Dual problem
Existence, strong duality, optimality condition and transport maps
RECAP

- Monge-Kantorovich Problem

$K_c(\mu, \nu) := \inf \left\{ \int c(x, y) \, d\pi(x, y) \mid \pi \in \Pi(\mu, \nu) \right\}

where $\Pi(\mu, \nu) := \left\{ \pi \in \mathcal{P}(X \times Y) \mid \pi_x \# \mu = \mu \text{ and } \pi_y \# \nu = \nu \right\}$

- Dual Problem

$D_c(\mu, \nu) := \sup \left\{ \int \psi(x) \mu(dx) + \int \varphi(y) \nu(dy) \mid (\psi, \varphi) \in C_b(X) \times C_b(Y) \text{ and } c(x, y) \geq \varphi(x) + \psi(y) \right\}$

AND $c(x, y) = \varphi(x) + \psi(y)$
TODAY PROGRAM

1. EXISTENCE OF AN OPTIMAL \((\Psi, \Phi)\)

2. STRONG DUALITY
   
   **Rmk** WEAK DUALITY \((K) \geq (D)\) SO FAR

3. OPTIMALITY CONDITIONS

4. CHARACTERIZATION OF OPTIMAL TRANSPORT MAPS VIA THE DUAL VARIABLE
EXISTENCE

*GOOD TO KNOW: C-TRANSFORM AND C-TRANSFORM*

THE C-TRANSFORM (C-TRANSFORM) OF A FUNCTION

\( \Psi: Y \rightarrow \mathbb{R} \cup \{+\infty\} \) \hspace{1cm} (\( \Psi: X \rightarrow \mathbb{R} \cup \{+\infty\} \)) IS

\[ \psi_c: x \mapsto \min_y \ c(x,y) - \psi(y) \]

\[ \bar{\psi}_c: y \in Y \mapsto \min_x \ c(x,y) - \psi(x) \]

• \( \Psi \) on \( X \) is CAUSAL C-CONCAVE if \( \Psi = \psi_c \) for some \( \psi \)
Properties of C-Transform

Let \( \mu : \mathbb{R}^+ \to \mathbb{R}^+ \) be a modulus of continuity for \( c \in C(x \times y) \) for the distance

\[
d_{x \times y}((x,y), (x',y')) = d_x(x,x') + d_y(y,y')
\]

Then \( \forall \phi \in C(x) \) and \( \forall \psi \in C(y) \)

(i) \( \phi^c \) and \( \psi^c \) also admit \( \mu \) as mod. of continuity

(ii) Given an admissible \( (\phi, \psi) \) for \( (D_c) \), we can always replace it with \( (\phi, \psi^c) \) and then \( (\phi^c, \psi^c) \). The constraints are preserved.
There exists a pair \((\psi^c, \psi^c)\) which solves (D).

Sketch: Given a maximizing sequence \((\psi_n, \bar{\psi}_n)\)

- Improve it obtaining \((\psi_n^c, \bar{\psi}_n^c)\)
- \(\psi_n^c\) is cont. on a compact set and hence bounded
- Subtract its min

- \(\psi_n^c\) is equibounded \(0 \leq \psi_n^c \leq \omega\) (\(\text{diam}(Y)\))

- We have also uniform bounds on \(\psi_n^c\)

- Apply Ascoli-Arzela \(\psi_n^c \rightarrow \overline{\psi}, \psi_n^c \rightarrow \overline{\psi}\)
- \(\psi^c > \overline{\psi}^c\)
STRONG DUALITY VIA THE DISCRETE CASE

**STEP 1** IF \( \mu \) AND \( \nu \) ARE FINITELY SUPP.

THEN \((D_t) = (K_t)\)

\(\mu = \sum^m \mu_i S_{x_i} \quad \nu = \sum^n \nu_j S_{y_j}\)

\[\begin{align*}
LP &= \min \left\{ \sum c_{ij} \mu_i \mid \mu_i \geq 0 \quad \sum \mu_{ij} = \mu_i \right\} \\
     &= \max \left\{ \sum \psi_i + \sum \varphi_j \mid \psi_i + \varphi_j \leq c_{ij}\right\}
\end{align*}\]

