\infty$ is possible

Interpreting (D-UQ) as (P-W)

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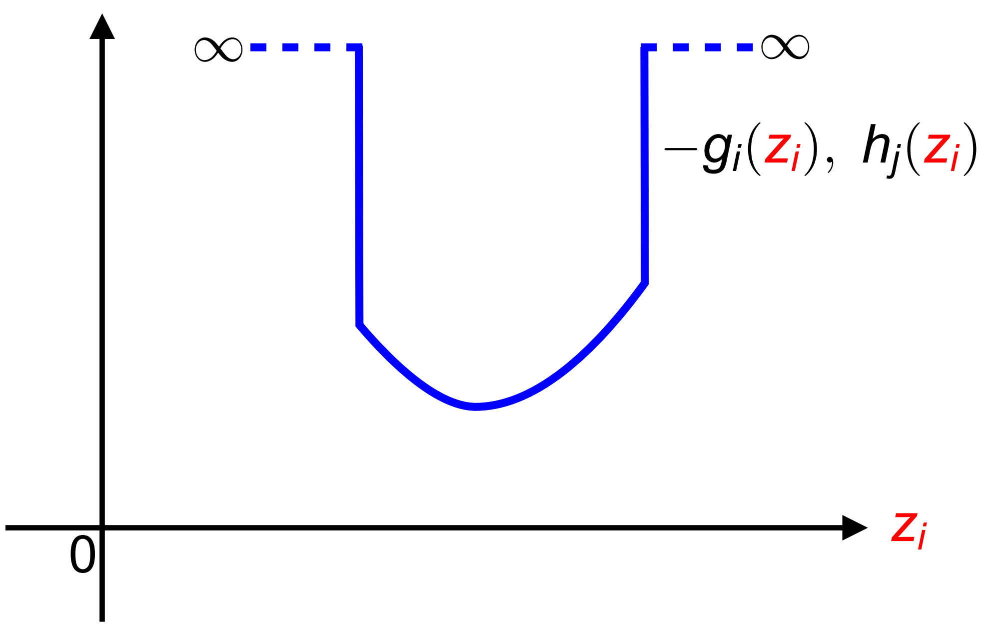


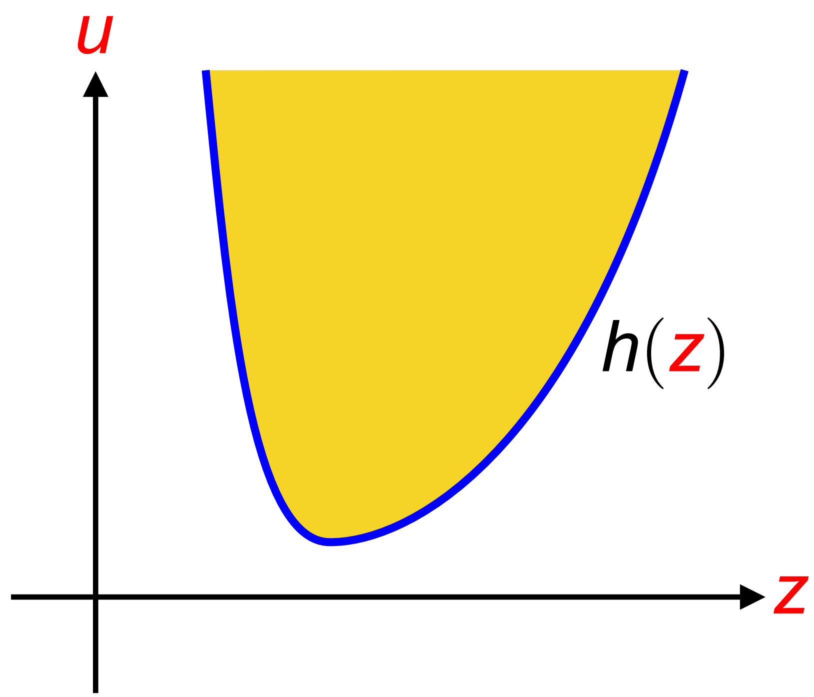
$\Longrightarrow$ $f_i((\alpha,\beta),z_i) = -\infty$ is possible

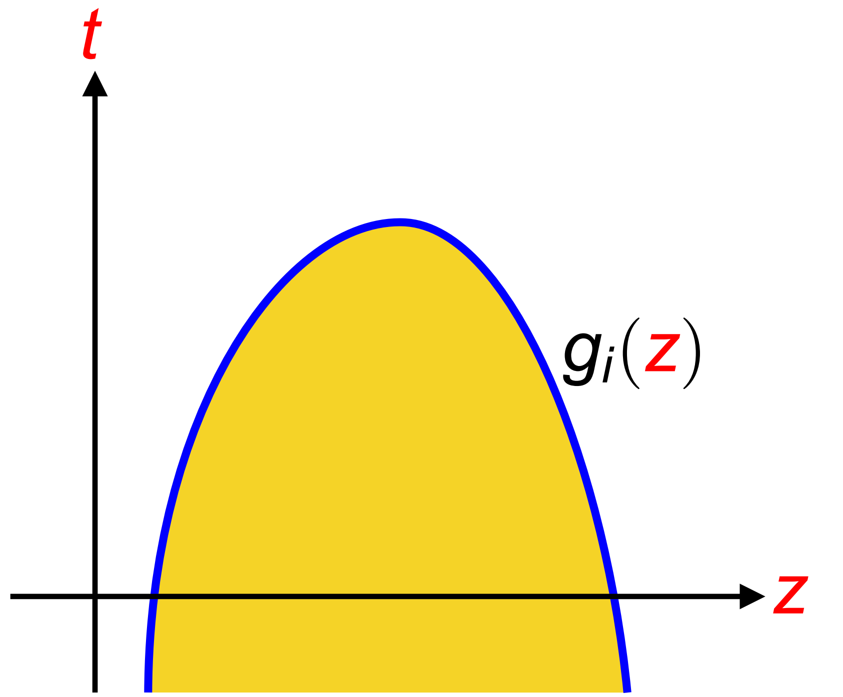
$\Longrightarrow$ $f_i((\alpha,\beta),z_i)$ is <u>*not*</u> pcc in (α,β)

Ambiguous Primal-Worst

$$\inf_{\alpha,\beta\geq 0} \quad \alpha + \mu^\top \beta$$
$$\text{s.t.} \quad t_i - \alpha - u_i^\top \beta \leq 0 \quad \forall (z_i, u_i, t_i) \in \mathcal{U}_i \quad \forall i = 1, \ldots, I \qquad \text{(AP-W)}$$

$$\mathcal{U}_i = \left\{ (z, u, t) : c_\ell(z) \leq 0 \ \forall \ell, \ h(z) \leq u, \ g_i(z) \geq t \right\} \quad \forall i$$





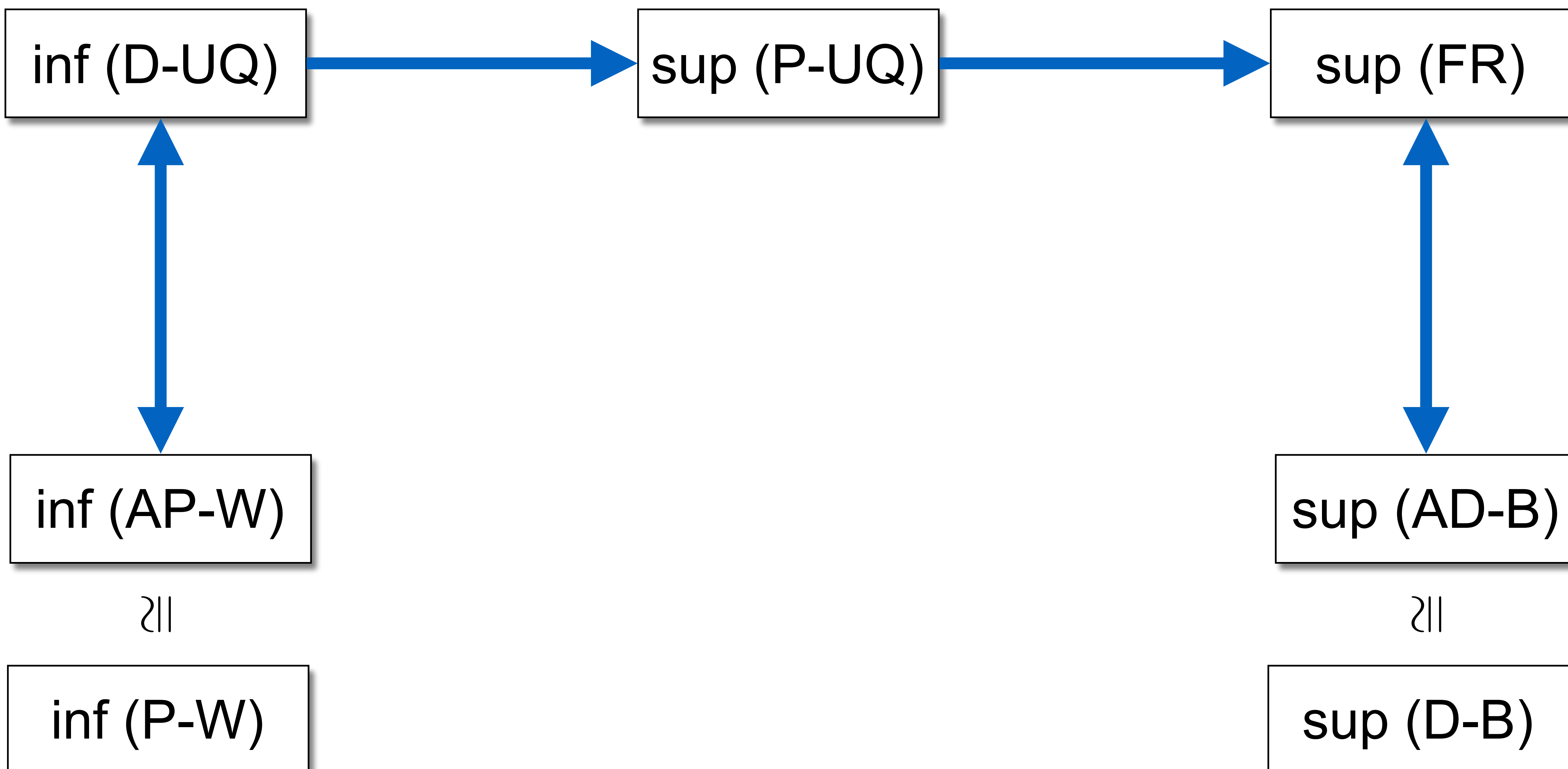
Interpreting (FR) as (D-B)

$$\begin{aligned}
\sup_{z,\lambda \geq 0} \quad & \sum_{i=1}^{l} \lambda_i \, g(z_i) \\
\text{s.t.} \quad & \sum_{i=1}^{l} \lambda_i = 1 \\
& \sum_{i=1}^{l} \lambda_i \, h(z_i) \leq \mu \\
& z_i \in \mathcal{S} \quad \forall i = 1, \ldots, l
\end{aligned}$$

Ambiguous Dual-Best

$$
\begin{aligned}
\sup_{z,u,t,\lambda \geq 0} \quad & \sum_{i=1}^{I} \lambda_i t_i \\
\text{s.t.} \quad & \sum_{i=1}^{I} \lambda_i = 1 \\
& \sum_{i=1}^{I} \lambda_i u_i \leq \mu \\
& (z_i, u_i, t_i) \in \mathcal{U}_i \quad \forall i = 1, \ldots, I
\end{aligned}
\qquad \text{(AD-B)}
$$

