Grushin problem method

Francis Nier, LAGA, Univ. Paris XIII

Cermics March 28th 2024

Warning:

- Semiclassical is more than semiclassical.
- A Grushin problem is not a Grushin problem.

A reference: J. Sjöstrand and M. Zworski "Elementary linear algebra for advanced spectral problems" (Ann. Inst. Fourier (2007)).

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  Hörmander FIO I (1971):*"The purpose of the present paper is not to extend the more or less formal methods used in geometrical optics but to extract from them a precise operator theory which can be applied to the theory of partial differential equations"*
- A Grushin problem is not a Grushin problem.
  The name and the notations were actually introduced in the early works of J. Sjöstrand (phD, 1973)

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Outline

- Schur complement and Grushin problem. First applications.
- Multiple wells, resonances
- Comparison of Langevin and overdamped Langevin.

Schur complement

The Grushin problem method can be viewed as a variation of the Schur complement formula of Linear Algebra (Numerical Analysis), like the Feshbach formula (Math. Phys.) or the Lyapunov-Schmidt method (Dynamical systems and non linear analysis).

Schur complement formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -A^{-1}B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -CA^{-1} & 1 \end{pmatrix},$$

whenever the block A is invertible as well as its Schur complement $(D - CA^{-1}B)$.

Grushin problem works in a different way

The block matrix is constructed in such a way that it is invertible and its relates the invertibility of A to the invertibility of a block of D-size (finite dimension for Fredholm theory).

Let $\theta \mapsto A_\theta$ be a continuous map from the metric space (X, d) to $\mathcal{L}(E; F)$.
Important example:
F Banach space; $E = D(A) \subset F$ the domain of a closed operator A;
(X, d) a Banach space of relatively bounded perturbations of A with bound less than 1 ($\Rightarrow A_\theta$ with $D(A_\theta) = D(A) = E$).

Definition

The operator $A \in \mathcal{L}(E, F)$ is Fredholm if $\dim \ker A = a_+ < \infty$, if its range $\mathrm{Ran} A$ is closed and if $\mathrm{codim}(\mathrm{Ran} A) = \dim \mathrm{coker} A = a_- < \infty$.
$\mathrm{Fred}(E; F)$ set of Fredholm operators.
$\mathrm{Ind}(A) = a_+ - a_-$.

Remember (checked below)

- $\mathrm{Fred}(E; F)$ is an open set of $\mathcal{L}(E; F)$;
- Ind is constant on connected components of $\mathrm{Fred}(E; F)$

Let us assume $A_{\theta_0} \in \mathrm{Fred}(E; F)$ (here $(X, d) = \mathcal{L}(E; F)$) with $\dim \ker A_{\theta_0} = a_+$ and $\dim \mathrm{coker} A_{\theta_0} = a_-$.

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Comparison of Langevin and overdamped Langevin

$$R_- : \mathbb{C}^{a-} \to F \quad \text{et} \quad R_+ : E \to \mathbb{C}^{a+}\,, \quad R_+ \in \mathcal{L}(E;\mathbb{C}^{a+})\,, \quad R_+\big|_{\ker A_{\theta_0}} \text{ bij.}$$

R_- (resp. R_+^{-1}) parametrizes $\text{coker}(A_{\theta_0})$ (resp. $\ker(A_{\theta_0})$) and

$$\mathcal{A}_\theta = \begin{pmatrix} A_\theta & R_- \\ R_+ & 0 \end{pmatrix} : \begin{matrix} E \\ \oplus \\ \mathbb{C}^{a-} \end{matrix} \to \begin{matrix} F \\ \oplus \\ \mathbb{C}^{a+} \end{matrix}\,.$$

is invertible for $\theta \sim \theta_0$ and we set

$$\mathcal{A}(\theta)^{-1} = \begin{pmatrix} E(\theta) & E_+(\theta) \\ E_-(\theta) & E_{-+}(\theta) \end{pmatrix} : \begin{matrix} F \\ \oplus \\ \mathbb{C}^{a+} \end{matrix} \to \begin{matrix} E \\ \oplus \\ \mathbb{C}^{a-} \end{matrix}\,.$$