STANDARD

**KKT** \(\Rightarrow\) \(\delta_{ij} (C_{ij} - \psi_i - \varphi_j) = 0\)

BUILD FEASIBLE PAIR \((\overline{x}, \overline{y})\)

\[\overline{x} = \{x_i \mid y_i = \overline{y}_i\}\]
\[ \tilde{Y} = \sum \gamma_{i,j} S(x_i, y_i) \]

\[ (D_2) \sim (K_u) \]

**STEP 2**

**Density of Discrete Measure**

If a sequence of finitely supported prob. measures weakly converging to a given \( \mu \in \mathcal{P}(X) \)
STEP 3 THE GENERAL CASE FOR $\nu \in \mathcal{E}(\mathcal{Y})$

- \exists \mu_k \to \mu \text{ with } \mu_k \text{ and } \nu_k \text{ finitely supported}

- \forall k \text{ take } \psi_k \text{ of STEP 1}

$\psi_k$ is supported on $S_k := \left\{ (x,y) \mid \psi_k(x) + \psi_k^c(y) = \mathcal{C}(x,y) \right\}$

AS FOR THE EXISTENCE WE CAN SHOW THAT

$(\psi_k, \psi_k^c) \to (\overline{\mu}, \overline{\nu})$ ADMISSIBLE FOR THE DUAL
By compactness of $\mathcal{B}(x, y) \ x_k \to x$

One can prove that $\forall (x, y) \in \text{supp}(\nu)$

$\exists (x_k, y_k) \in \text{supp}(\nu_k) \ s.t. \ (x_k, y_k) \to (x, y)$

$c(x_k, y_k) = \nu_k(x_k) + \psi_k^n(x_k) \to c(x, y) = \varphi(x) + \psi^n(y)$

$(\kappa_c) \leq \int c \, d\nu = \int (\varphi(x) + \psi^n(y)) \, d\nu = \int \varphi \, d\nu + \int \psi^n \, d\nu \leq (I_2)$
\section*{Consequences of Duality}

- \textbf{Stability} \quad \mu_k \rightarrow \mu \\
  \nu_k \rightarrow \varphi \\
  c_k \rightarrow C

- If $y_k$ is a min for (K) between $\mu_k$ and $\nu_k$ with cost $c_k$, then $\nu_k \Rightarrow \varphi$ min for (K) between $\mu$ and $\varphi$ with cost $c$

- Same for the optimal dual variables
OPTIMALITY CONDITIONS

Let \( \gamma \in \Pi(x, y) \) and \((y, \gamma) \in \mathcal{E}(x) \times \mathcal{E}(y)\)

s.t.

\[
\gamma(x) + \gamma(y) \leq C(x, y) \quad \text{THE FOLLOWING ARE EQUIVALENT}
\]

(i) \[
\gamma(x) + \gamma(y) = C(x, y) \quad \gamma \text{ a.e.}
\]

(ii) \( \gamma \) is a minimizer

\((y, \gamma) \text{ is a maximizer}\)
A set $S \subseteq \mathbb{X} \times \mathbb{Y}$ is said $C$-cyclically monotone if for all $n \in \mathbb{N}^*$ and $(x_i, y_i)_{i=1}^n \in S$ it holds

$$\sum_{i=1}^{n} c(x_i, y_i) \leq \sum_{i=1}^{n} c(x_i, y_{i+1})$$

$y_{n+1} = y_n$

If $\gamma$ is the optimal coupling between $\mu$ and $\nu$ for the cost $c \in C(\mathbb{X} \times \mathbb{Y})$ then $\text{supp}(\gamma)$ is $C$-cyclically monotone.
Let \( (x_i, y_i)_{i=1}^n \) be points in \( \text{supp}(\varphi) \):

\[
\sum \varphi(x_i) \geq \sum (\varphi(x_i) + \varphi(y_{i+1})) - \sum (\varphi(x_i) + \varphi(y_i))
\]

