Grushin problem method

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Warning:
- Semiclassical is more than semiclassical.
- A Grushin problem is not a Grushin problem.

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  Hörmander FIO I (1971): "The purpose of the present paper is not to extend the more or less formal methods used in geometrical optics but to extract from them a precise operator theory which can be applied to the theory of partial differential equations"

- A Grushin problem is not a Grushin problem.
  The name and the notations were actually introduced in the early works of J. Sjöstrand (phD, 1973)

Outline

- Schur complement and Grushin problem. First applications.
- Multiple wells, resonances
- Comparison of Langevin and overdamped Langevin.
The Grushin problem method can be viewed as a variation of the Schur complement formula of Linear Algebra (Numerical Analysis), like the Feshbach formula (Math. Phys.) or the Lyapunov-Schmidt method (Dynamical systems and non linear analysis).

Schur complement formula

\[
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^{-1}
= \begin{pmatrix}
1 & -A^{-1}B \\
0 & 1
\end{pmatrix} \begin{pmatrix}
A^{-1} & 0 \\
0 & (D - CA^{-1}B)^{-1}
\end{pmatrix} \begin{pmatrix}
1 & 0 \\
-CA^{-1} & 1
\end{pmatrix},
\]

whenever the block $A$ is invertible as well as its Schur complement $(D - CA^{-1}B)$.

Grushin problem works in a different way

The block matrix is constructed in such a way that it is invertible and its relates the invertibility of $A$ to the invertibility of a block of $D$-size (finite dimension for Fredholm theory).
Let $\theta \mapsto A_\theta$ be a continuous map from the metric space $(X, d)$ to $\mathcal{L}(E; F)$.

**Important example:**

$F$ Banach space; $E = D(A) \subset F$ the domain of a closed operator $A$; $(X, d)$ a Banach space of relatively bounded perturbations of $A$ with bound less than 1 ($\Rightarrow A_\theta$ with $D(A_\theta) = D(A) = E$).

**Definition**

The operator $A \in \mathcal{L}(E, F)$ is Fredholm if $\dim \ker A = a_+ < \infty$, if its range $\text{Ran} A$ is closed and if $\text{codim}(\text{Ran} A) = \dim \text{coker} A = a_- < \infty$.

$\text{Fred}(E; F)$ set of Fredholm operators.

$\text{Ind}(A) = a_+ - a_-$. 

Remember (checked below)

- $\text{Fred}(E; F)$ is an open set of $\mathcal{L}(E; F)$;
- $\text{Ind}$ is constant on connected components of $\text{Fred}(E; F)$

Let us assume $A_{\theta_0} \in \text{Fred}(E; F)$ (here $(X, d) = \mathcal{L}(E; F)$) with

$\dim \ker A_{\theta_0} = a_+$ and $\dim \text{coker} A_{\theta_0} = a_-$. 

Let $\theta \mapsto A_\theta$ be a continuous map from the metric space $(X, d)$ to $\mathcal{L}(E; F)$. Important example: $F$ Banach space; $E = D(A) \subset F$ the domain of a closed operator $A$; $(X, d)$ a Banach space of relatively bounded perturbations of $A$ with bound less than $1$ ($\Rightarrow A_\theta$ with $D(A_\theta) = D(A) = E$).

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\[ R_- : \mathbb{C}^a- \to F \quad \text{et} \quad R_+ : E \to \mathbb{C}^a+, \quad R_+ \in \mathcal{L}(E; \mathbb{C}^a), \quad R_+|_{\ker A_\theta} \text{ bij}. \]

\( R_- \) (resp. \( R_-^{-1} \)) parametrizes \( \text{coker}(A_\theta) \) (resp. \( \ker(A_\theta) \)) and

\[ A_\theta = \begin{pmatrix} A_\theta & R_- \\ R_+ & 0 \end{pmatrix} : \begin{pmatrix} E \\ \mathbb{C}^a- \end{pmatrix} \to \begin{pmatrix} F \\ \mathbb{C}^a+ \end{pmatrix}. \]

is invertible for \( \theta \sim \theta_0 \) and we set

\[ A(\theta)^{-1} = \begin{pmatrix} E(\theta) & E_+(\theta) \\ E_-(\theta) & E_-(\theta) \end{pmatrix} : \begin{pmatrix} F \\ \mathbb{C}^a+ \end{pmatrix} \to \begin{pmatrix} E \\ \mathbb{C}^a- \end{pmatrix}. \]

For \( \theta = \theta_0 \) solving \( A_{\theta_0}(u) = (v) \) gives

\[ \begin{pmatrix} E(\theta_0) & E_+(\theta_0) \\ E_-(\theta_0) & E_-(\theta_0) \end{pmatrix} = \begin{pmatrix} (A_{\theta_0}|_{\ker R_+})^{-1} \pi \text{Ran} A_{\theta_0} & (R_+|_{\ker A_{\theta_0}})^{-1} \\ (R_-)^{-1} \pi \text{Ran} R_- & 0 \end{pmatrix}. \]

In particular \( E_-(\theta_0) \) (resp. \( E_+(\theta_0) \)) has the maximal rank \( a_- \) (resp. \( a_+ \)), which is an open condition \( \Rightarrow \) the same holds true for \( E_-(\theta) \) and \( E_+(\theta) \) when \( \theta \sim \theta_0 \).