Universal Inequalities



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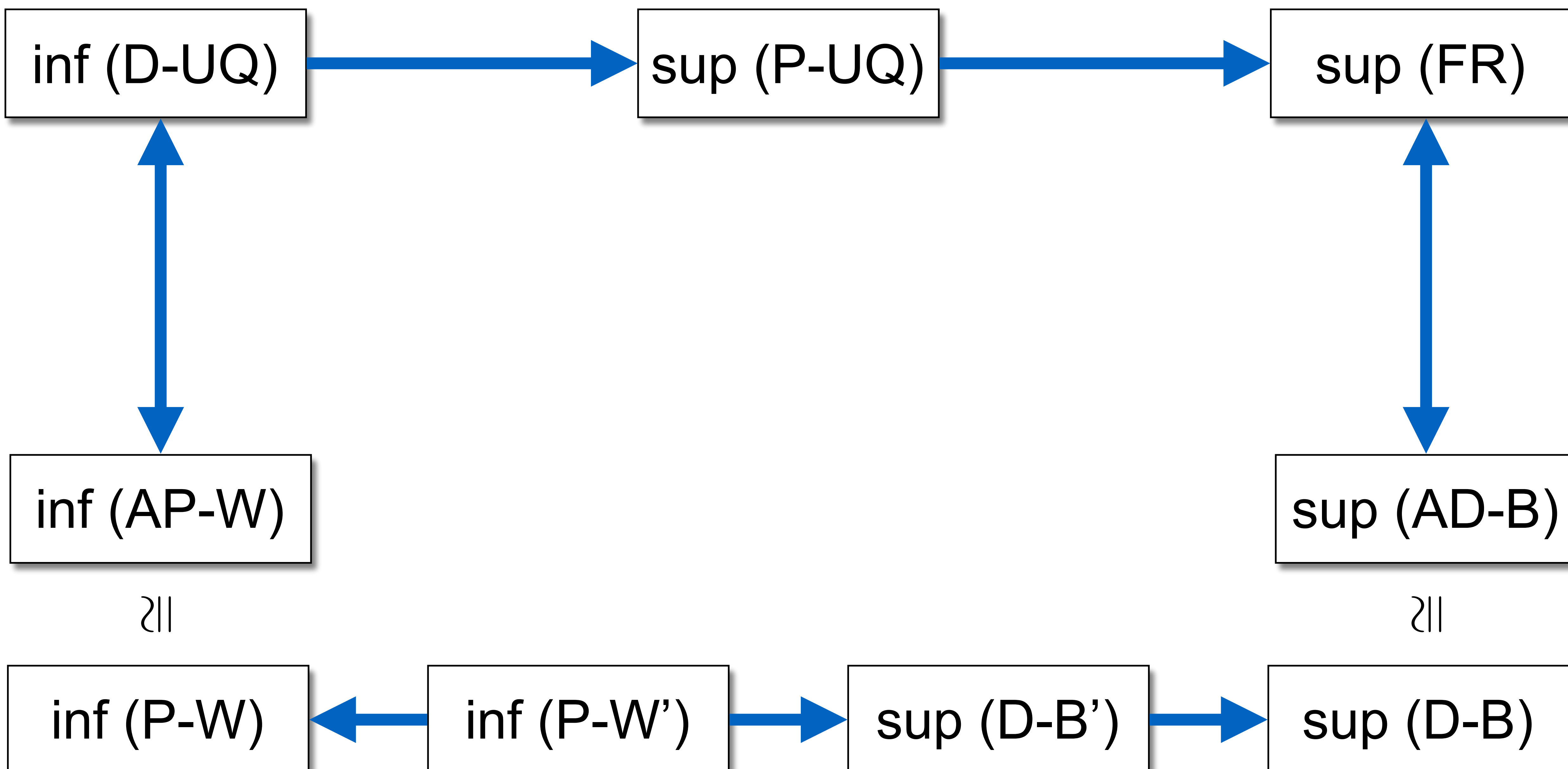

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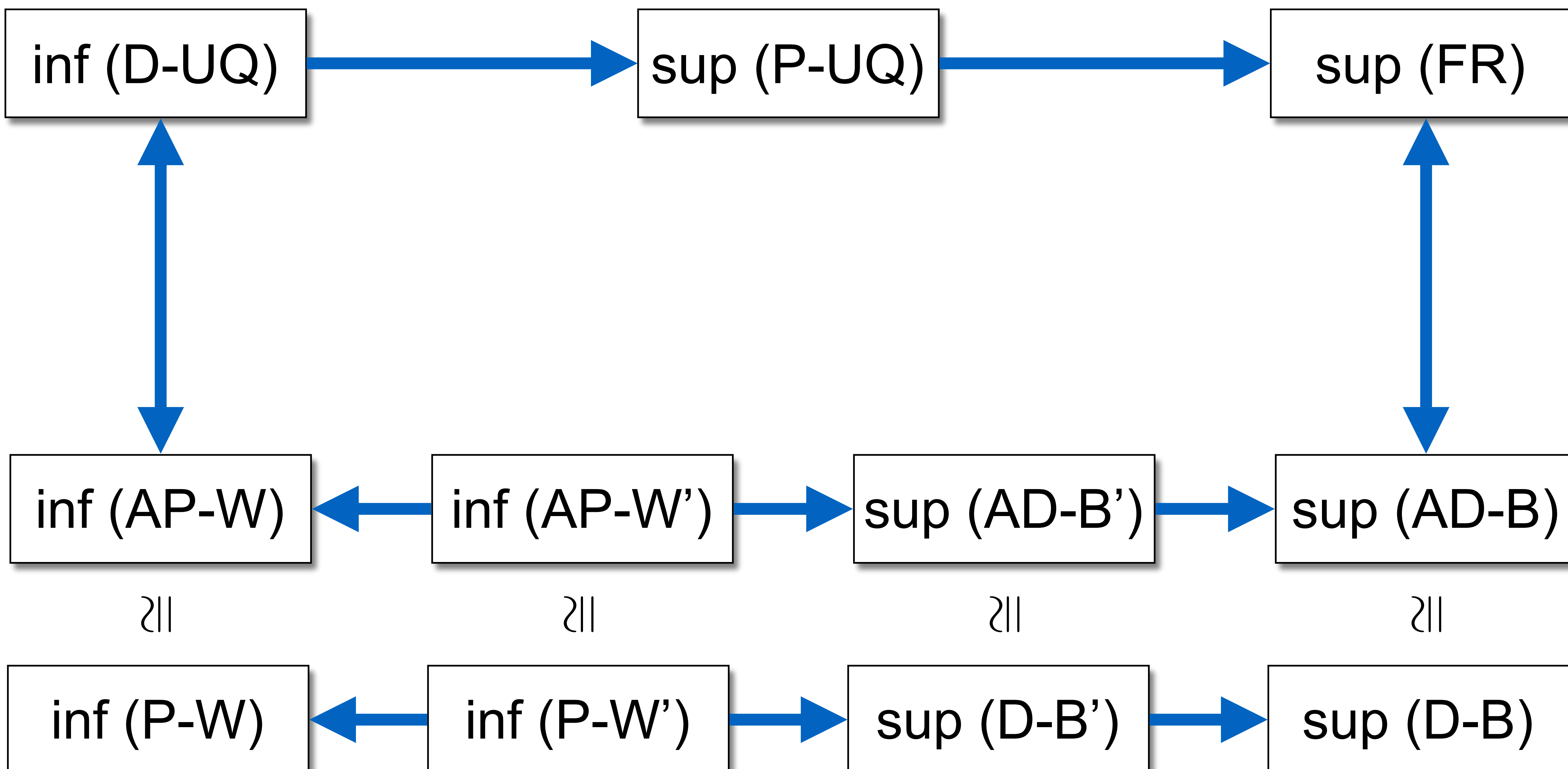


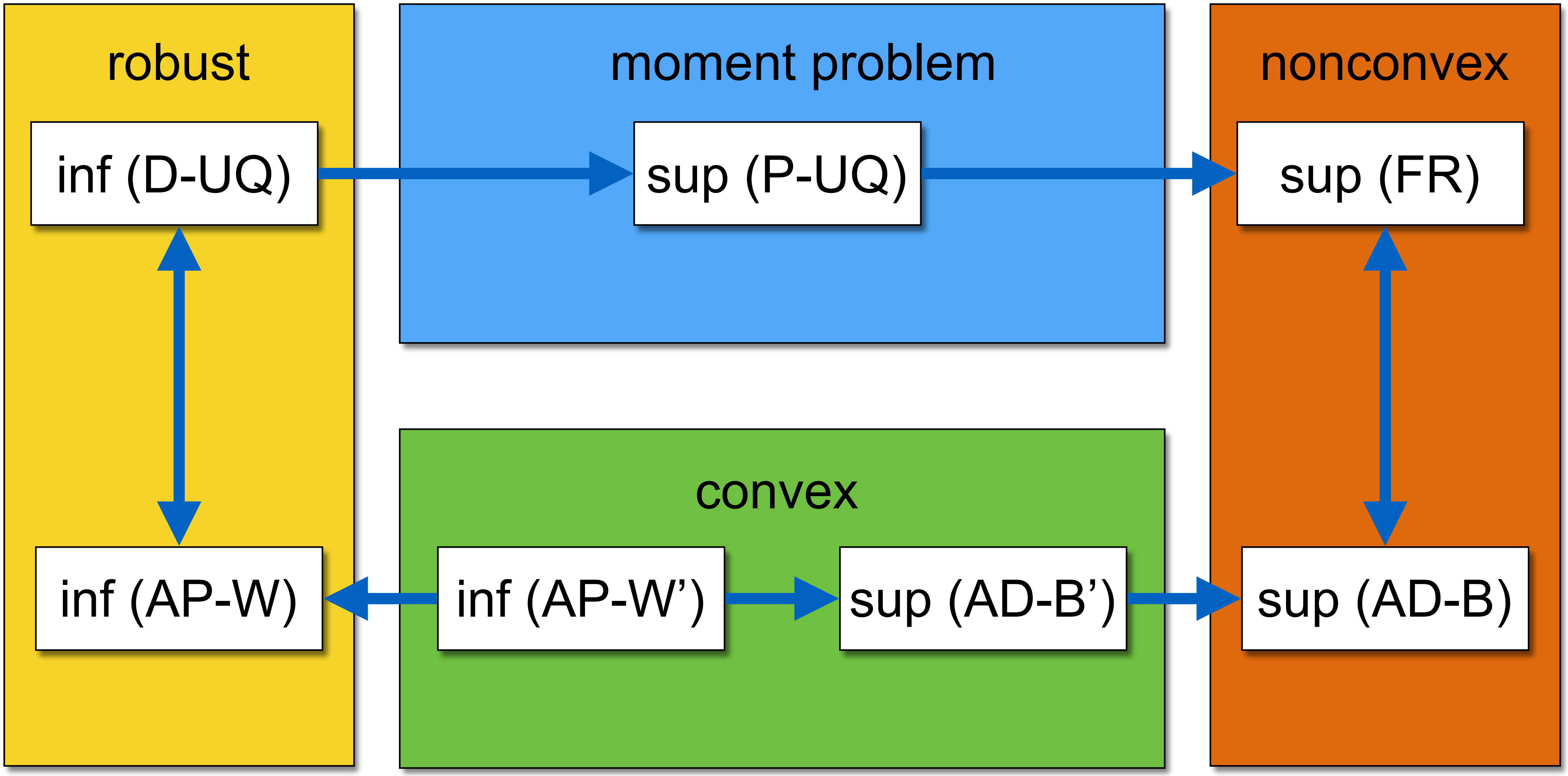
Convex Reformulation of (AP-W)

$$
\begin{aligned}
\inf_{\substack{\alpha,\beta\geq 0\\ y,v\geq 0}} \quad & \alpha + \mu^\top \beta \\
\text{s.t.} \quad & (-g_i)^*\left(y_i^{(0)}\right) + \sum_{j=1}^{J} \beta_j h_j^*\left(y_{ij}^{(1)}/\beta_j\right) \\
& \quad + \sum_{\ell=1}^{L} v_{i\ell} c_\ell^*\left(y_{i\ell}^{(2)}/v_{i\ell}\right) \leq \alpha \qquad \forall i = 1,\ldots,I \\
& y_i^{(0)} + \sum_{j=1}^{J} y_{ij}^{(1)} + \sum_{\ell=1}^{L} y_{i\ell}^{(2)} = 0 \qquad \forall i = 1,\ldots,I
\end{aligned}
\qquad \text{(AP-W')}
$$

