Learning Solution Operators For PDEs

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Algorithms, Analysis and Applications

Andrew Stuart

Computing and Mathematical Sciences
California Institute of Technology

Various Locations in Paris

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Table of Contents

Talk Outline

Operator Learning

Supervised Learning

Determine $\Psi^\dagger : \mathcal{U} \to \mathcal{V}$ from samples

$$\{u_n, \Psi^\dagger(u_n)\}_{n=1}^N, \quad u_n \sim \mu.$$

Probability measure μ supported on $\mathcal{U}$.

In standard supervised learning $\mathcal{U} = \mathbb{R}^{d_x}$ and $\mathcal{V} = \mathbb{R}^{d_y}$ (regression) or $\mathcal{V} = \{1, \cdots K\}$ (classification).

Supervised Learning Of Operators

Separable Banach spaces $\mathcal{U}, \mathcal{V}$ of vector-valued functions:

$$\mathcal{U} = \{u : D_u \to \mathbb{R}^{d_i}\}, \quad D_u \subseteq \mathbb{R}^{d_u}$$
$$\mathcal{V} = \{v : D_v \to \mathbb{R}^{d_o}\}, \quad D_v \subseteq \mathbb{R}^{d_v}.$$

Operator Learning

Training

Consider a family of parameterized functions from $\mathcal{U}$ into $\mathcal{V}$:

$$\Psi : \mathcal{U} \times \Theta \mapsto \mathcal{V}.$$

Here $\Theta \subseteq \mathbb{R}^p$ denotes the parameter space.

$$\mu_N = \frac{1}{N}\sum_{n=1}^{N} \delta_{u_n}, \quad \mathrm{RE}(v, w) = \frac{\|v - w\|_{\mathcal{V}}}{\max\{1, \|v\|_{\mathcal{V}}\}},$$

$$\theta^* = \mathrm{argmin}_{\theta\in\Theta}\, \mathcal{R}_N(\theta), \quad \mathcal{R}_N(\theta) := \mathbb{E}^{u\sim\mu_N}\, \mathrm{RE}\big(\Psi^\dagger(u), \Psi(u;\theta)\big).$$

Example (Porous Medium Flow)

Darcy Law

Mass conservation	$-\nabla \cdot (a\nabla v) = f, \quad x \in D$
Boundary condition	$v = 0, \quad x \in \partial D$

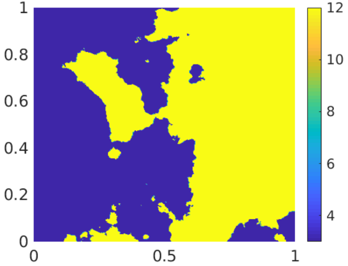
Operator Of Interest

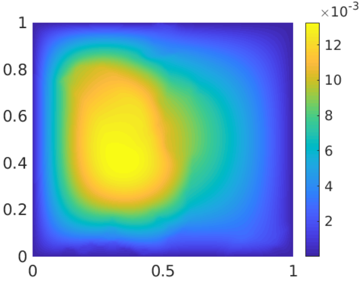
Parametric Dependence $\quad \Psi^{\dagger} : a \mapsto v$

Example (Porous Medium Flow)

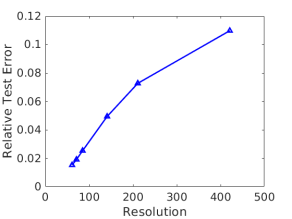
Input-Output

$$
\begin{aligned}
\text{Input:} \quad & a \in L^\infty(D) \quad (\textit{Left}), \\
\text{Output:} \quad & v \in H_0^1(D). \quad (\textit{Right}),
\end{aligned}
$$



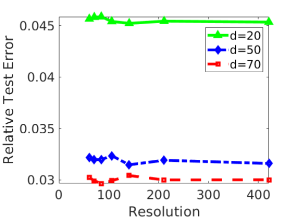


Example (Porous Medium Flow): Discretize Then Learn



Zhu and Zabaras 2018 [21], Khoo et al [8]

Example (Porous Medium Flow): Learn Then Discretize



Bhattacharya et al 2021 [2]

Examples: (Homogenized Constitutive Models)

Material Properties Dependence

- $\mathcal{U}$: material properties $A(\cdot)$.
- $\mathcal{V}$: stress σ.
- $\sigma = \Psi^\dagger(A)\big(\nabla u + \nabla u^\top\big)$.
- Approximate $\Psi^\dagger \approx \Psi(\cdot;\theta^\star)$