Now by setting \( R_+ u = v_+ \), the equivalences

\[ (A_\theta u = v) \iff \begin{cases} A_\theta u + R_- 0 \\ R_+ u \end{cases} = \begin{cases} v \\ v_+ \end{cases} \iff \begin{cases} u = E(\theta)v + E_+(\theta)v_+ \\ 0 = E_-(\theta)v + E_-(\theta)v_+ \end{cases}, \]

lead to \( (v \in \text{Ran} A_\theta) \iff (E_-(\theta)v \in \text{Ran} E_-(\theta)) \)

and \( (v = 0) \iff (u \in \ker A_\theta) \iff (u \in E_+(\theta) \ker E_+(\theta)) \).
Grushin pb and Fredholm thy

\[ R_- : \mathbb{C}^a \to F \quad \text{et} \quad R_+ : E \to \mathbb{C}^{a+}, \quad R_+ \in \mathcal{L}(E; \mathbb{C}^{a+}), \quad R_+|_{\ker A_{\theta_0}} \text{ bij.} \]

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\[ A_{\theta} = \begin{pmatrix} A_{\theta} & R_- \\ R_+ & 0 \end{pmatrix} : \begin{array}{c} E \\ \mathbb{C}^a \end{array} \to \begin{array}{c} F \\ \mathbb{C}^{a+} \end{array}. \]

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\begin{pmatrix} E(\theta_0) & E_+(\theta_0) \\ E_-(\theta_0) & E_-(\theta_0) \end{pmatrix} = \begin{pmatrix} (A_{\theta_0} \mid_{\ker R_+})^{-1} \pi_{\text{Ran} A_{\theta_0}} & (R_+|_{\ker A_{\theta_0}})^{-1} \\ (R_-)^{-1} \pi_{\text{Ran} R_-} & 0 \end{pmatrix}.
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(A_{\theta} u = v) \Leftrightarrow \begin{cases} A_{\theta} u + R_- 0 = v \\ R_+ u = \mathbf{v}_+ \end{cases} \Leftrightarrow \begin{cases} u = E(\theta)v + E_+(\theta)v_+ \\ 0 = E_-(\theta)v + E_-+(\theta)v_+ \end{cases}.
\]

lead to

\[
(v \in \text{Ran} A_{\theta}) \Leftrightarrow \left( E_-(\theta)v \in \text{Ran} E_-(\theta) \right)
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and \( (v = 0) \)

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(u \in \ker A_{\theta}) \Leftrightarrow \left( u \in E_+(\theta) \ker E_+(\theta) \right).
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Grushin problem and Fredholm thy

\[ \mathcal{A}(\theta)^{-1} = \begin{pmatrix} E(\theta) & E_+(\theta) \\ E_-(\theta) & E_-(\theta) \end{pmatrix} : \bigoplus_{\mathbb{C}^a^+} F \to \bigoplus_{\mathbb{C}^a^-} E, \]

\((v \in \text{Ran} A_\theta) \iff (E_-(\theta)v \in \text{Ran} E_+(\theta))\)

\((u \in \ker A_\theta) \iff (u \in E_+(\theta)\ker E_+(\theta))\),

\(\text{rank} E_-(\theta) = a_-, \quad \text{rank} E_+(\theta) = a_+\) when \(\theta \sim \theta_+\).

\(\dim \text{Ran} E_+(\theta) < \infty\) and \(E_+(\theta) \in \mathcal{L}(F; \mathbb{C}^a_-) \Rightarrow \text{Ran} A_\theta\) closed.

Because \(E_-(\theta)\) is onto and \(E_+(\theta)\) is one to one, we deduce
- \(\dim \ker A_\theta = \dim \ker E_+(\theta)\);
- \(\dim \ker A_\theta = \dim \ker E_+(\theta)\).