Convex Reformulation of (AD-B)

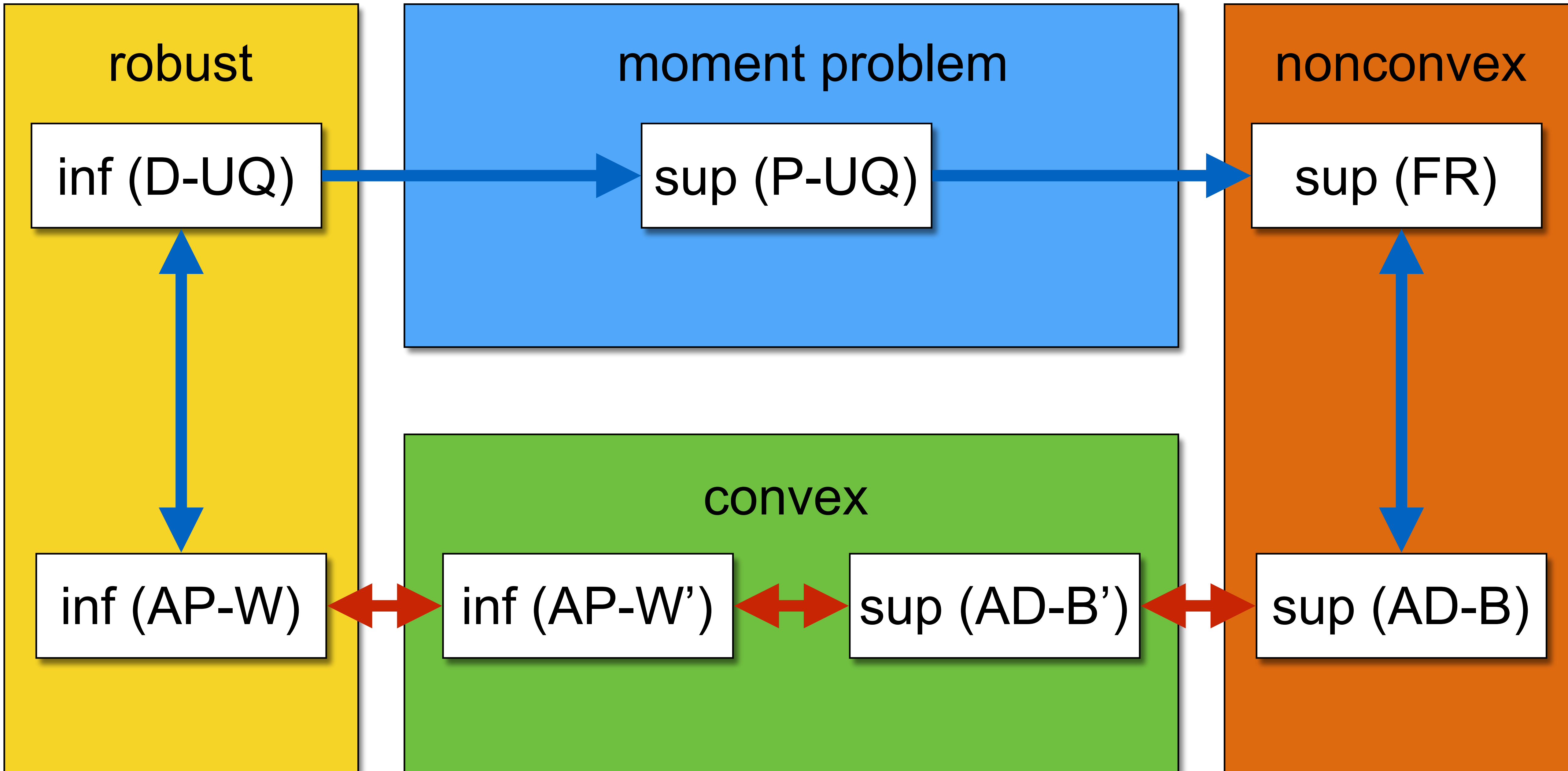
$$
\begin{aligned}
\sup_{v,\omega,\tau,\lambda\geq 0} \quad & \textstyle\sum_{i=1}^{I} \tau_i \\
\text{s.t.} \quad & \textstyle\sum_{i=1}^{I} \lambda_i = 1 \\
& \textstyle\sum_{i=1}^{I} \omega_i \leq \mu \qquad\qquad \text{(AD-B')} \\
& \lambda_i c_\ell(v_i/\lambda_i) \leq 0 \quad \forall i = 1,\ldots,I,\ \forall \ell = 1,\ldots,L \\
& \lambda_i h(v_i/\lambda_i) \leq \omega_i \quad \forall i = 1,\ldots,I \\
& \lambda_i g_i(v_i/\lambda_i) \geq \tau_i \quad \forall i = 1,\ldots,I
\end{aligned}
$$

