Dual problem

Existence, strong duality, optimality condition and transport maps

Luca Nenna (LMO, Université Paris-Saclay) 18/01/2024

RECAP

- MONGE-KANTOROVICH PROBLEM

$$K_c(\mu,\nu) := \inf \left\{ \int c(x,y)\, d\gamma(x,y) \;\middle|\; \gamma \in \Pi(\mu,\nu) \right\}$$

WHERE $\Pi(\mu,\nu) := \left\{ \gamma \in \mathcal{P}(X \times Y) \;\middle|\; \pi_{X\#}\gamma = \mu \text{ and } \pi_{Y\#}\gamma = \nu \right\}$

- DUAL PROBLEM

$$D_c(\mu,\nu) := \sup \left\{ \int \varphi(x)\, d\mu(x) + \int \psi(y)\, d\nu(y) \;\middle|\; (\varphi,\psi) \in \mathcal{C}_b(X) \times \mathcal{C}_b(Y) \right.$$

AND

$$c(x,y) \geq \varphi(x) + \psi(y)$$

TODAY PROGRAM

1. EXISTENCE OF AN OPTIMAL (φ, ψ)

2. STRONG DUALITY

   Rmk WEAK DUALITY $(K) \geq (D)$ SO FAR

3. OPTIMALITY CONDITIONS

4. CHARACTERIZATION OF OPTIMAL TRANSPORT MAPS VIA THE DUAL VARIABLE

EXISTENCE

GOOD TO KNOW: C-TRANSFORM AND $\bar{C}$-TRANSFORM

THE C-TRANSFORM ($\bar{C}$-TRANSFORM) OF A FUNCTION $\psi : Y \to \mathbb{R} \cup \{+\infty\}$ ($\varphi : X \to \mathbb{R} \cup \{+\infty\}$) IS

$$\psi^c : x \in X \mapsto \min_y c(x,y) - \psi(y)$$

$$\varphi^{\bar{c}} : y \in Y \mapsto \min_x c(x,y) - \varphi(x)$$

- φ on X IS CALLED C-CONCAVE if $\varphi = \psi^c$ FOR SOME ψ

PROPERTIES OF C-TRANSFORM

LET $\omega : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ BE A MODULUS OF CONTINUITY FOR $c \in \mathcal{C}(X \times Y)$ FOR THE DISTANCE

$$d_{X \times Y}((x,y),(x',y')) = d_X(x,x') + d_Y(y,y')$$

THEN $\forall \varphi \in \mathcal{C}(X)$ AND $\forall \psi \in \mathcal{C}(Y)$

(i) φ^c AND ψ^c ALSO ADMITS ω AS MOD. OF CONTINUITY

(ii) GIVEN AN ADMISSIBLE (φ, ψ) FOR (D_c) WE CAN ALWAYS REPLACE IT WITH (φ, φ^c) AND THEN $(\varphi^{cc}, \varphi^c)$

- THE CONSTRAINTS ARE PRESERVED
- THE INTEGRALS

THM $\exists$ A PAIR $(\varphi^{c\bar{c}}, \varphi^{c})$ WHICH SOLVES (D)

SKETCH: GIVING A MAXIMISING SEQUENCE (φ_n, ψ_n)

- IMPROVE IT OBTAINING $(\varphi_n^{c\bar{c}}, \varphi_n^{c})$
- φ_n^{c} IS CONT. ON A COMPACT SET AND HENCE BOUNDED

  SUBSTRACT ITS MIN
- φ_n^{c} IS EQUIBOUNDED $0 \leq \varphi_n^{c} \leq \omega(\mathrm{diam}(Y))$
- WE HAVE ALSO UNIFORM BOUNDS ON $\varphi_n^{c\bar{c}}$
- APPLY ASCOLI-ARZELA $\varphi_{n_k}^{c\bar{c}} \to \bar{\varphi}$, $\varphi_{n_k}^{c} \to \bar{\varphi}$
- $(\bar{\varphi}^{c\bar{c}}, \bar{\varphi}^{c})$

STRONG DUALITY VIA THE DISCRETE CASE

STEP 1 IF μ AND ν ARE FINITELY SUPP.