History Dependence

- $\mathcal{U}$: time histories of strain $\{\nabla u\}$.
- $\mathcal{V}$: time histories of stress $\{\sigma\}$.
- $\{\sigma\} = \Psi^\dagger(\{\nabla u\})$.
- Approximate $\Psi^\dagger \approx \Psi(\cdot;\theta^\star)$

Talk Outline

Finding Latent Structure

In A Picture

$$
\begin{tikzcd}
\mathcal{U} \arrow[r, "F_{\mathcal{U}}"] \arrow[d, "\Psi^{\dagger}"'] & \mathbb{R}^{d_{\mathcal{U}}} \arrow[r, "G_{\mathcal{U}}"] \arrow[d, "\varphi"'] & \mathcal{U} \arrow[d, "\Psi^{\dagger}"'] \\
\mathcal{V} \arrow[r, "F_{\mathcal{V}}"] & \mathbb{R}^{d_{\mathcal{V}}} \arrow[r, "G_{\mathcal{V}}"] & \mathcal{V}
\end{tikzcd}
$$

$$G_{\mathcal{U}} \circ F_{\mathcal{U}} \approx Id$$

$$G_{\mathcal{V}} \circ F_{\mathcal{V}} \approx Id$$

$$G_{\mathcal{V}} \circ \varphi \circ F_{\mathcal{U}} \approx \Psi^{\dagger}$$

PCA-NET

Architecture Bhattacharya, Hosseini, Kovachki and AMS '19 [2]

$$\Psi_{PCA}(u;\theta)(x) = \sum_{j=1}^{m} \alpha_j(Lu;\theta)\psi_j(x), \quad \forall u \in \mathcal{U} \qquad x \in D_v.$$

Details

- $\{\phi_j\}$ are PCA basis functions under μ.
- $Lu = \{\langle \phi_j, u \rangle\}$ maps to PCA coefficients under μ.
- $\{\psi_j\}$ are PCA basis functions under $(\Psi^\dagger)^\sharp \mu$.
- $\{\alpha_j\}$ are finite dimensional neural networks.

DEEPONET Shallow approximation of functionals (Chen and Chen [5]) made deep

Architecture Lu, Jin, Pang, Zhang, and Karniadakis'19 [18]

$$\Psi_{DEEP}(u;\theta)(x) = \sum_{j=1}^{m} \alpha_j(Lu;\theta_\alpha)\psi_j(y;\theta_\psi), \quad \forall u \in \mathcal{U} \qquad x \in D_v.$$

Details

- Lu maps to linear functionals on $\mathcal{U}$.
- e.g. PCA coefficients under μ; or pointwise $\{u(x_\ell)\}$.
- $\{\alpha_j, \psi_j\}$ are finite dimensional neural networks.
- $\theta = (\theta_\alpha, \theta_\psi)$.

Fourier Neural Operator (FNO) DNN (Goodfellow et al [7]) Extended to Operators

Architecture Li, Kovachki, Azizzadenesheli, Liu, Bhattacharya, AMS and Anandkumar et al '20 [16, 11]

$\mathcal{U} = \mathcal{V}$ Hilbert

$$\Psi_{FNO}(u;\theta) = \mathcal{Q} \circ \mathcal{L}_L \circ \cdots \mathcal{L}_2 \circ \mathcal{L}_1 \circ \mathcal{R}(u), \; \forall u \in \mathcal{U},$$

$$\mathcal{L}_l(v)(x;\theta) = \sigma\big(W_l v(x) + b_l + \mathcal{K}(v)(x;\gamma_l)\big),$$

$$\mathcal{K}(v)(x;\gamma) = \sum_{m=1}^{M} \alpha_m\big(\gamma(m)\big)\langle \mathrm{f}_m, v\rangle_{\mathcal{U}^{\otimes d_c}} \mathrm{g}_m(x).$$

Details

- $\mathcal{Q}, \mathcal{R}$ pointwise NNs or linear transformations.
- (W_l, b_l) define pointwise affine transformations.
- $\mathcal{K}$ eg FFT as convolutional integral operator.
- θ collects parameters from previous three bullets.

Talk Outline

Set-Up

Algorithms

Analysis

Application I: Material-Dependent Constitutive Models

Application II: History-Dependent Constitutive Models

Conclusions

Universal Approximation (Latent) over compact sets

Definition

A Banach space has the **approximation property (AP)** if every compact operator is a limit of finite-rank operators.

Theorem 1 Kovacvhki '22 [10, 11]

Assume

- $\mathcal{U}$, $\mathcal{V}$ Banach spaces with the AP.
- $\Psi^\dagger : \mathcal{U} \to \mathcal{V}$ continuous, $K \subset \mathcal{U}$ compact.