Strong Semi-Infinite Duality



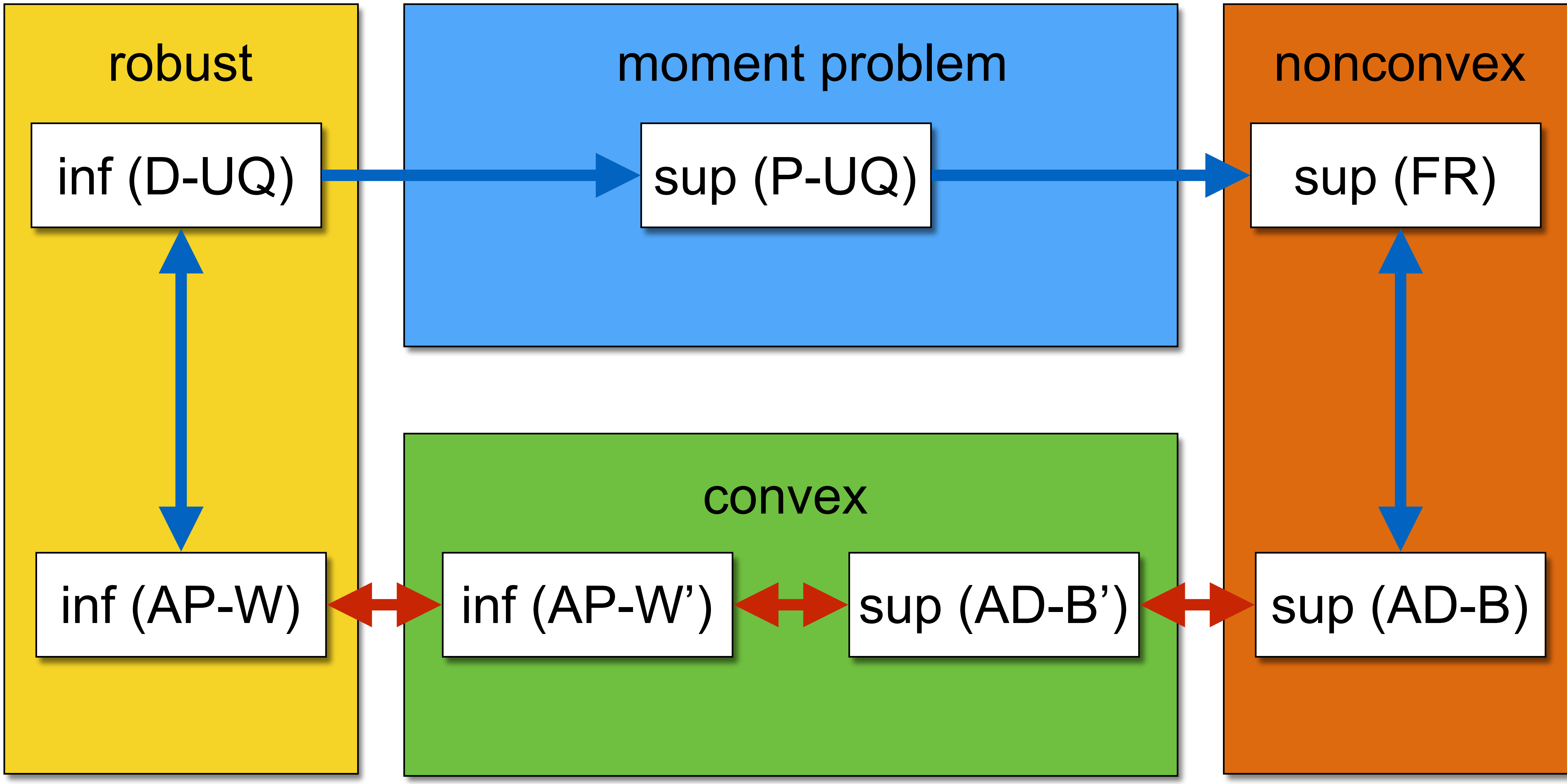
Strong Semi-Infinite Duality

If (AD-B) has a Slater point…



Strong Semi-Infinite Duality

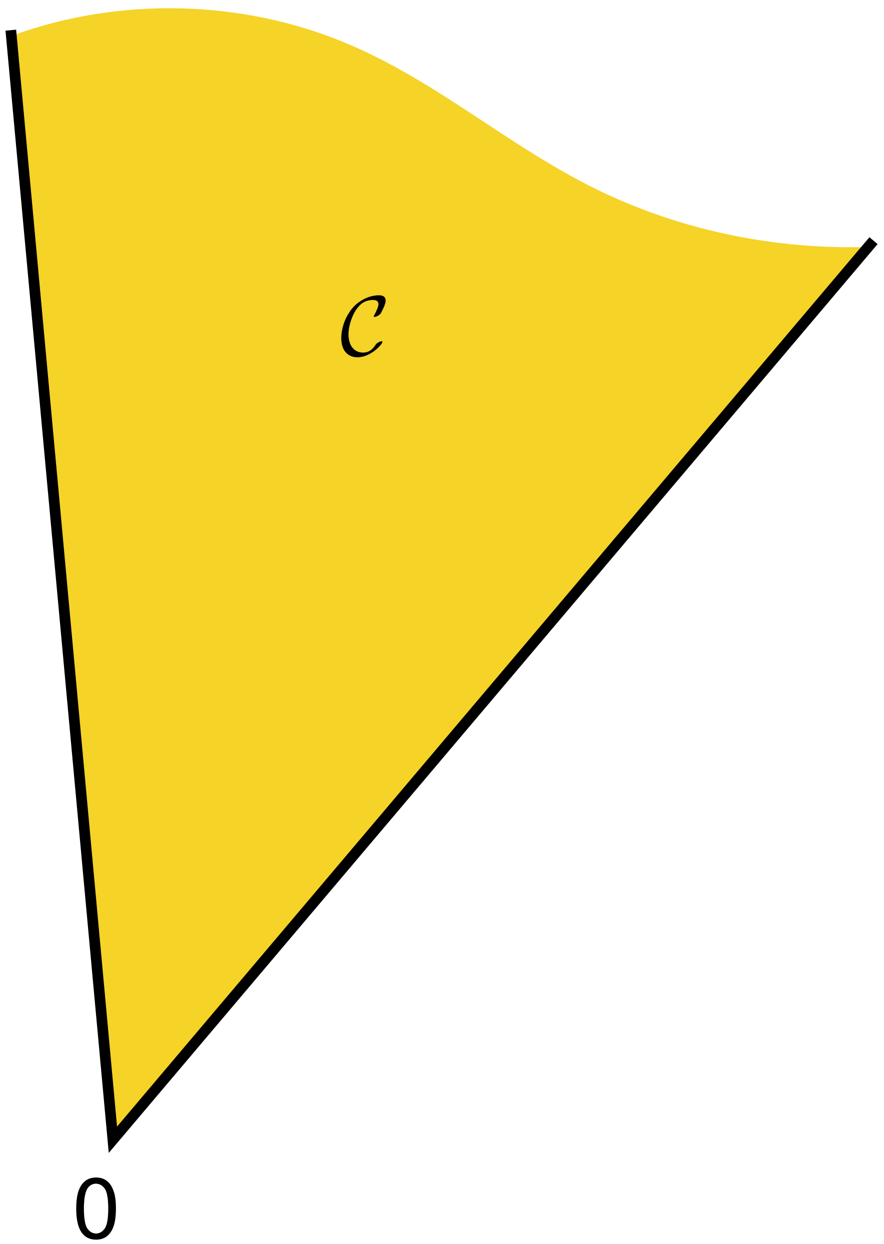
If (AP-W) has a Slater point and $\mathcal{S}$ is bounded…



Extensions

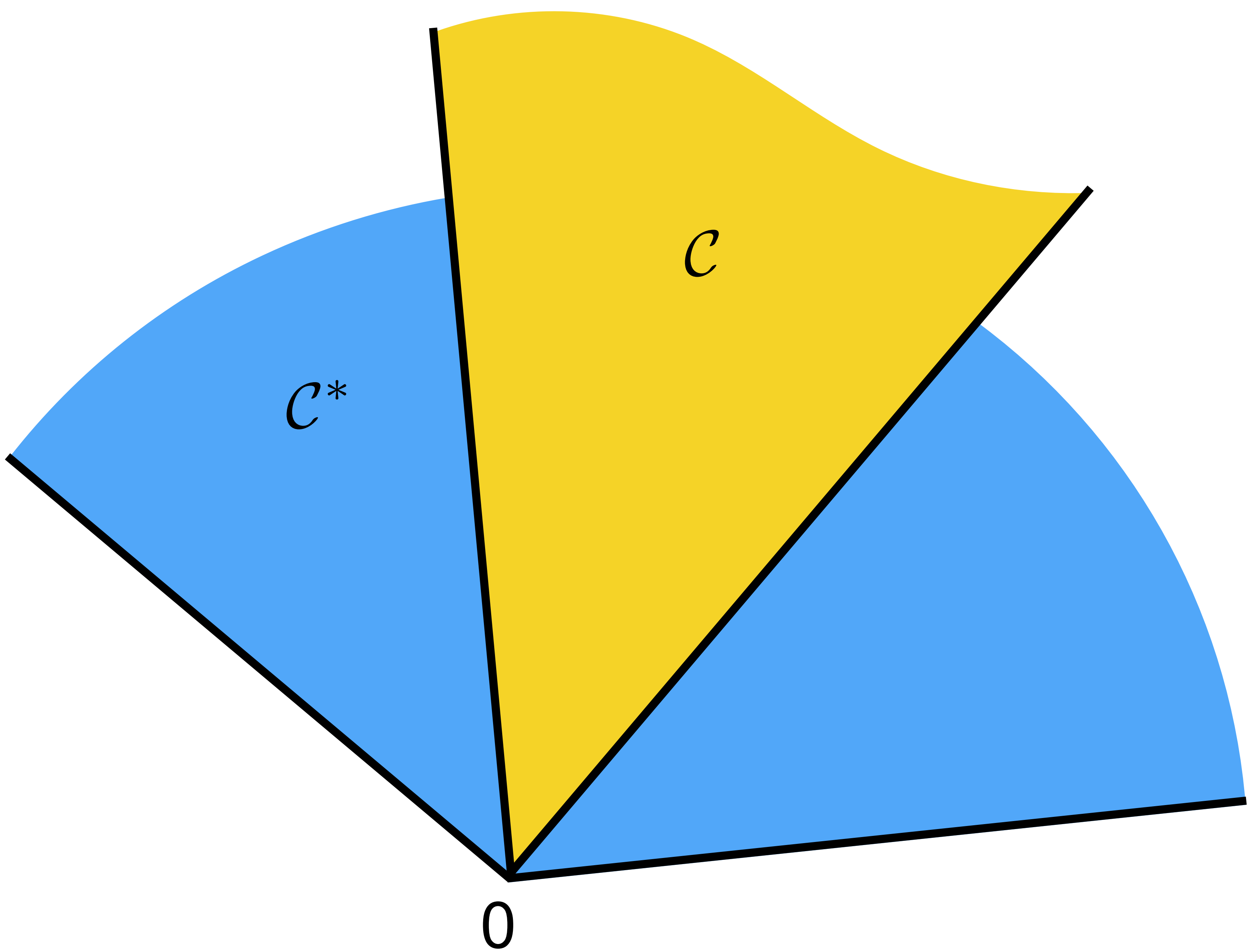
Dual Cones

$\mathcal{C} =$ proper convex cone



Dual Cones

$\mathcal{C} =$ proper convex cone $\qquad \mathcal{C}^* = \{\lambda : \lambda^\top x \geq 0 \ \forall x \in \mathcal{C}\}$



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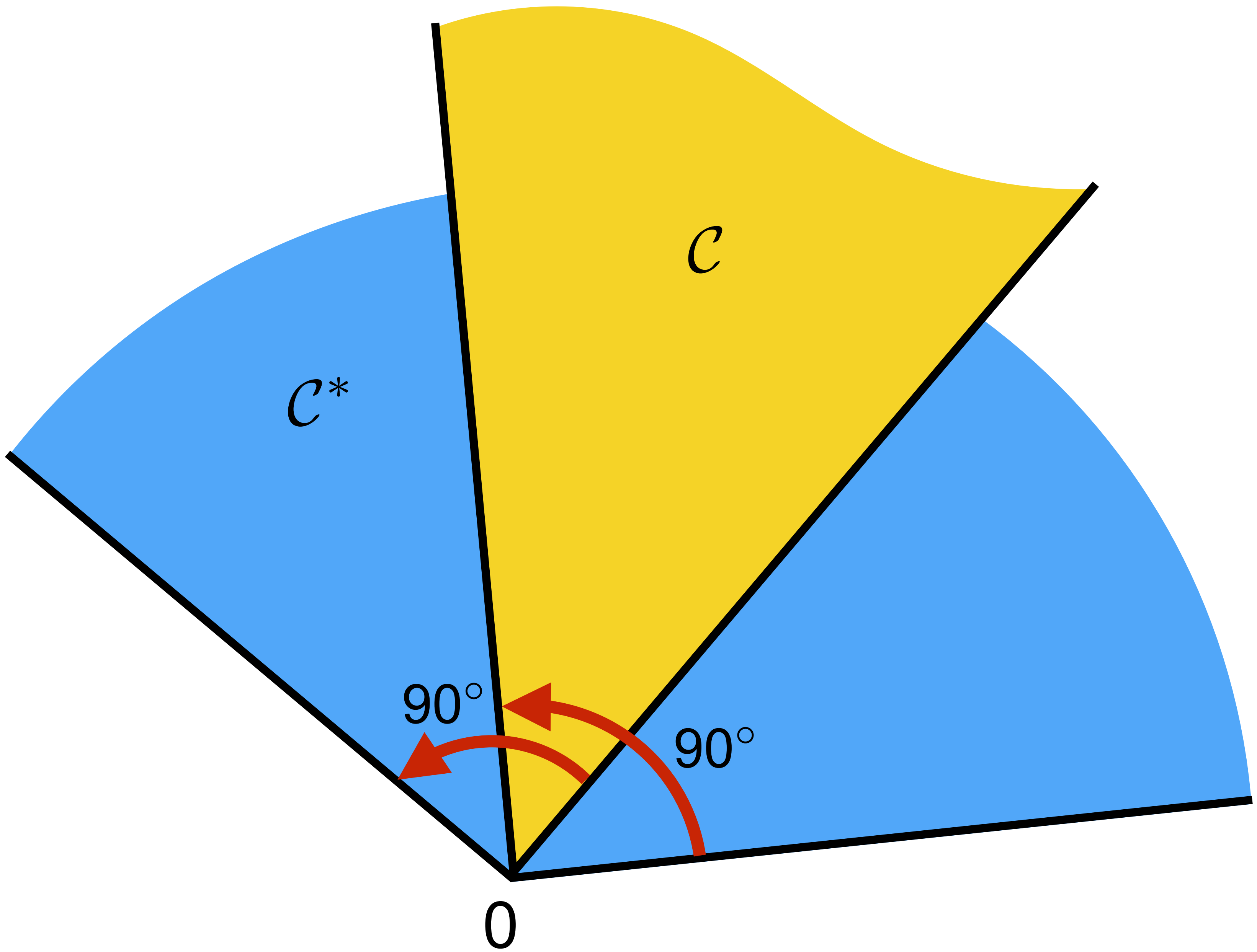

Generalized PCC Functions

Definition: Generalized convexity

$f(x)$ is $\mathcal{C}$-convex if $\lambda^\top f(x)$ is convex $\forall \lambda \in \mathcal{C}^*$

Generalized PCC Functions

Definition: Generalized convexity

$f(\boldsymbol{x})$ is $\mathcal{C}$-convex if $\boldsymbol{\lambda}^\top f(\boldsymbol{x})$ is convex $\forall \boldsymbol{\lambda} \in \mathcal{C}^*$

Definition: Generalized lower semicontinuity

$f(\boldsymbol{x})$ is star-$\mathcal{C}$ lsc if $\boldsymbol{\lambda}^\top f(\boldsymbol{x})$ is lsc $\forall \boldsymbol{\lambda} \in \mathcal{C}^*$

Generalized PCC Functions

Definition: Generalized convexity

$f(x)$ is $\mathcal{C}$-convex if $\lambda^\top f(x)$ is convex $\forall \lambda \in \mathcal{C}^*$

Definition: Generalized lower semicontinuity

$f(x)$ is star-$\mathcal{C}$ lsc if $\lambda^\top f(x)$ is lsc $\forall \lambda \in \mathcal{C}^*$