THEN $(D_c) = (K_c)$

- $\mu = \sum^n \mu_i \delta_{x_i}$ $\quad \nu = \sum^m \nu_j \delta_{y_j}$

$$LP = \min \left\{ \sum c_{ij} \gamma_{ij} \;\middle|\; \gamma_{ij} \geq 0 \quad \sum_j \gamma_{ij} = \mu_i, \ \sum_i \gamma_{ij} = \nu_j \right\}$$

$$\underset{\text{STANDARD}}{=} \max \left\{ \sum \varphi_i \mu_i + \sum \psi_j \nu_j \;\middle|\; \varphi_i + \psi_j \leq c_{ij} \right\}$$

- KKT $\Rightarrow$ $\gamma_{ij}(c_{ij} - \varphi_i - \psi_j) = 0$

- BUILD FEASIBLE PAIR $(\bar{\varphi}, \bar{\psi})$ $\quad \bar{\psi}(y) = \begin{cases} \psi_j & y = y_j \\ -\infty & \end{cases}$

$\bar{\varphi} = \bar{\psi}^{\bar{c}}$

- $\bar{\gamma} = \sum \gamma_{ij} \delta_{(x_i, y_j)}$
- $(D_c) = (K_c)$

<u>STEP 2</u> DENSITY OF DISCRETE MEASURE

$\exists$ A SEQUENCE OF FINITELY SUPPORTED PROB. MEASURES WEAKLY CONVERGING TO A GIVEN $\mu \in \mathcal{P}(X)$

- STEP 3 THE GENERAL CASE FOR $\mu \in \mathcal{P}(X)$, $\nu \in \mathcal{P}(Y)$

- $\exists\ \mu_k \to \mu$, $\nu_k \to \nu$ with μ_k AND ν_k FINITELY SUPPORTED

- $\forall k$ take γ_k (φ_k, φ_k^c) OF $\boxed{\text{STEP 1}}$

γ_k is SUPPORTED ON $S_k := \{(x,y) \mid \varphi_k(x) + \varphi_k^c(y) = c(x,y)\}$

- AS FOR THE EXISTENCE WE CAN SHOW THAT

$(\varphi_k, \varphi_k^c) \to (\bar{\varphi}, \bar{\psi})$ ADMISSIBLE FOR THE DUAL

- BY COMPACTNESS OF $\mathcal{P}(X \times Y)$ $\gamma_k \to \gamma$

- ONE CAN PROVE THAT $\forall (x,y) \in \mathrm{supp}(\gamma)$

$\exists (x_k, y_k) \in \mathrm{supp}(\gamma_k)$ s.t. $(x_k, y_k) \to (x,y)$

$c(x_k, y_k) = \varphi_k(x_k) + \psi_k^c(x_k) \longrightarrow c(x,y) = \varphi(x) + \psi^c(y)$

$$(K_c) \leq \int c \, d\gamma = \int \left(\varphi(x) + \psi^c(y)\right) d\gamma = \int \varphi \, d\mu + \int \psi^c \, d\nu \leq (D_c)$$

⚠ CONSEQUENCES OF DUALITY

- <u>STABILITY</u>

$$\mu_k \longrightarrow \mu$$

$$\nu_k \longrightarrow \nu$$

$$c_k \xrightarrow{u} c$$

- IF γ_k IS A MIN FOR (K) BETWEEN μ_k AND ν_k WITH COST c_k THEN $\gamma_k \to \gamma$ MIN FOR (K_c) BETWEEN μ AND ν WITH COST c

- SAME FOR THE OPTIMAL DUAL VARIABLES

OPTIMALITY CONDITIONS

LET $\gamma \in \Pi(\mu,\nu)$ AND $(\varphi,\psi) \in \mathcal{C}(X)\times\mathcal{C}(Y)$ s.t. $\varphi(x)+\psi(y) \le c(x,y)$. THE FOLLOWING ARE EQUIVALENT

(i) $\varphi(x)+\psi(y) = c(x,y)$ γ-a.e.