For any $\epsilon > 0$ $\exists$ bounded linear $F_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}^{d_{\mathcal{U}}}$, $G_{\mathcal{V}} : \mathbb{R}^{d_{\mathcal{V}}} \to \mathcal{V}$, and a continuous map $\varphi \in C(\mathbb{R}^{d_{\mathcal{U}}}; \mathbb{R}^{d_{\mathcal{V}}})$ such that

$$\sup_{u \in K} \|\Psi^\dagger(u) - (G_{\mathcal{V}} \circ \varphi \circ F_{\mathcal{U}})(u)\|_{\mathcal{V}} \leq \epsilon.$$

Universal Approximation (Latent) Bochner integration

Theorem 2 Kovacvhki '22 [10, 11]

Assume

- $\mathcal{U}$ Banach space with AP, $\mathcal{V}$ separable Hilbert space.
- μ probability measure on $\mathcal{U}$.
- $\Psi^\dagger \in L^p_\mu(\mathcal{U};\mathcal{V})$ for $1 \le p < \infty$.

For any $\epsilon > 0$ $\exists$ bounded linear $F_{\mathcal{U}} : \mathcal{U} \to \mathbb{R}^{d_{\mathcal{U}}}$, $G_{\mathcal{V}} : \mathbb{R}^{d_{\mathcal{V}}} \to \mathcal{V}$, and a continuous map $\varphi \in C(\mathbb{R}^{d_{\mathcal{U}}};\mathbb{R}^{d_{\mathcal{V}}})$ such that

$$\|\Psi^\dagger - G_{\mathcal{V}} \circ \varphi \circ F_{\mathcal{U}}\|_{L^p_\mu(\mathcal{U};\mathcal{V})} \le \epsilon.$$

Lanthaler, Mishra and Karniadakis '21 [14] (DeepONet and complexity, NSE)

Lanthaler '23 [12] (PCA-Net and complexity, Darcy and NSE)

Marcati and Schwab '23 [19] (Complexity estimates, Darcy)

Lanthaler and AMS '23 [15] (Complexity estimates, Hamilton-Jacobi)

Universal Approximation (FNO)

Theorem 3 Lanthaler, Li and AMS '23 [13]

Assume

- $\mathcal{U} = C(\overline{D}, \mathbb{R}^d)$.
- $\mathcal{V} = C(\overline{D}, \mathbb{R}^{d'})$.
- $\Psi^\dagger : \mathcal{U} \to \mathcal{V}$ continuous, $K \subset \mathcal{U}$ compact.

For any $L, M > 0$ and any $\epsilon > 0$ $\exists$ an FNO $\Psi(\cdot; \theta^\star) : \mathcal{U} \to \mathcal{V}$ such that

$$\sup_{u \in K} \|\Psi^\dagger(u) - \Psi(u; \theta)\|_{\mathcal{V}} \leq \epsilon.$$

Kovachki, Lanthaler and Mishra '21 [9] (Complexity, FNO)

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Multiscale Problem

Canonical Elliptic Multiscale Problem

$$
\begin{aligned}
-\nabla \cdot (A^\epsilon \nabla u^\epsilon) &= f, \quad x \in \Omega \\
u^\epsilon &= 0, \quad x \in \partial\Omega \\
A^\epsilon(x) = A\left(\frac{x}{\epsilon}\right), &\quad A \in \mathcal{U} := L^\infty(\mathbb{T}^d, \mathbb{R}^{d\times d}).
\end{aligned}
$$

Standing Asssumption on A

$$
\begin{aligned}
PD_{\alpha,\beta} = &\{A \in L^\infty(\mathbb{T}^d; \mathbb{R}^{d\times d}) : \\
&\forall (y,\xi) \in \mathbb{T}^d \times \mathbb{R}^d,\ \alpha|\xi|^2 \le \langle \xi, A(y)\xi\rangle \le \beta|\xi|^2\}.
\end{aligned}
$$

Operator Learning

Homogenized Elliptic Problem; $u_0 \approx u^\epsilon$

$$
\begin{aligned}
-\nabla \cdot (A_0 \nabla u_0) &= f, \quad x \in \Omega \\
u_0 &= 0, \quad x \in \partial\Omega \\
A_0(x) &= A_0, \quad \text{constant.}
\end{aligned}
$$

Constitutive Model Bensoussan, Lions, Papanicolaou '78 [1], Pavliotis and AMS '08 [20][Ch12]