Definition: Generalized pcc functions

$f(x)$ is $\mathcal{C}$-pcc if it is proper, $\mathcal{C}$-convex and $\mathcal{C}$-lsc

Uncertainty Quantification

$$\sup_{\mathbb{P} \in \mathcal{P}_g} \mathbb{E}_{\mathbb{P}} \left[g(\tilde{z}) \right] \qquad \text{(P-UQ}_g\text{)}$$

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$$\mathcal{P}_g = \left\{ \mathbb{P} \text{ distribution on } \cup_{k=1}^{K} \mathcal{S}_k \;\middle|\; \begin{array}{ll} \mathbb{E}_{\mathbb{P}} \left[h_j(\tilde{z}) \right] \preceq_{\mathcal{H}_j} \mu_j & \forall j \\ \mathbb{P} \left[\tilde{z} \in \mathcal{S}_k \right] = p_k & \forall k \end{array} \right\}$$

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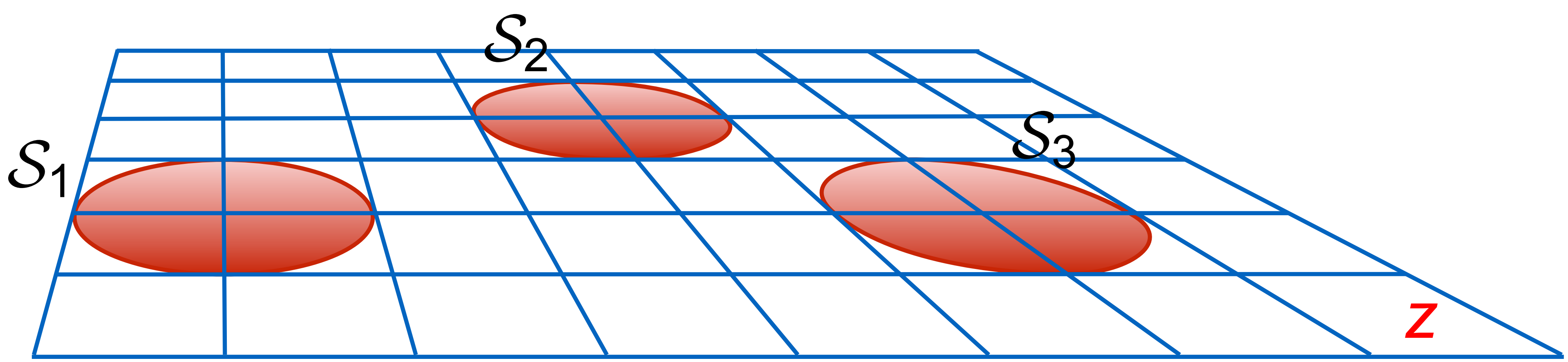

$$\mathcal{S}_k = \{z : c_{\ell k}(z) \preceq_{\mathcal{C}_{\ell k}} 0 \;\forall \ell\}$$

Uncertainty Quantification

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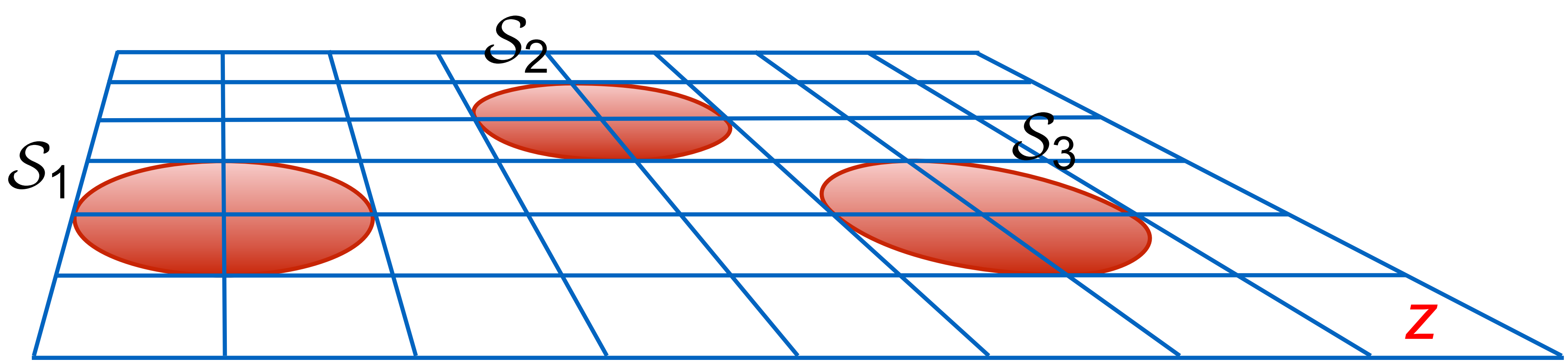

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$\mathcal{C}_{\ell k}$-pcc

Uncertainty Quantification

$$\sup_{\mathbb{P} \in \mathcal{P}_g} \mathbb{E}_{\mathbb{P}} \left[g(\tilde{z}) \right] \qquad \text{(P-UQ}_\text{g}\text{)}$$

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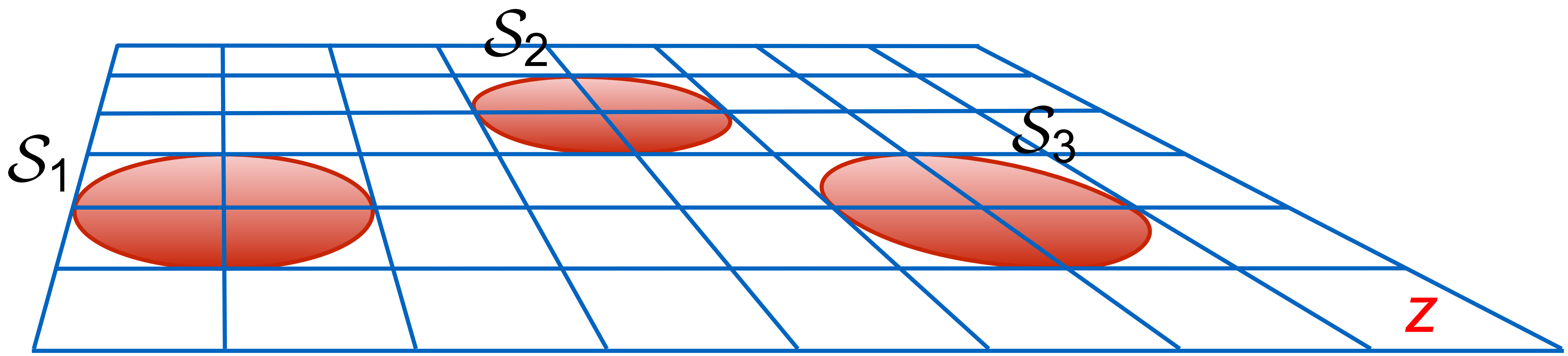

$$\mathcal{S}_k = \{ z : c_{\ell k}(z) \preceq_{\mathcal{C}_{\ell k}} 0 \; \forall \ell \} \qquad h_j(z) = h_{jk}(z) \quad \forall z \in \mathcal{S}_k, \; \forall j, \forall k$$

Uncertainty Quantification

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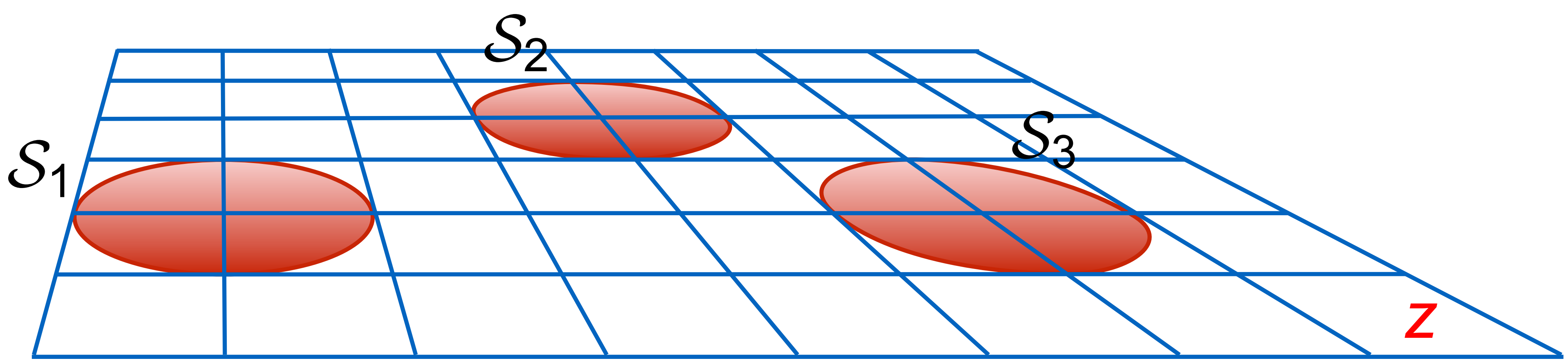

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$\mathcal{H}_j$-pcc

Uncertainty Quantification

$$\begin{aligned}
\inf_{\alpha,\beta} \quad & \sum_{k=1}^{K} p_k \alpha_k + \sum_{j=1}^{J} \mu_j^\top \beta_j \\
\text{s.t.} \quad & g(z_k) - \alpha - \sum_{j=1}^{J} h_{jk}(z_k)^\top \beta_j \leq 0 \quad \forall z_k \in \mathcal{S}_k,\ \forall k \qquad \text{(D-UQ}_\text{g}\text{)} \\
& \beta_j \in \mathcal{H}_j^* \quad \forall j
\end{aligned}$$

Finite Reduction

$$\begin{array}{llll} \sup\limits_{z,\lambda \geq 0} & \sum_{k=1}^{K} \sum_{i=1}^{I} \lambda_{ik}\, g(z_{ik}) & & \\ \text{s.t.} & \sum_{i=1}^{I} \lambda_{ik} = p_k & \forall k & \quad (\mathrm{FR_g}) \\ & \sum_{k=1}^{K} \sum_{i=1}^{I} \lambda_{ik}\, h_{jk}(z_{ik}) \preceq_{\mathcal{H}_j} \mu_j & \forall j & \\ & z_{ik} \in \mathcal{S}_k & \forall i,\ \forall k & \end{array}$$

(FR_g) restricts $\mathcal{P}_g$ to distributions of the form $\mathbb{P} = \sum_{k=1}^{K} \sum_{l=1}^{I} \lambda_{ik} \delta_{z_{ik}}$

Convex Reformulation of (AP-W_g)