**Proposition**

*With the above notations and with \(\theta \in \mathcal{V}_{\theta_0}\) small neighborhood of \(\theta_0\), \(A_\theta \in \text{Fred}(E; F)\) and

\(\text{Ind} A_\theta = \text{Ind} E_+(\theta) = a_+ - a_-\)

which does not depend on \(\theta \in \mathcal{V}_{\theta_0}\). Additionally when \(\text{Ind}(A_{\theta_0}) = 0\), \(a_+ = a_-\), we have

\((A_\theta \text{ invertible}) \iff (E_+(\theta) \text{ invertible}) \iff (\det E_+(\theta) \neq 0),\)

with the formula

\[ A_\theta^{-1} = E(\theta) - E_+(\theta)E_+(\theta)^{-1}E_+(\theta). \]
Grushin problem and Fredholm theory

$$\mathcal{A}(\theta)^{-1} = \begin{pmatrix} E(\theta) & E_+(\theta) \\ E_-(\theta) & E_-^+(\theta) \end{pmatrix} \colon F \oplus \mathbb{C}^{a+} \to E \oplus \mathbb{C}^{a-},$$

$$(v \in \text{Ran} A_\theta) \iff (E_-(\theta)v \in \text{Ran} E_-^+(\theta))$$
$$(u \in \ker A_\theta) \iff (u \in E_+(\theta) \ker E_-^+(\theta)),$$

$$\text{rank} E_-(\theta) = a_-, \quad \text{rank} E_+(\theta) = a_+ \quad \text{when} \ \theta \sim \theta_+.$$

$$\dim \text{Ran} E_-^+(\theta) < \infty \text{ and } E_-^+(\theta) \in \mathcal{L}(F; \mathbb{C}^{a_-}) \Rightarrow \text{Ran} A_\theta \text{ closed.}$$

Because $E_-^+(\theta)$ is onto and $E_+(\theta)$ is one to one, we deduce

- $\dim \ker A_\theta = \dim \ker E_-^+(\theta);$  
- $\dim \ker A_\theta = \dim \ker E_+(\theta).$

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which does not depend on $\theta \in \mathcal{V}_{\theta_0}$. Additionally when $\text{Ind}(A_{\theta_0}) = 0$, $a_+ = a_-$, we have

$$(A_\theta \text{ invertible}) \iff (E_-^+(\theta) \text{ invertible}) \iff (\det E_-^+(\theta) \neq 0),$$

with the formula

$$A_\theta^{-1} = E(\theta) - E_+(\theta)E_-^+(\theta)^{-1}E_+(\theta).$$*
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A(\theta)^{-1} = \begin{pmatrix}
E(\theta) & E_+(\theta) \\
E_-(\theta) & E_-(\theta)
\end{pmatrix} : \bigoplus_{\mathbb{C}^{a+}} F \to \bigoplus_{\mathbb{C}^{a-}} E,
\]

\( (v \in \text{Ran}A_\theta) \iff (E_-(\theta)v \in \text{Ran}E_+(\theta)) \)
\( (u \in \text{ker} A_\theta) \iff (u \in E_+(\theta) \text{ker} E_+(\theta)) \),

\( \text{rank}E_-(\theta) = a_- \) , \( \text{rank}E_+(\theta) = a_+ \) when \( \theta \sim \theta_+ \).

\( \dim \text{Ran}E_+(\theta) < \infty \) and \( E_-(\theta) \in \mathcal{L}(F; \mathbb{C}^{-a-}) \Rightarrow \text{Ran}A_\theta \) closed.

Because \( E_-(\theta) \) is onto and \( E_+(\theta) \) is one to one, we deduce
- \( \dim \text{coker}A_\theta = \dim \text{coker}E_+(\theta) \),
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Grushin problem and Fredholm theory

\[ A(\theta)^{-1} = \begin{pmatrix} E(\theta) & E_{+}(\theta) \\ E_{-}(\theta) & E_{-+}(\theta) \end{pmatrix} : \mathbb{F} \oplus \mathbb{C}^{a+} \to \mathbb{E} \oplus \mathbb{C}^{a-} , \]

\( (v \in \text{Ran} A_\theta) \Leftrightarrow (E_{-}(\theta)v \in \text{Ran} E_{-+}(\theta)) \)

\( (u \in \text{ker} A_\theta) \Leftrightarrow (u \in E_{+}(\theta) \text{ker} E_{-+}(\theta)) , \)

\( \text{rank} E_{-}(\theta) = a_- , \quad \text{rank} E_{+}(\theta) = a_+ \quad \text{when} \ \theta \sim \theta_+ . \)

\( \text{dim} \ \text{Ran} E_{-+}(\theta) < \infty \ \text{and} \ E_{-}(\theta) \in \mathcal{L}(\mathbb{F} ; \mathbb{C}^{a-}) \Rightarrow \text{Ran} A_\theta \ \text{closed.} \)

Because \( E_{-}(\theta) \) is onto and \( E_{+}(\theta) \) is one to one, we deduce

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with the formula \( A_{\theta}^{-1} = E(\theta) - E_{+}(\theta)E_{-+}(\theta)^{-1}E_{+}(\theta) .\)
Examples of non s.a. operators

When $A : D(A) \to F$ is closed and the imbedding $D(A) \to F$ is compact then either $\sigma(A) = \mathbb{C}$ or $\sigma(A) = \sigma_{\text{disc}}(A)$. 