A_0 determined by $\chi \in \mathcal{V} := H^1_{\mathrm{per}}(\mathbb{T}^d, \mathbb{R}^d)$

$$
\begin{aligned}
-\nabla_y \cdot (\nabla_y \chi A) &= \nabla_y \cdot A, \quad y \in \mathbb{T}^d, \\
A_0 &= \int_{\mathbb{T}^d} \left(A(y) + A(y) \nabla \chi(y)^T \right) dy.
\end{aligned}
$$

Universal Approximation (Cell Problem Solution Operator)

Goal: Supervised Learning (FNO)

Bhattacharya, Kovachki, Rajan, AMS, Trautner '23 [3]

- Learn map $A(\cdot) \in \mathcal{U} \mapsto \chi(\cdot) \in \mathcal{V} = H^1_{\mathrm{per}}(\mathbb{T}^d, \mathbb{R}^d)$.
- How to choose $\mathcal{U}$?

Theorem 4 Bhattacharya, Kovachki, Rajan, AMS, Trautner '23 [3]

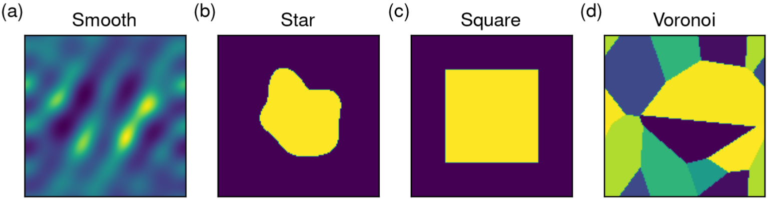
Define the mapping $\Psi^\dagger : PD_{\alpha,\beta} \to \dot{H}^1(\mathbb{T}^d; \mathbb{R}^d)$ from the solution map $A \mapsto \chi$ given by

$$-\nabla_y \cdot (\nabla_y \chi A) = \nabla_y \cdot A, \quad y \in \mathbb{T}^d.$$

Then, for any $\epsilon > 0$ and $K \subset PD_{\alpha,\beta}$ compact in $L^2(\mathbb{T}^d; \mathbb{R}^{d\times d})$, there exists an FNO $\Psi(\cdot; \theta^*) : K \to \dot{H}^1(\mathbb{T}^d; \mathbb{R}^d)$ such that

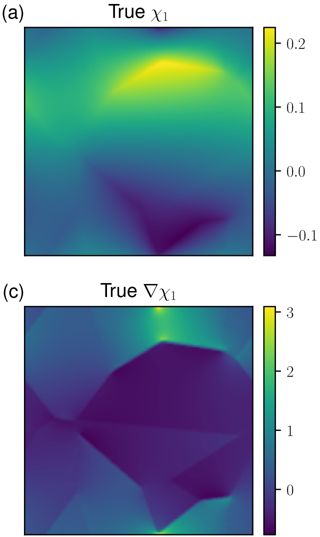
$$\sup_{A\in K} \|\Psi^d(A) - \Psi(A; \theta^\star)\|_{\dot{H}^1} < \epsilon.$$

Varying Microstructures

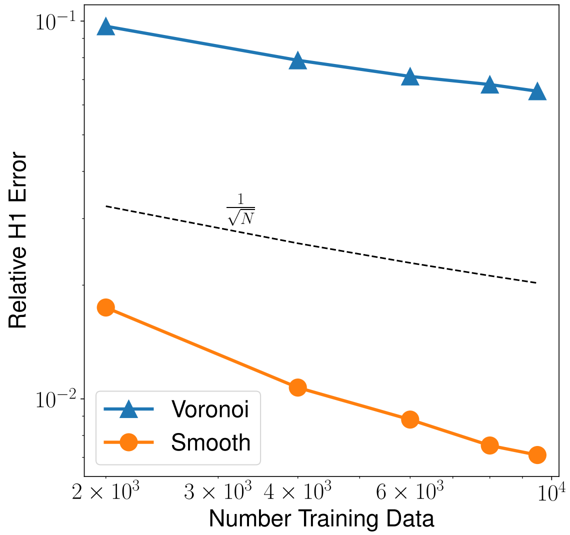


$$K \subset \mathrm{BV}(\mathbb{T}^d; \mathbb{R}^{d\times d}) \cap PD_{\alpha,\beta} \Subset L^2(\mathbb{T}^d; \mathbb{R}^{d\times d}).$$