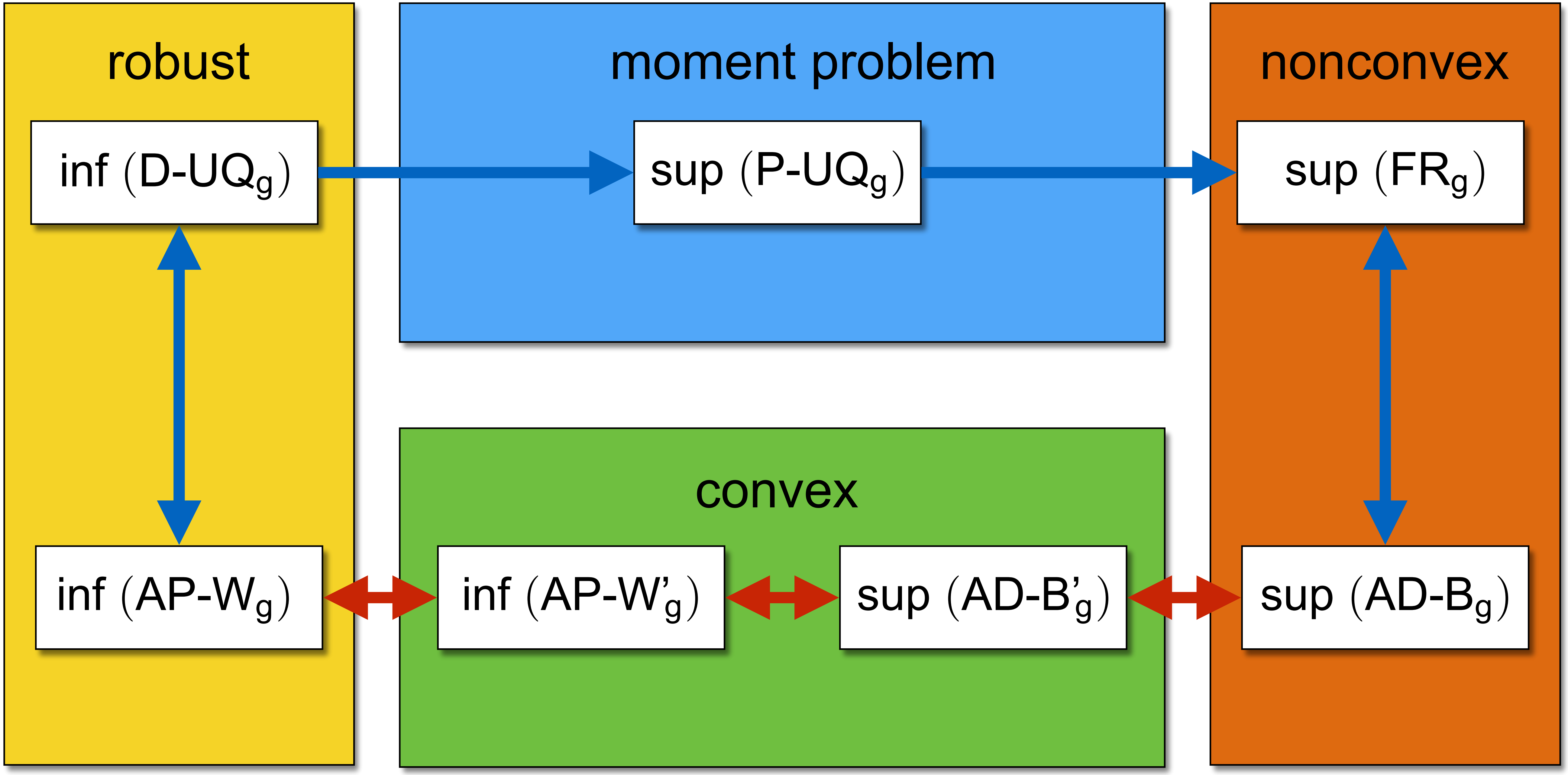
$$
\begin{aligned}
\inf_{\substack{\alpha,\beta\\ y,v}} \quad & \sum_{k=1}^{K} p_k \alpha_k + \mu^\top \beta \\
\text{s.t.} \quad & (-g_{ik})^* \left(y_{ik}^{(0)} \right) + \sum_{j=1}^{J} (\beta_j{}^\top h_{jk})^* \left(y_{ijk}^{(1)} \right) \\
& \qquad + \sum_{\ell=1}^{L} (v_{i\ell k}{}^\top c_{\ell k})^* \left(y_{i\ell k}^{(2)} \right) \leq \alpha_k \quad \forall i,\ \forall k \\
& y_{ik}^{(0)} + \sum_{j=1}^{J} y_{ijk}^{(1)} + \sum_{\ell=1}^{L} y_{i\ell k}^{(2)} = 0 \quad \forall i,\ \forall k \\
& \beta_j \in \mathcal{H}_j^*, \quad v_{i\ell k} \in \mathcal{C}_{\ell k}^* \quad \forall i,\ \forall j,\ \forall k,\ \forall \ell
\end{aligned}
\qquad (\text{AP-W}_g')
$$

Convex Reformulation of (AD-B$_g$)

$$
\begin{array}{lll}
\sup\limits_{v,\omega,\tau,\lambda\geq 0} & \sum_{k=1}^{K}\sum_{i=1}^{I} \tau_{ik} & \\
\text{s.t.} & \sum_{i=1}^{I} \lambda_{ik} = p_k & \forall k \\
& \sum_{k=1}^{K}\sum_{i=1}^{I} \omega_{ijk} \preceq_{\mathcal{H}_j} \mu_j & \forall j \\
& \lambda_{ik} c_{\ell k}(v_{ik}/\lambda_{ik}) \preceq_{\mathcal{C}_{\ell k}} 0 & \forall i,\ \forall k,\ \forall \ell \\
& \lambda_{ik} h(v_{ik}/\lambda_{ik}) \preceq_{\mathcal{H}_j} \omega_{ijk} & \forall i,\ \forall j,\ \forall k \\
& \lambda_{ik} g_{ik}(v_{ik}/\lambda_{ik}) \geq \tau_{ik} & \forall i,\ \forall k
\end{array}
\qquad (\text{AD-B}'_g)
$$

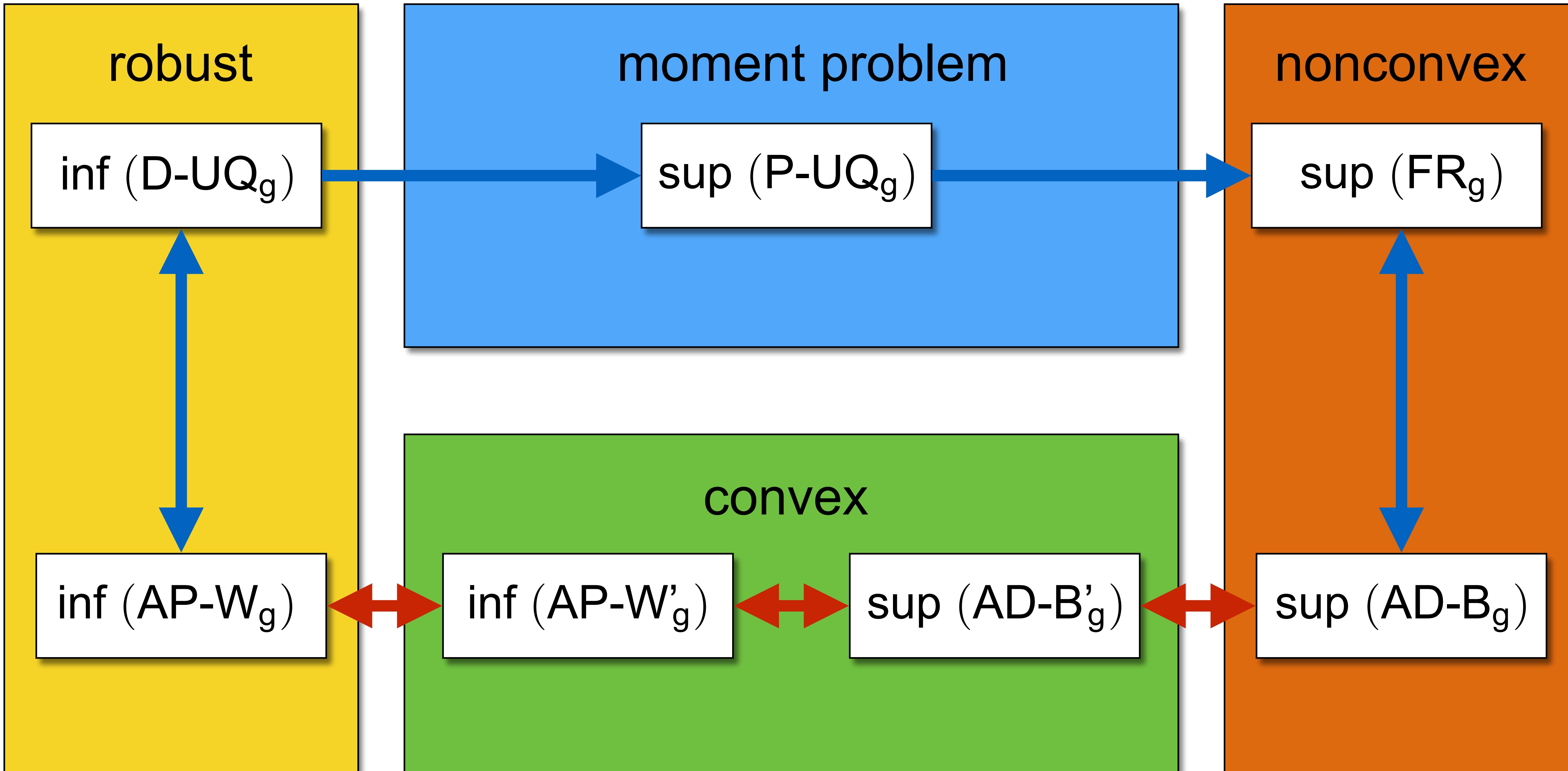
Strong Semi-Infinite Duality

If (AD-B_g) has a Slater point…



Strong Semi-Infinite Duality

If (AP-W_g) has a Slater point and $\mathcal{S}_k$ is bounded for all k…



Optimal Transport-Based
Distributionally Robust Optimization

OT-Based Uncertainty Quantification

$$\sup_{\mathbb{P} \in \mathbb{B}_{\varepsilon}(\hat{\mathbb{P}})} \mathbb{E}_{\mathbb{P}}\left[g(\tilde{z})\right] \qquad \text{(OT)}$$

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$$\mathbb{B}_{\varepsilon}(\hat{\mathbb{P}}) = \left\{ \mathbb{P} \text{ distribution on } \mathcal{S} \;\middle|\; D(\mathbb{P}, \hat{\mathbb{P}}) \leq \varepsilon \right\}$$