For $\theta = \theta_0$ solving $\mathcal{A}_{\theta_0}\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}$ gives

$$\begin{pmatrix} E(\theta_0) & E_+(\theta_0) \\ E_-(\theta_0) & E_{-+}(\theta_0) \end{pmatrix} = \begin{pmatrix} \left(A_{\theta_0}\big|_{\ker R_+}\right)^{-1}\pi_{\mathrm{Ran}A_{\theta_0}} & \left(R_+\big|_{\ker A_{\theta_0}}\right)^{-1} \\ (R_-)^{-1}\pi_{\mathrm{Ran}R_-} & 0 \end{pmatrix}.$$

In particular $E_-(\theta_0)$ (resp. $E_+(\theta_0)$) has the maximal rank a_- (resp. a_+), which is an open condition ($\Rightarrow$ the same holds true for $E_-(\theta)$ and $E_+(\theta)$ when $\theta \sim \theta_0$).

Now by setting $R_+ u = v_+$, the equivalences

$$(A_\theta u = v) \Leftrightarrow \left\{ \begin{matrix} A_\theta u + R\_0 & = & v \\ R_+ u & = & v_+ \end{matrix} \right. \Leftrightarrow \left\{ \begin{matrix} u = E(\theta)v + E_+(\theta)v_+ \\ 0 = E_-(\theta)v + E_{-+}(\theta)v_+ \end{matrix} \right.,$$

lead to $\quad (v \in \mathrm{Ran}A_\theta) \Leftrightarrow \left(E_-(\theta)v \in \mathrm{Ran}E_{-+}(\theta)\right)$

and $(v = 0)$ $\quad (u \in \ker A_\theta) \Leftrightarrow \left(u \in E_+(\theta)\ker E_{-+}(\theta)\right)$.

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$$\begin{pmatrix} E(\theta_0) & E_+(\theta_0) \\ E_-(\theta_0) & E_{-+}(\theta_0) \end{pmatrix} = \begin{pmatrix} (A_{\theta_0}\big|_{\ker R_+})^{-1}\pi_{\mathrm{Ran}A_{\theta_0}} & (R_+\big|_{\ker A_{\theta_0}})^{-1} \\ (R_-)^{-1}\pi_{\mathrm{Ran}R_-} & 0 \end{pmatrix}.$$

In particular $E_-(\theta_0)$ **(resp.** $E_+(\theta_0)$**) has the maximal rank** a_- **(resp.** a_+**), which is an open condition (**$\Rightarrow$ **the same holds true for** $E_-(\theta)$ **and** $E_+(\theta)$ **when** $\theta \sim \theta_0$**).**
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$$(A_\theta u = v) \Leftrightarrow \left\{ \begin{array}{lcl} A_\theta u + R\_0 & = & v \\ R_+ u & = & v_+ \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} u = E(\theta)v + E_+(\theta)v_+ \\ 0 = E_-(\theta)v + E_{-+}(\theta)v_+ \end{array} \right.,$$

$$\textbf{lead to} \qquad (v \in \mathrm{Ran}A_\theta) \Leftrightarrow \big(E_-(\theta)v \in \mathrm{Ran}E_{-+}(\theta)\big)$$

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Grushin problem method
Francis Nier, LAGA, Univ. Paris XIII
Schur complement and Grushin problem
Grushin problem and multiscale analysis
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$$(v \in \mathrm{Ran} A_\theta) \Leftrightarrow (E_-(\theta) v \in \mathrm{Ran} E_{-+}(\theta))$$

$$(u \in \ker A_\theta) \Leftrightarrow (u \in E_+(\theta) \ker E_{-+}(\theta)) ,$$

$$\mathrm{rank} E_-(\theta) = a_- \quad , \quad \mathrm{rank} E_+(\theta) = a_+ \quad \text{when } \theta \sim \theta_+ .$$

$\dim \mathrm{Ran} E_{-+}(\theta) < \infty$ and $E_-(\theta) \in \mathcal{L}(F; \mathbb{C}^{a_-}) \Rightarrow \mathrm{Ran} A_\theta$ closed.
Because $E_-(\theta)$ is onto and $E_+(\theta)$ is one to one, we deduce

- $\dim \mathrm{coker} A_\theta = \dim \mathrm{coker} E_{-+}(\theta)$;
- $\dim \ker A_\theta = \dim \ker E_{-+}(\theta)$.