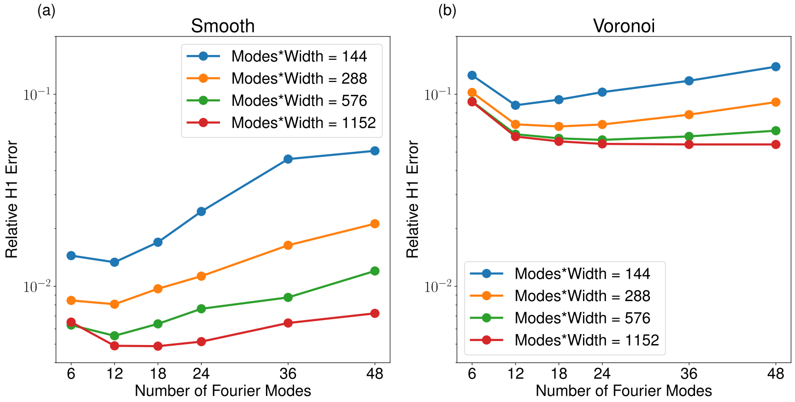
Learning Error for Voronoi Microstructure



Test Error versus Data Size



Test Error versus Model Size



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Big Picture

Multiscale Problem Pavliotis and AMS '08 [20][Ch12]

Displacement $u^\epsilon(x,t)$, stress $\sigma^\epsilon(x,t)$, $0 < \epsilon \ll 1$. F=MA:

$$\rho\,\partial_t^2 u^\epsilon = \nabla \cdot (\sigma^\epsilon) + f, \quad \sigma^\epsilon = \Psi^\epsilon\left(\{\nabla u^\epsilon\}, \frac{x}{\epsilon}\right).$$

Homogenized Problem Bensoussan, Lions, Papanicolaou [1]

Approximate $u^\epsilon = u_0 + \epsilon u_1 + \epsilon^2 u_2 + \dots$

Determine map Ψ, so that small scales are removed in u_0. F=MA:

$$\rho\,\partial_t^2 u_0 = \nabla \cdot (\sigma) + f, \quad \sigma = \Psi(\{\nabla u_0\}).$$

Operator Learning

- $\mathcal{U}$ histories of strain $\{\nabla u_0\}; V histories of stress \{\sigma\}$.
- Approximate $\Psi \approx \Psi_{NN}$

Viscoelasticity I Francfort and Suquet '86 [6]

Quasi-Static Viscoelasticity Multiscale Problem

$$
\begin{aligned}
-\nabla \cdot (\sigma^{\epsilon}) &= f, \quad x \in \Omega \\
\sigma^{\epsilon} &= \nu^{\epsilon}\partial_t \nabla u^{\epsilon} + E^{\epsilon}\nabla u^{\epsilon} \\
E^{\epsilon}(x) = E\left(\frac{x}{\epsilon}\right), \quad \nu^{\epsilon}(x) &= \nu\left(\frac{x}{\epsilon}\right), \quad E, \nu : \mathbb{T}^d \to \mathbb{R}.
\end{aligned}
$$

Laplace Transform, Homogenize, Invert

$$
\begin{aligned}
-\nabla \cdot (\overline{\sigma}^{\epsilon}) &= f, \quad x \in \Omega \\
\overline{\sigma}^{\epsilon} &= (s\nu^{\epsilon} + E^{\epsilon})\nabla \overline{u}^{\epsilon}.
\end{aligned}
$$

- Introduces memory.

Viscoelasticity II Bhattacharya, Liu, AMS, Trautner '22 [4]

Theorem (Piecewise-Constant Homogenization: Memory)

In piecewise-constant case and in dimension $d = 1$ homogenized equation for u_0 is Markovian:

$$
\begin{aligned}
-\nabla \cdot (\sigma) &= f, \quad x \in \Omega,\\
\sigma &= \nu' \partial_t \nabla u_0 + E' \nabla u_0 + \langle \mathbb{1}, r \rangle\\
\partial_t r_\ell &= -\alpha_\ell r_\ell + \beta_\ell \nabla u_0, \quad \ell \in \{1, 2, \cdots, L\},
\end{aligned}
$$

for some choice of $E' \in \mathbb{R}_+$, $\nu' \in \mathbb{R}_+$, $\alpha \in \mathbb{R}_+^L$, $\beta \in \mathbb{R}^L$, $L \in \mathbb{Z}_+$.

- Homogenization introduces memory.
- In $d = 1$ (approximate) Markovian structure.
- Dimension $d > 1$?