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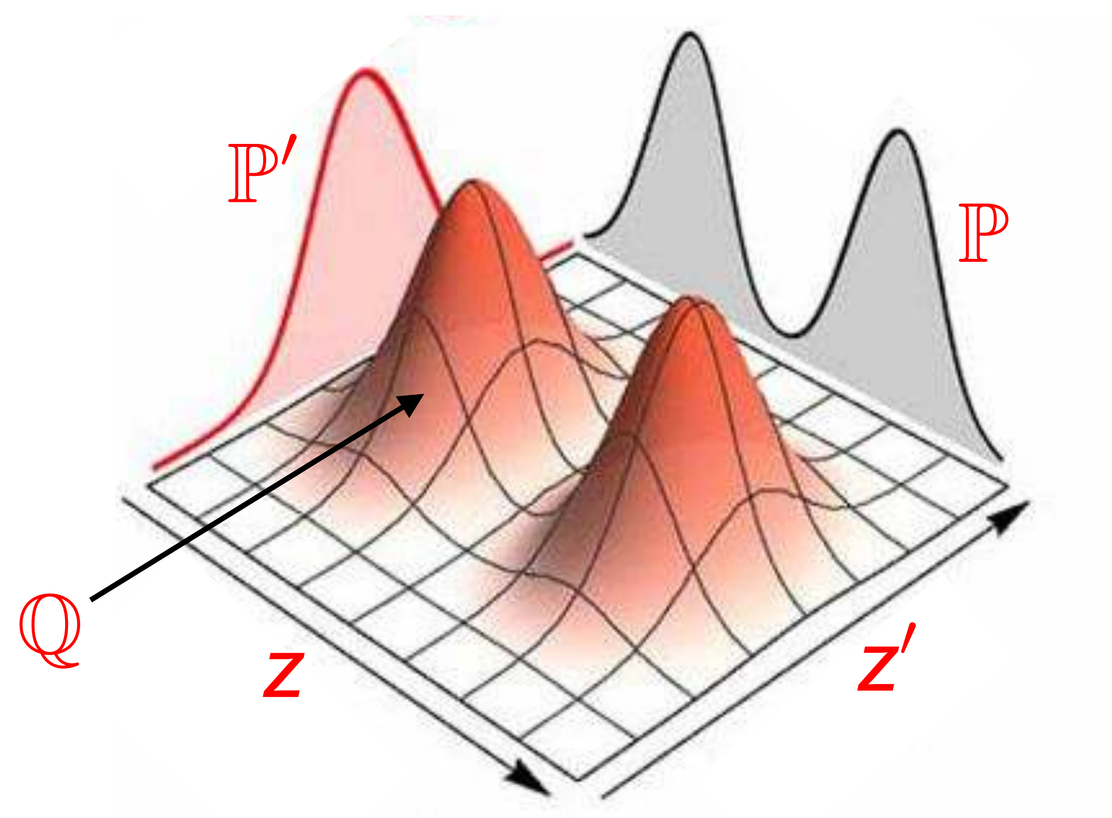
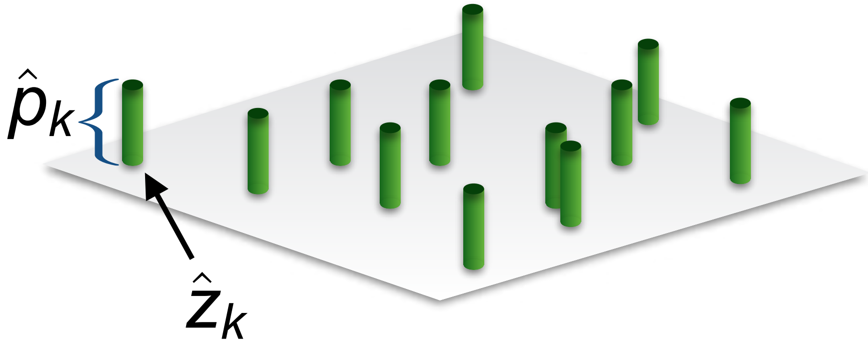
Optimal transport distance:

$$D(\mathbb{P}, \mathbb{P}') = \begin{cases} \min_{\mathbb{Q}} & \mathbb{E}_{\mathbb{Q}}\left[d(\tilde{z}, \tilde{z}')\right] \\ \text{s.t.} & \mathbb{Q} \text{ is a coupling} \\ & \text{of } \mathbb{P} \text{ and } \mathbb{P}' \end{cases}$$



Reduction of (OT) to (P-UQ$_g$)

$$\sup_{\mathbb{P}\in\mathbb{B}_\varepsilon(\hat{\mathbb{P}})} \mathbb{E}_{\mathbb{P}}\left[g(\tilde{z})\right] \qquad \text{(OT)}$$



Nominal distribution: $\hat{\mathbb{P}} = \sum_{k=1}^{K} \hat{p}_k \, \delta_{\hat{z}_k}$

Reduction of (OT) to (P-UQ$_g$)

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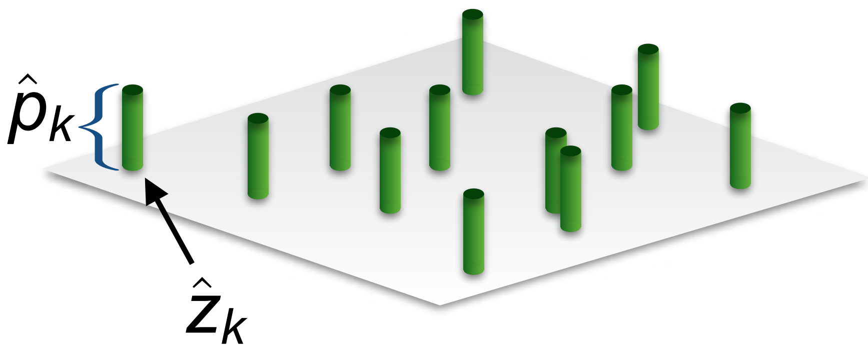

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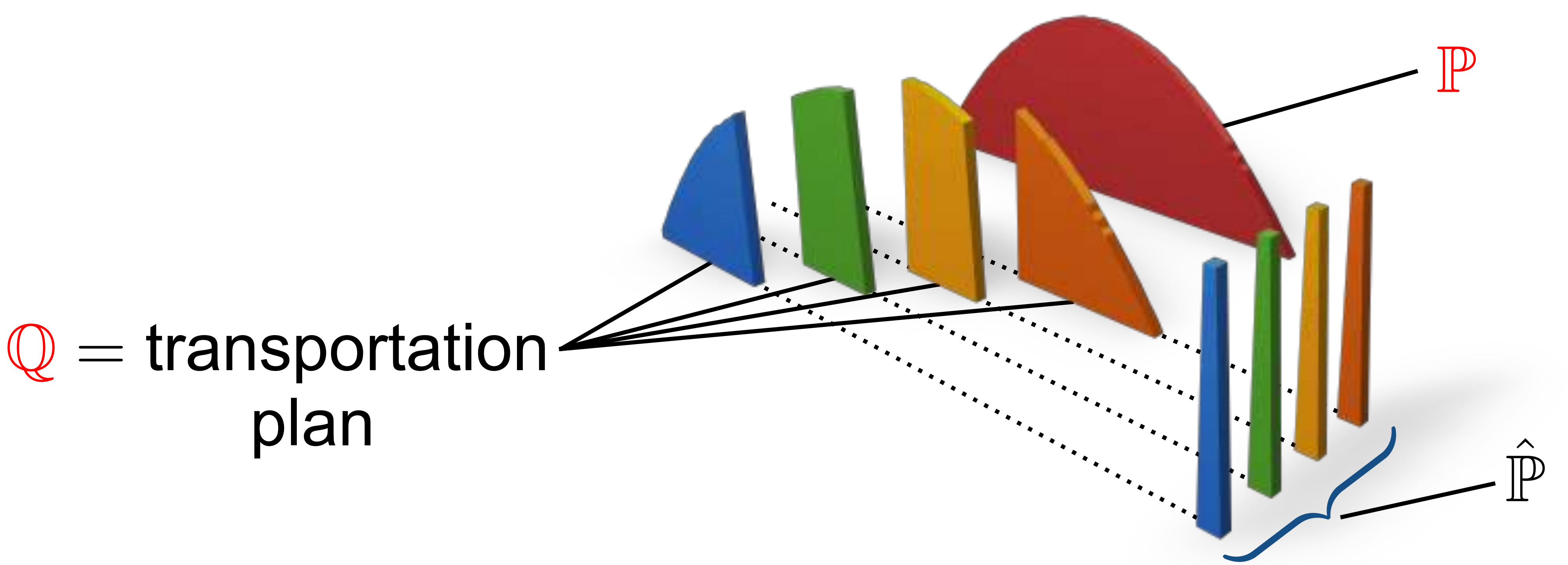

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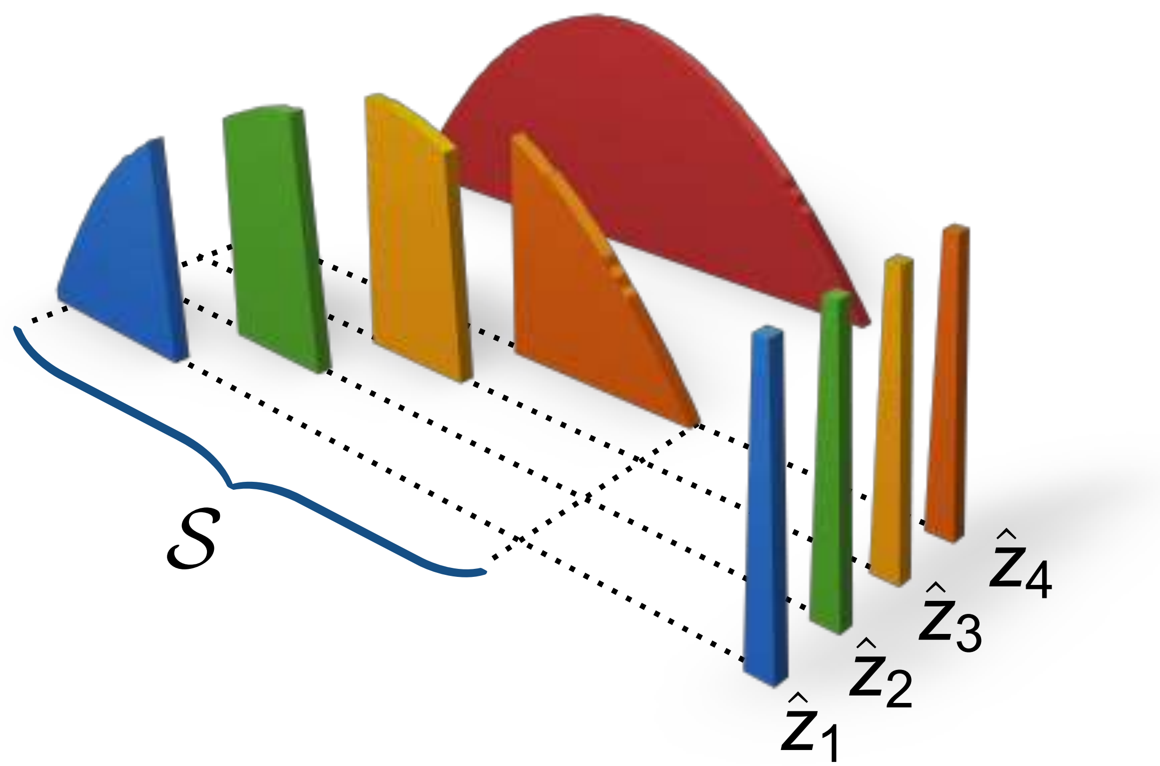
$$\sup_{\mathbb{P} \in \mathbb{B}_\varepsilon(\hat{\mathbb{P}})} \mathbb{E}_{\mathbb{P}}\left[g(\tilde{z})\right] \qquad \text{(OT)}$$