Proposition

With the above notations and with $\theta \in \mathcal{V}_{\theta_0}$ small neighborhood of θ_0, $A_\theta \in \mathrm{Fred}(E; F)$ and

$$\mathrm{Ind} A_\theta = \mathrm{Ind} E_{-+}(\theta) = a_+ - a_-$$

which does not depend on $\theta \in \mathcal{V}_{\theta_0}$. Additionally when $\mathrm{Ind}(A_{\theta_0}) = 0$, $a_+ = a_-$, we have

$$(A_\theta \text{ invertible}) \Leftrightarrow (E_{-+}(\theta) \text{ invertible}) \Leftrightarrow (\det E_{-+}(\theta) \neq 0) ,$$

with the formula

$$A_\theta^{-1} = E(\theta) - E_+(\theta) E_{-+}(\theta)^{-1} E_+(\theta) .$$

Grushin problem method
Francis Nier, LAGA, Univ. Paris XIII
Schur complement and Grushin problem
Grushin problem and multiscale analysis
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Schur complement and Grushin problem

Grushin problem and multiscale analysis

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Grushin problem method

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Schur complement and Grushin problem

Grushin problem and multiscale analysis

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When $A : D(A) \to F$ is closed and the imbedding $D(A) \to F$ is compact then either $\sigma(A) = \mathbb{C}$ or $\sigma(A) = \sigma_{disc}(A)$.

For $a = \frac{\partial_x + x}{\sqrt{2}}$ with $D(a) = D(\mathcal{O}^{1/2})$, $\mathcal{O} = \frac{-\Delta + x^2}{2}$, compactly included in $L^2(\mathbb{R}, dx)$, $\sigma(a) = \sigma(a^*) = \mathbb{C}$.

$\sigma(-\Delta_x + ix) = \emptyset$ for the complex Airy operator $-\Delta_x + ix$ with $D(A) = \left\{u \in L^2(\mathbb{R}, dx)\,, \quad -\Delta_x u \text{ and } xu \in L^2(\mathbb{R}, dx)\right\}$.
Subelliptic consequence:

$$\forall u \in \mathcal{C}_K^\infty(\mathbb{R}^2), \quad \|(-\Delta_{x_1} + x_1\partial_{x_2})u\|_{H^s} \geq C_K \|u\|_{H^{s+2/3}}$$

Weyl's theorem when $B : D(A) \to F$ is compact the set $\{(t, z) \in \mathbb{C} \times (\mathbb{C} \setminus \sigma_{ess}(A)), z \in \sigma(A + tB)\}$ is an analytic set. (locally $\det(E_{-+}(z, t)) = 0$)
When $F = \ell^2(\mathbb{Z})$ consider the operators A et $C \in \mathcal{L}(F)$ given by

$$(A\varphi)_n = \varphi_{n+1} \quad \text{et} \quad (C\varphi)_n = \delta_{n,0}\varphi_1$$

Then $\sigma_{ess}(A) = \mathbb{S}^1$ and

- $\sigma_{ess}(A - tC) = \mathbb{S}^1$ for $t \neq 1$;
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Grushin problem method

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Schur complement and Grushin problem

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Examples of non s.a. operators

When $A : D(A) \to F$ is closed and the imbedding $D(A) \to F$ is compact then either $\sigma(A) = \mathbb{C}$ or $\sigma(A) = \sigma_{disc}(A)$.

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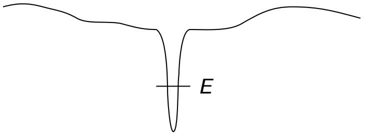
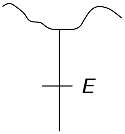
Quantum/semiclassical separation





Spectrum around E of $-\Delta + U(x - \frac{x_0}{h}) + \tilde{V}(hx)$ or $-h^2\Delta + U(\frac{x}{h}) + \tilde{V}(x_0 + x)$

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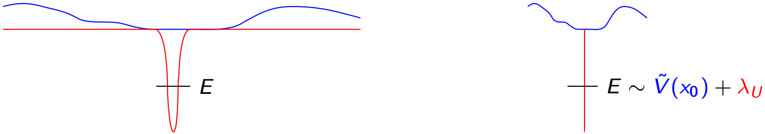