(ii) γ IS A MINIMIZER

(φ,ψ) IS A MAXIMIZER

<u>def</u> (CYCLICAL MONOTONICITY) A SET $S \subset X \times Y$ IS SAID C-CYCLICALLY MONOTONE IF $\forall n \in \mathbb{N}^*$ AND $(x_i, y_i)_{i=1..n} \in S$ IT HOLDS

$$\sum^{n} c(x_i, y_i) \leq \sum c(x_i, y_{i+1})$$

$y_{n+1} = y_1$

IF γ IS THE OPTIMAL COUPLING BETWEEN μ AND ν FOR THE COST $c \in \mathcal{C}(X \times Y)$ THEN SUPP(γ) IS C-CYCLICALLY MONOTONE

Let $(x_i, y_i)_{i=1}^{n}$ n points in $\mathrm{supp}(\gamma)$

$$\sum c(x_i, y_{i+1}) - \sum c(x_i, y_i) \geq \sum (\varphi(x_i) + \psi(y_{i+1})) - \sum (\varphi(x_i) + \psi(y_i))$$

$c(x_i, y_{i+1}) \geq \varphi(x_i) + \psi(y_{i+1})$ (x_i, y_{i+1}) may not be in $\mathrm{supp}(\gamma)$

$c(x_i, y_i) = \varphi(x_i) + \psi(y_i)$ $(x_i, y_i) \in \mathrm{supp}(\gamma)$

$$= 0$$

CHARACTERIZATION OF OPTIMAL TRANSPORT PLANS

X, Y COMPACT, $c \in \mathcal{C}(X \times Y)$ $\mu \in \mathcal{P}(X)$ $\nu \in \mathcal{P}(Y)$

THE FOLLOWING STATEMENTS ARE EQUIVALENT:

(i) γ IS AN OPTIMAL PLAN FOR $K_c(\mu, \nu)$

(ii) $\mathrm{supp}(\gamma)$ IS c-CYC. MONOTONE

(iii) $\exists$ c-CONCAVE φ s.t.

$$\mathrm{supp}\,\gamma \subset \{(x,y) \mid \varphi(x) + \varphi^c(y) = c(x,y)\}$$

$\hookrightarrow$ (c-ROCKAFELLAR THM)

OPTIMAL TRANSPORT MAPS

<u>Lemma</u> $\gamma \in \Pi(\mu,\nu)$ AND $T : X \to Y$ s.t.

$\gamma(\{(x,y) \mid T(x) \neq y\}) = 0$ THEN $\gamma := \gamma_T = (Id, T)_\# \mu$

AND SO?

- GIVEN A MINIMIZER γ FOR $K_c(\mu,\nu)$ AND A MAXIMIZER (φ, φ^c) FOR $D_c(\mu,\nu)$ WE KNOW

$$\varphi + \varphi^c = c \quad \gamma\text{-A.E.}$$

IDEA: SHOW ~~THAT~~ $\{\varphi + \varphi^c = c\}$ IS CONTAINED IN THE GRAPH OF A FUNCTION !!

Def (TWISTED COST) A COST FCT $c \in \mathcal{C}^1(X \times Y)$ IS SAID TO BE TWISTED IF

$$\forall x_0 \quad y \longmapsto D_x c(x_0, y) \text{ IS INJECTIVE}$$

Thm ASSUME μ IS A.C. WITH RESPECT TO LEBESGUE THEN $\exists\ \varphi$ C-CONCAVE THAT IS DIFFERENTIABLE ALMOST EVERYWHERE S.T. $\nu = T_\# \mu$ WHERE $T(x) = D_x^{-1} c(x, \cdot) \circ D\varphi(x)$. MOREOVER THE ONLY OPTIMAL TRANSPORT PLAN IS $\gamma = \gamma_T$

REMARKS

1) [THE QUADRATIC CASE]

$c(x,y) = |x-y|^2$

1.a) THE C-TRANSFORM IS THE LEGENDRE TRANSFORM

1.b) BRENIER'S THM

$$T(x) = \nabla u(x) = \nabla\left(\frac{x^2}{2} - \varphi(x)\right)$$

u is a CVX FCT

↳ the optimal potential

1.c) MONGE - AMPÈRE EQ

$d\mu = \bar{\mu}\, dx \qquad \nu = \bar{\nu}\, dy$

$$\bar{\nu}(T(x)) \det(DT(x)) = \bar{\mu}(x)$$

$$\bar{\nu}(\nabla u(x)) \det(D^2 u(x)) = \bar{\mu}(x)$$

ANOTHER WAY TO FIND A SOL. TO OPTIMAL TRANSPORT

2) [THE 1-D CASE]

$d\mu = \bar{\mu}\, dx \qquad F_\mu(x) = \mu([-\infty, x])$

$$F_\mu^{-1}(x) = \inf\{t \mid F_\mu(x) \leq t\}$$

$$T = F_\nu^{-1}(F_\mu(x))$$