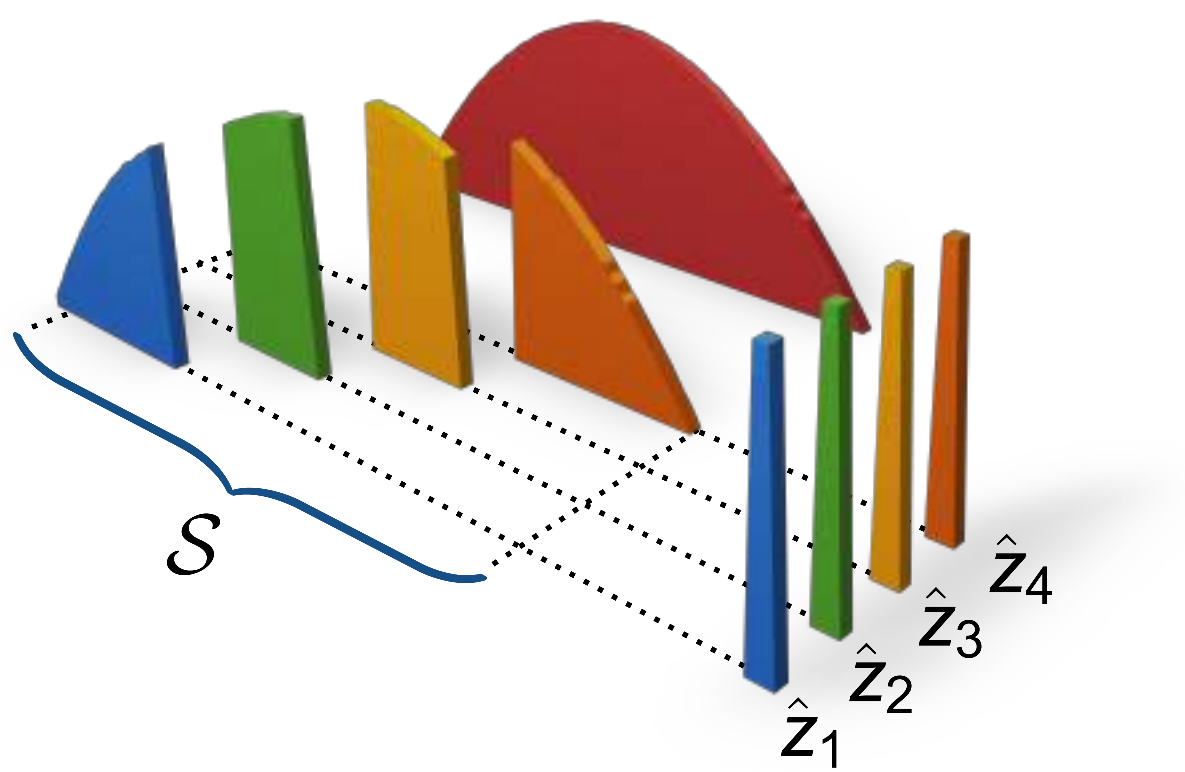
✓ Coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\hat{\mathbb{P}}$ supported on $\cup_{k=1}^{K} (\mathcal{S} \times \{\hat{z}_k\})$



Reduction of (OT) to (P-UQ$_g$)

$$\sup_{\mathbb{P}\in\mathbb{B}_\varepsilon(\hat{\mathbb{P}})} \mathbb{E}_{\mathbb{P}}[g(\tilde{z})] \qquad \text{(OT)}$$

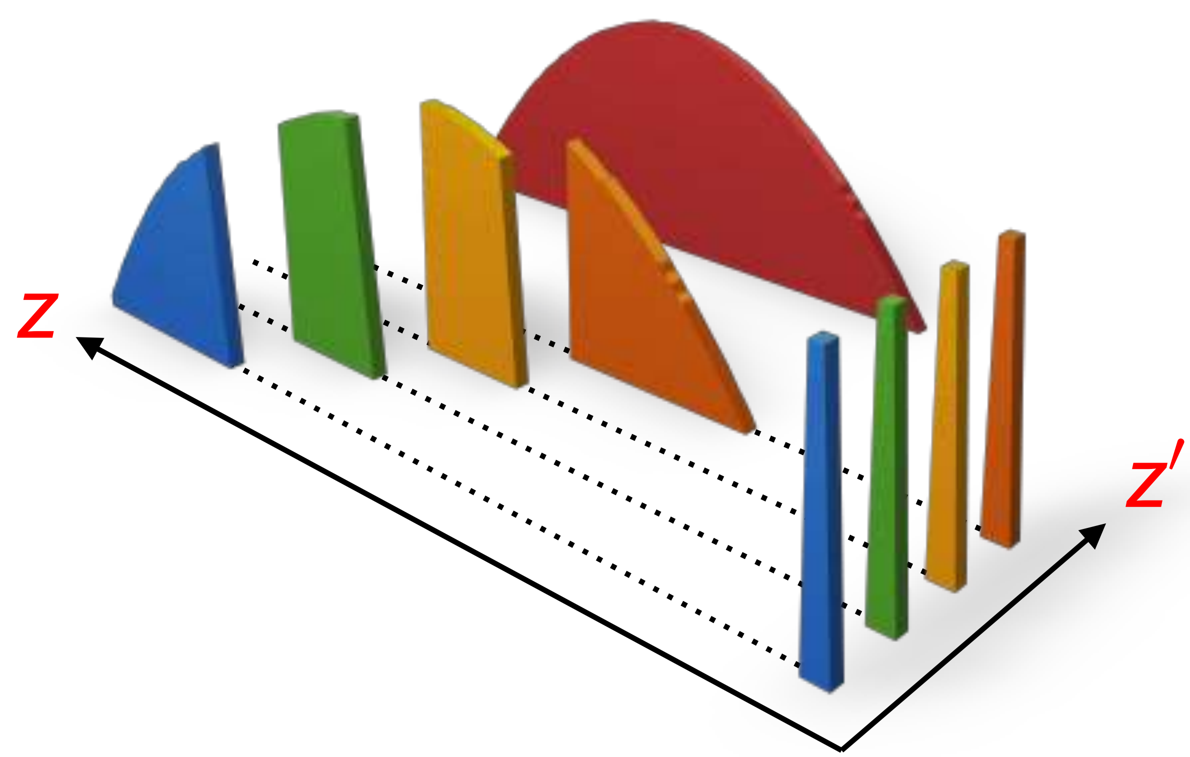
✓ Coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\hat{\mathbb{P}}$ supported on $\cup_{k=1}^{K} \underbrace{(\mathcal{S}\times\{\hat{z}_k\})}_{=\mathcal{S}_k}$



Reduction of (OT) to (P-UQ$_g$)

$$\sup_{\mathbb{P} \in \mathbb{B}_\varepsilon(\hat{\mathbb{P}})} \mathbb{E}_{\mathbb{P}} [g(\tilde{z})] \qquad \text{(OT)}$$

- Coupling $\mathbb{Q}$ of $\mathbb{P}$ and $\hat{\mathbb{P}}$ supported on $\cup_{k=1}^{K} \underbrace{(\mathcal{S} \times \{\hat{z}_k\})}_{= \mathcal{S}_k}$
- $\mathbb{Q}\left[(\tilde{z}, \tilde{z}') \in \mathcal{S}_k\right] = \hat{p}_k \quad \forall k$



Reduction of (OT) to (P-UQ$_g$)

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- $\mathbb{Q}\left[(\tilde{z},\tilde{z}')\in\mathcal{S}_k\right]=\hat{p}_k \quad \forall k$
- $\mathbb{E}_{\mathbb{Q}}\left[h(\tilde{z},\tilde{z}')\right]\leq\varepsilon$

Reduction of (OT) to (P-UQ$_g$)

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- $\mathbb{E}_{\mathbb{Q}}\left[\underbrace{h(\tilde{z}, \tilde{z}')}\right] \leq \varepsilon$

$$h(z, z') = \begin{cases} d(z, z') & \text{if } z' = \hat{z}_k \text{ for some } k \\ +\infty & \text{otherwise} \end{cases}$$

Reduction of (OT) to (P-UQ$_g$)

$$\sup_{\mathbb{P} \in \mathbb{B}_\varepsilon(\hat{\mathbb{P}})} \mathbb{E}_{\mathbb{P}}[g(\tilde{z})] \qquad \text{(OT)}$$

$\Longrightarrow \quad \mathbb{Q} \in \mathcal{P}_g$ (generalized moment ambiguity set)

Reduction of (OT) to (P-UQ$_g$)

$$\sup_{\mathbb{P} \in \mathbb{B}_{\varepsilon}(\hat{\mathbb{P}})} \mathbb{E}_{\mathbb{P}}\left[g(\tilde{z})\right] \qquad \text{(OT)}$$

$$\Updownarrow$$

(P-UQ$_g$)

Convex Reformulation of OT

$$\sup_{\mathbb{P}\in\mathbb{B}_\varepsilon(\hat{\mathbb{P}})} \quad \mathbb{E}_{\mathbb{P}}\left[g(\tilde{z})\right] \qquad \text{(OT)}$$

$\Updownarrow$ (regularity conditions)

$$\begin{aligned}
\inf_{\substack{\alpha,\beta\geq 0\\ y,v\geq 0}} \quad & \sum_{k=1}^{K} \hat{p}_k \alpha_k + \varepsilon\beta \\
\text{s.t.} \quad & (-g_i)^*(y_{ik}^{(0)}) + \beta d^{*1}\left(y_{ik}^{(1)}/\beta, \hat{z}_k\right) \\
& \quad + \sum_{\ell=1}^{L} v_{i\ell k} c_\ell^* \left(y_{i\ell k}^{(2)}/v_{i\ell k}\right) \leq \alpha_k \quad \forall i,\ \forall k \\
& y_{ik}^{(0)} + y_{ik}^{(1)} + \sum_{\ell=1}^{L} y_{i\ell k}^{(2)} = 0 \qquad \forall i,\ \forall k
\end{aligned} \qquad (\text{AP-W}'_{\text{g}})$$

Convex Reformulation of OT

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\sup\limits_{v,\omega,\tau,\lambda\geq 0} & \sum_{k=1}^{K}\sum_{i=1}^{I}\tau_{ik} & \\
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& \lambda_{ik} c_{\ell k}(v_{ik}/\lambda_{ik}) \leq 0 & \forall i,\ \forall k,\ \forall \ell \\
& \lambda_{ik} h(v_{ik}/\lambda_{ik}) \leq \omega_{ijk} & \forall i,\ \forall j,\ \forall k \\
& \lambda_{ik} g_{ik}(v_{ik}/\lambda_{ik}) \geq \tau_{ik} & \forall i,\ \forall k
\end{array} \qquad \text{(AD-B}'_{\text{g}}\text{)}$$

Conclusions

Summary

Convex Optimization

- Explicit representation of dual problem (D) using conjugates, perspectives and infimal convolutions

Robust Optimization

- (P-W) convexified to (P-W') by dualizing subproblems
- (D-B) convexified to (D-B') by change of variables
- "Primal-worst-equals-dual-best" duality

Distributionally Robust Optimization

- Finite restriction (FR) of (P-UQ) akin to "dual best" (AD-B)
- Dual problem (D-UQ) akin to "primal worst" (AP-W)
- Semi-infinite duality from first principles of convex analysis

Summary

Extensions

- Support as union of convex sets
- Conic support and moment constraints

Applications

- Convex reformulations of distributionally robust problems with optimal-transport-based ambiguity sets, e.g.,
  - single-stage problems
  - two-stage problems
  - chance-constrained problems

The Devil is in the Details

This Talk is Based on…

[1] D. Kuhn, P. Mohajerin Esfahani, V. Nguyen and S. Shafieezadeh Abadeh. **Wasserstein Distributionally Robust Optimization: Theory and Applications in Machine Learning**. INFORMS TutORials in Operations Research. 2019.

[2] P. Mohajerin Esfahani and D. Kuhn. **Data-Driven Distributionally Robust Optimization using the Wasserstein Metric: Performance Guarantees and Tractable Reformulations**. *Mathematical Programming* 171(1–2), 115–166, 2018.

[3] J. Zhen, D. Kuhn and W. Wiesemann. **Mathematical Foundations of Robust and Distributionally Robust Optimization**. Working paper, 2021.

https://www.epfl.ch/labs/rao/