**Claim:**

\[
\sum \varphi(x_i) \geq \sum \varphi(x_i) + \varphi(y_{i+1}) \quad (x_i, y_{i+1}) \text{ may not be in } \text{supp}(\varphi)
\]

\[
\sum \varphi(x_i) = \sum \varphi(x_i) + \varphi(y_i) \quad (x_i, y_i) \in \text{supp}(\varphi)
\]

\[
\implies 0
\]
CHARACTERIZATION OF OPTIMAL TRANSPORT PLANS

\( X, Y \) COMPACT, \( C \in C(X \times Y) \) \( \mu \in PC(X) \) \( \nu \in PC(Y) \)

THE FOLLOWING STATEMENTS ARE EQUIVALENT:

(i) \( \gamma \) IS AN OPTIMAL PLAN FOR \( \kappa = (\mu, \nu) \)

(ii) \( \text{supp}(\gamma) \) IS C-CYC. MONOTONE

(iii) \( \exists \) C-CONCAVE \( \varphi \) s.t.

\[ \text{supp} \gamma \subseteq \{(x, y) \mid \varphi(x) + \varphi^*(y) = \kappa(x, y)\} \]

\( \Rightarrow \) (C-ROCKEFELLAR THM)
Lemma \( \nu \ll (\mu, \nu) \) and \( T : X \to Y \) s.t.
\[
\nu \left( \{(x,y) \mid T(x) \neq y\} \right) = 0 \quad \text{then} \quad \lambda := \lambda_T := (\text{Id}, T) \# \mu
\]

AND SO?

* Given a minimizer \( \gamma \) for \( \mathcal{K}_c(\mu, \nu) \) and a maximizer \( (\psi, \psi^c) \) for \( \mathcal{D}_c(\mu, \nu) \) we know

\[
\psi + \psi^c \leq c \quad \lambda - \text{a.e.}
\]

IDEA: SHOW THAT \( \{ \psi + \psi^c = c \} \) IS CONTAINED IN THE GRAPH OF A FUNCTION!!
Def (Twisted Cost) A cost $f(x, z, y)$ is said to be twisted if

\[ \forall x_0, y \rightarrow D_x c(x_0, y) \text{ is injective} \]

Thm Assume $\mu$ is a.c. with respect to Lebesgue then $\exists y \in C$ concave that is differentiable almost everywhere s.t. $V = T\mu$ where

\[ T(x) = D_x^{-1} c(x, \cdot) \cdot D_y (x) \]. Moreover the only optimal transport plan is $\gamma = \delta_{x_0}$
REMARKS

1) [THE QUADRATIC CASE]

\[ C(x, y) = |x - y|^2 \]

1.a) THE C-TRANSFORM IS THE LEGENDRE TRANSFORM

1.b) BRENIER'S THM

\[ T(x) = \nabla w(x) = \nabla \left( \frac{x^2}{2} - \psi(x) \right) \]

\[ w \text{ is a convex potential} \]
1. C) MONGE - AMPÈRE EQ

\[ d\mu = \bar{\mu} \, dx \quad \nu = \bar{\nu} \, dy \]

\[ \bar{\nabla}(T(x)) \det(DT(x)) = \bar{\mu}(x) \]

\[ \bar{\nabla}(\nabla u(x)) \det(D^2u(x)) = \mu(x) \]

ANOTHER WAY TO FIND A SOL. TO OPTIMAL TRANSPORT

2) THIS 1-D CASE

\[ d\mu = \bar{\mu} \, dx \]

\[ F_\mu(x) = \mu \left((0, x)\right) \]

\[ F_\mu^{-1}(x) = \inf \left\{ t \mid F_\mu(t) \leq x \right\} \]

\[ T = F_\nu^{-1}(F_\mu(x)) \]