Viscoelasticity III: Operator Learning

True Solution Map

Let $\Psi : \mathcal{U} \to \mathcal{V}$ be the map such that the homogenized constitutive relation is

$$\sigma = \Psi(\{\nabla u_0\}).$$

Goal: Supervised Learning (RNO-NET)

Learn map $\Psi_{RNO} : \mathcal{U} \to \mathcal{V}$ approximating Ψ with the form

$$\begin{aligned} \sigma &= F\left(\nabla u_0, \partial_t \nabla u_0, r\right) \\ \partial_t r &= G(r, \nabla u_0), \quad r(0) = 0. \end{aligned}$$

- RNO – Recurrent neural operator.
- Dimension of memory variable r has to be learned.

Viscoelasticity IV: Learning the Constitutive Map

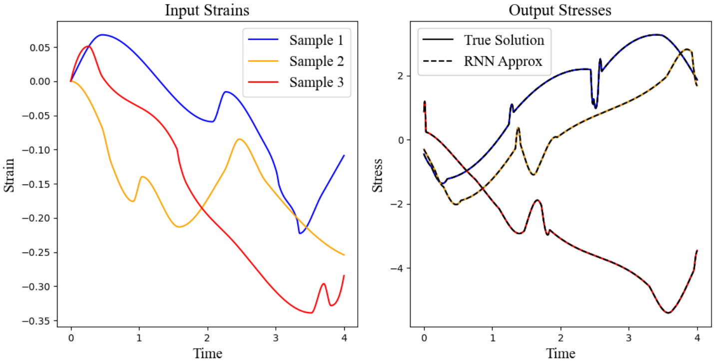


Figure: Viscoelasticity: trained model performs well on test samples

Viscoelasticity V: Choosing the Number of Hidden Variables

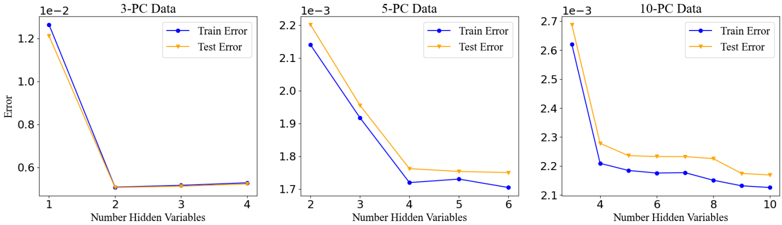


Figure: Absolute L^2 error of RNNs trained with different numbers of hidden variables on different piecewise-constant viscoelastic materials.

Viscoelasticity VI: Time Discretization Invariance

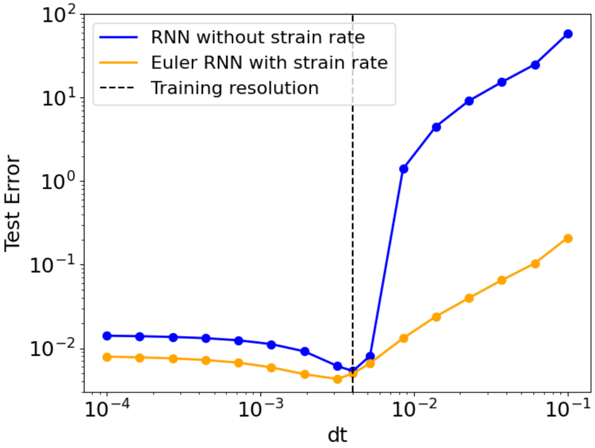


Figure: Viscoelasticity models trained with and without access to the strain rate variable: prefered model exhibits more invariance to time discretization of test trajectories

Talk Outline

Conclusions

1. Algorithms:
   - define on function space;
   - then learn;
   - leads to models which transfer between discretizations.
2. Analysis:
   - universal approximation theory well-developed;
   - complexity (cost verus error) incompletely understood;
   - what solution is found via optimization?
3. Applications:
   - cheap surrogates;
   - scientific discovery;
   - constitutive laws.

References I

[1] A. Bensoussan, J.-L. Lions, and G. Papanicolaou.
Asymptotic Analysis for Periodic Structures, volume 374.
American Mathematical Soc., 2011.

[2] K. Bhattacharya, B. Hosseini, N. B. Kovachki, and A. M. Stuart.
Model reduction and neural networks for parametric pdes.
The SMAI journal of computational mathematics, 7:121–157, 2021.

[3] K. Bhattacharya, N. Kovachki, A. Rajan, A. M. Stuart, and M. Trautner.
Learning homogenization for elliptic operators.
arXiv:2306.12006, 2023.