There is an eigenvalue $E^h = \tilde{V}(x_0) + \lambda_U + o(h^0)$ where $\lambda_U \in \sigma(-\Delta + U)$
The spectral elements (eigenvectors, spectral projectors...) of
$H^h = -\Delta + U(x) + \tilde{V}(hx)$ are well approximated by the ones of H_U

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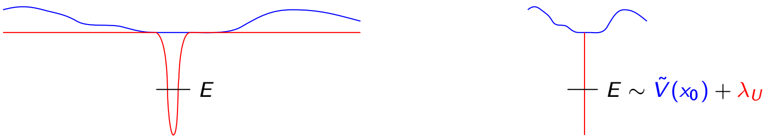

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Exponential decay:

Quantum Hamiltonian

$$H_U = -\Delta + U(x)$$

$$H_U\psi_U = \lambda_U\psi_U\,,\ \lambda_U \leq E/2 < 0$$

$$\psi_U(x) = \tilde{\mathcal{O}}(e^{-\alpha_E|x|})$$

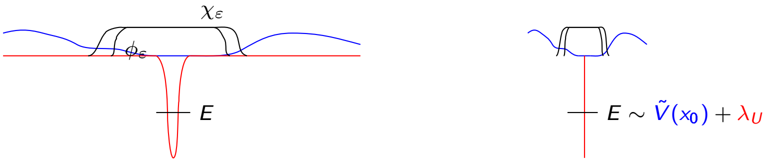
Filled well Hamiltonian (semiclassical)

$$\tilde{H}^h = -\Delta + \tilde{V}(hx)$$

$$\tilde{V} \geq \frac{E}{4} \text{ and } \mathbb{R}\mathrm{e}\ z \leq \frac{E}{2}\,,$$

$$(z - \tilde{H}^h)^{-1}(x,y) = \tilde{\mathcal{O}}(e^{-\alpha_E|x-y|})$$

Quantum/semiclassical separation



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Exponential decay:

Quantum Hamiltonian	Filled well Hamiltonian (semiclassical)
$H_U = -\Delta + U(x)$	$\tilde{H}^h = -\Delta + \tilde{V}(hx)$
$H_U\psi_U = \lambda_U\psi_U\,,\ \lambda_U \leq E/2 < 0$	$\tilde{V} \geq \frac{E}{4}$ and $\Re e\ z \leq \frac{E}{2}$,
$\psi_U(x) = \tilde{\mathcal{O}}(e^{-\alpha_E\lvert x\rvert})$	$(z - \tilde{H}^h)^{-1}(x,y) = \tilde{\mathcal{O}}(e^{-\alpha_E\lvert x-y\rvert})$

Cut-off: $\phi, \chi \in \mathcal{C}_0^\infty\,,\ \phi \preceq \chi\,,$

$$1 = \underbrace{\phi\left(\frac{hx - x_0}{\varepsilon}\right)}_{\phi_\varepsilon(hx)} + (1-\phi)\left(\frac{hx - x_0}{\varepsilon}\right) = \underbrace{\chi\left(\frac{hx - x_0}{\varepsilon}\right)}_{\chi_\varepsilon(hx)} + (1-\chi)\left(\frac{hx - x_0}{\varepsilon}\right)$$

Grushin problem for H_U:

$$\mathcal{A}^0(z) = \begin{pmatrix} H_U - z & R_-^0 \\ R_+^0 & 0 \end{pmatrix} : \begin{matrix} D(H_U) \\ \oplus \\ \mathbb{C} \end{matrix} \to \begin{matrix} L^2 \\ \oplus \\ \mathbb{C} \end{matrix}$$

With $R_-^0 = |\psi_U\rangle$ and $R_+^0 = \langle\psi_U|$, $\mathcal{A}^0(\lambda_U)$ is invertible $\Rightarrow$ so $\mathcal{A}^0(z)$ is invertible for $z \in \mathcal{V}(\lambda_U)$:

$$\mathcal{G}^0(z) = [\mathcal{A}^0(z)]^{-1} = \begin{pmatrix} E_0^0(z) & E_+^0 \\ E_-^0 & E_{-+}^0(z) \end{pmatrix} .$$