[4] K. Bhattacharya, B. Liu, A. M. Stuart, and M. Trautner.
Learning Markovian homogenized models in viscoelasticity.
Multiscale Modeling & Simulation, 21:641–679, 2023.
arXiv:2205.14139.

[5] T. Chen and H. Chen.
Approximations of continuous functionals by neural networks with application to dynamic systems.
IEEE Transactions on Neural networks, 4(6):910–918, 1993.

[6] G. A. Francfort and P. M. Suquet.
Homogenization and mechanical dissipation in thermoviscoelasticity.
Archive for Rational Mechanics and Analysis, 96(3):265–293, 1986.

[7] I. Goodfellow, Y. Bengio, and A. Courville.
Deep learning.
MIT press, 2016.

[8] Y. Khoo, J. Lu, and L. Ying.
Solving parametric pde problems with artificial neural networks.
European Journal of Applied Mathematics, 32(3):421–435, 2021.

References II

[9] N. Kovachki, S. Lanthaler, and S. Mishra.
On universal approximation and error bounds for Fourier neural operators.
The Journal of Machine Learning Research, 22(1):13237–13312, 2021.

[10] N. B. Kovachki.
Machine Learning and Scientific Computing.
PhD thesis, California Institute of Technology, 2022.

[11] N. B. Kovachki, Z. Li, B. Liu, K. Azizzadenesheli, K. Bhattacharya, A. M. Stuart, and A. Anandkumar.
Neural operator: Learning maps between function spaces with applications to pdes.
J. Mach. Learn. Res., arXiv:2108.08481, 24(89):1–97, 2023.

[12] S. Lanthaler.
Operator learning with pca-net: upper and lower complexity bounds.
arXiv preprint arXiv:2303.16317, 2023.

[13] S. Lanthaler, Z. Li, and A. M. Stuart.
The nonlocal neural operator: Universal approximation.
arXiv:2304.13221, 2023.

[14] S. Lanthaler, S. Mishra, and G. E. Karniadakis.
Error estimates for deeponets: a deep learning framework in infinite dimensions.
Transactions of Mathematics and Its Applications, 6(1):tnac001, 2022.

[15] S. Lanthaler and A. M. Stuart.
The curse of dimensionality in operator learning.
arXiv preprint arXiv:2306.15924, 2023.

[16] Z. Li, N. Kovachki, K. Azizzadenesheli, B. Liu, K. Bhattacharya, A. M. Stuart, and A. Anandkumar.
Fourier neural operator for parametric partial differential equations.
ICLR, arXiv:2010.08895, 2021.

References III

[17] B. Liu, N. Kovachki, Z. Li, K. Azizzadenesheli, A. Anandkumar, A. M. Stuart, and K. Bhattacharya.
A learning-based multiscale method and its application to inelastic impact problems.
Journal of the Mechanics and Physics of Solids, 158:104668, 2022.

[18] L. Lu, P. Jin, G. Pang, Z. Zhang, and G. E. Karniadakis.
Learning nonlinear operators via deeponet based on the universal approximation theorem of operators.
Nature Machine Intelligence, 3(3):218–229, 2021.

[19] C. Marcati and C. Schwab.
Exponential convergence of deep operator networks for elliptic partial differential equations.
SIAM Journal on Numerical Analysis, 61(3):1513–1545, 2023.

[20] G. Pavliotis and A. M. Stuart.
Multiscale Methods: Averaging and Homogenization.
Springer Science & Business Media, 2008.

[21] Y. Zhu and N. Zabaras.
Bayesian deep convolutional encoder–decoder networks for surrogate modeling and uncertainty quantification.
Journal of Computational Physics, 366:415–447, 2018.

Plasticity I Liu, Kovachki, Li, Azizzadenesheli, Anandkumar, AMS, Bhattacharya '22 [17]

Plasticity Multiscale Problem

$$
\begin{aligned}
\rho\,\partial_t^2 u^\epsilon &= \nabla \cdot \sigma^\epsilon + f, \qquad x \in \Omega \\
\partial_t \xi^\epsilon &= K(\xi^\epsilon, \nabla u^\epsilon), \quad x \in \Omega \\
\sigma^\epsilon &= \Psi^\epsilon\left(\nabla u^\epsilon, \xi^\epsilon, \frac{x}{\epsilon}\right)
\end{aligned}
$$

Plasticity II

Homogenized Plasticity Problem

$$\rho\,\partial_t^2 u_0 = \nabla\cdot\sigma_0 + f, \qquad x\in\Omega$$
$$\sigma_0 = \Psi\Big(\{\nabla u_0\}\Big)$$

Plasticity III: Operator Learning

Goal: Supervised Learning (PCA-NET)

Learn map $\Psi_{PCA} : \mathcal{U} \to \mathcal{V}$ approximating Ψ.
In particular causality must be learned.