We consider now for z close to $\tilde{V}(x_0) + \lambda_U$

$$\mathcal{A}(z) = \begin{pmatrix} H^h - z & R_- = \chi_\varepsilon R_-^0 \\ R_+ = R_+^0 & 0 \end{pmatrix}$$

and take

$$\mathcal{F}(z) = \begin{pmatrix} \chi_\varepsilon E_0^0(z - \tilde{V}(x_0))\phi_\varepsilon + (\tilde{H}^h - z)^{-1}(1 - \phi_\varepsilon) & \chi_\varepsilon E_+^0 \\ E_-^0 \phi_\varepsilon & E_{-+}^0(z - \tilde{V}(x_0)) \end{pmatrix}$$

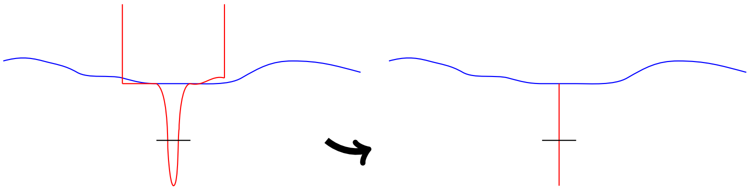
A direct computation shows (with $h \leq \varepsilon^2$)

$$\mathcal{A}(z)\mathcal{F}(z) = \mathrm{Id} + \mathcal{O}(\varepsilon) + \tilde{O}(e^{-\frac{\varepsilon}{h}}) \quad \text{and} \quad E_{-+}(z) = E_{-+}^0(z - \tilde{V}(x_0)) + \mathcal{O}(\varepsilon) + \tilde{O}(e^{-\frac{\varepsilon}{h}}))$$

Combined with $(H^h - z)^{-1} = E(z) - E_+(z)(E_{-+}(z))^{-1}E_-(z)$ this allows to compare spectral quantities (eigenvalues, projector) around E.

Changing H_U, changing $\tilde{H}^h$

In the construction of the approximate inverse $\mathcal{F}(z)$, H_U can be replaced by any quantum local problem (think of ϕ_ε and χ_ε-truncations)



The long range exponential decay of eigenvector with energy $\sim E$ comes from the exponential decay estimate of $\tilde{H}^h$, expressed here in terms of

$$d_{Ag}(x', y'; E) = \inf_{\substack{\gamma \\ \gamma(0)=x' \\ \gamma(1)=y'}} \int_0^1 \sqrt{\tilde{V}(\gamma(t)) - E}|\dot{\gamma}(t)|\, dt\,,\ x' = hx\,,\ y' = hy\,.$$

A perturbative analysis of $E_{-+}(z)^{-1}$ for two different choices $\tilde{V}_1$ and $\tilde{V}_2$,

$\tilde{V}_1 = \tilde{V}_2$ in $|h(x - x_0)| \leq R$, leads to an error of size $\tilde{O}(e^{-2\frac{S_R}{h}})$ where $S_R = \inf_{|x'-x_0|=R} d_{Ag}(x', x_0)$ as long as $\inf_{x\in\mathbb{R}} \tilde{V}_k(x) > E$.

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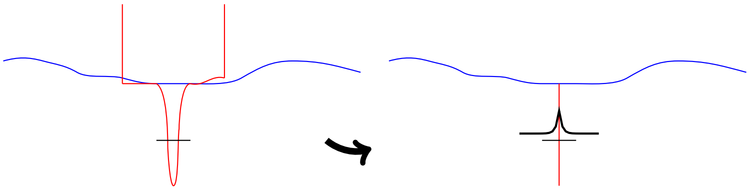

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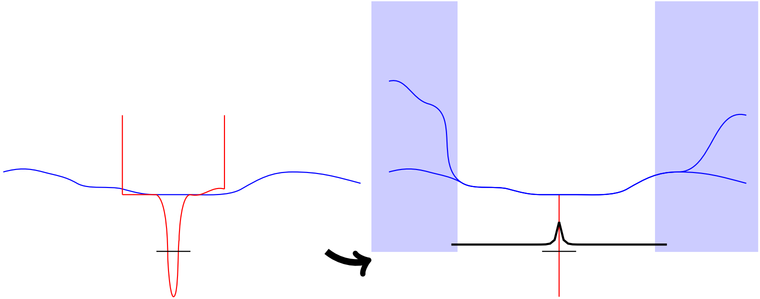

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