Goal: Supervised Learning (RNO-NET)

Learn map $\Psi_{RNO} : \mathcal{U} \to \mathcal{V}$ approximating Ψ with the form

$$\begin{aligned} \sigma &= F\left(\nabla u_0, \partial_t \nabla u_0, r\right) \\ \partial_t r &= G(r, \nabla u_0), \quad r(0) = 0. \end{aligned}$$

Plasticity IV: Learning the Constitutive Map

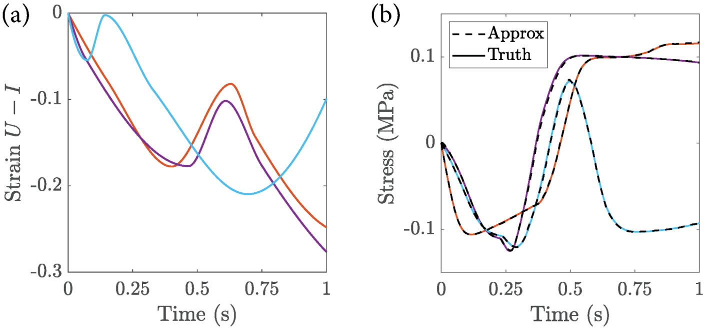


Figure: Viscoplasticity: Trained model performs well on test samples. Left: Input strains. Right: Output truth and approximation.

Plasticity V:
Choosing the Number of Hidden Variables

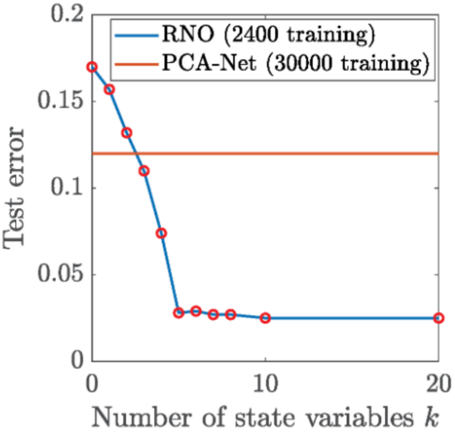


Figure: Viscoplasticity: 3D polycrystal (different hidden variable counts)

Plasticity VI: Time Discretization Invariance

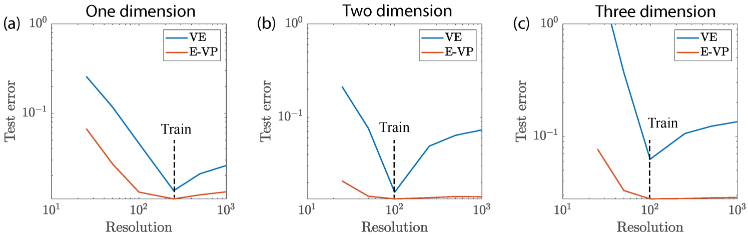
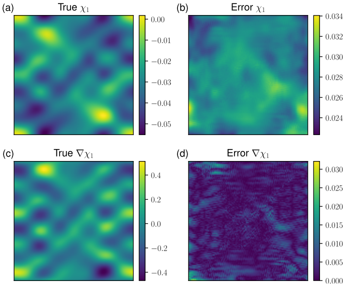


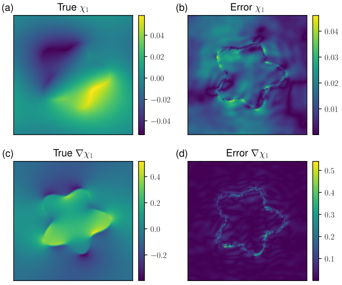
Figure: Models with viscoplastic (VP) and elasto-viscoplastic (E-VP) architecture trained on data from an E-VP material: prefered model exhibits more time discretization invariance

Learning Error for Varying Microstructures (I)

Smooth Microstructure

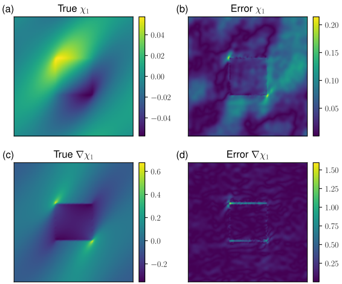


Star Inclusion Microstructure



Learning Error for Varying Microstructures (II)

Square Inclusion Microstructure



Voronoi Microstructure