For $a = \frac{\partial x + x}{\sqrt{2}}$ with $D(a) = D(\mathcal{O}^{1/2})$, $\mathcal{O} = \frac{-\Delta + x^2}{2}$, compactly included in $L^2(\mathbb{R}, dx)$, $\sigma(a) = \sigma(a^*) = \mathbb{C}$. 

$\sigma(-\Delta_x + ix) = \emptyset$ for the complex Airy operator $-\Delta_x + ix$ with $D(A) = \{ u \in L^2(\mathbb{R}, dx), \ -\Delta_x u \text{ and } xu \in L^2(\mathbb{R}, dx) \}$. 

Subelliptic consequence: 

$$\forall u \in C^\infty_K(\mathbb{R}^2), \quad \|(-\Delta_{x_1} + x_1 \partial_{x_2})u\|_{H^s} \geq C_K \|u\|_{H^{s+2/3}}$$ 

Weyl’s theorem when $B : D(A) \to F$ is compact the set 

$$\{(t, z) \in \mathbb{C} \times (\mathbb{C} \setminus \sigma_{\text{ess}}(A)), z \in \sigma(A + tB) \}$$ is an analytic set. (locally $\det(E_{-+}(z, t)) = 0$) 

When $F = \ell^2(\mathbb{Z})$ consider the operators $A$ et $C \in \mathcal{L}(F)$ given by 

$$(A\varphi)_n = \varphi_{n+1} \quad \text{et} \quad (C\varphi)_n = \delta_{n,0}\varphi_1$$ 

Then $\sigma_{\text{ess}}(A) = S^1$ and 

- $\sigma_{\text{ess}}(A - tC) = S^1$ for $t \neq 1$; 
- $\sigma(A - C) = \overline{D(0, 1)}$. 


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Quantum/semiclassical separation

Spectrum around $E$ of $-\Delta + U(x - \frac{x_0}{\hbar}) + \tilde{V}(hx)$ or $-\hbar^2 \Delta + U(\frac{x}{\hbar}) + \tilde{V}(x_0 + x)$
There is an eigenvalue $E^h = \tilde{V}(x_0) + \lambda_U + o(h^0)$ where $\lambda_U \in \sigma(-\Delta + U)$.

The spectral elements (eigenvectors, spectral projectors...) of $H^h = -\Delta + U(x) + \tilde{V}(hx)$ are well approximated by the ones of $H_U$. 

Quantum/semiclassical separation

Grushin problem and multiscale analysis

Comparison of Langevin and overdamped Langevin
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**Exponential decay:**

**Quantum Hamiltonian**

$H_U = -\Delta + U(x)$

$H_U \psi_U = \lambda_U \psi_U$, $\lambda_U \leq E/2 < 0$

$\psi_U(x) = \tilde{O}(e^{-\alpha_E|x|})$

**Filled well Hamiltonian (semiclassical)**

$\tilde{H}^h = -\Delta + \tilde{V}(hx)$

$\tilde{V} \geq \frac{E}{4}$ and $\Re z \leq \frac{E}{2}$, $z - \tilde{H}^h)^{-1}(x, y) = \tilde{O}(e^{-\alpha_E|x-y|})$
Quantum/semiclassical separation

The eigenvalue is given by $E^h = \tilde{V}(x_0) + \lambda_U + o(h^0)$ where $\lambda_U \in \sigma(-\Delta + U)$

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\[ H_U \psi_U = \lambda_U \psi_U, \; \lambda_U \leq E/2 < 0 \]

\[ \psi_U(x) = \mathcal{O}(e^{-\alpha_E|x|}) \]

**Cut-off:** $\phi, \chi \in C_0^\infty$, $\phi \preceq \chi$,

\[ 1 = \phi \left( \frac{hx - x_0}{\varepsilon} \right) + (1 - \phi) \left( \frac{hx - x_0}{\varepsilon} \right) = \chi \left( \frac{hx - x_0}{\varepsilon} \right) + (1 - \chi) \left( \frac{hx - x_0}{\varepsilon} \right) \]

\[ \phi_\varepsilon(hx) \]

\[ \chi_\varepsilon(hx) \]

**Filled well Hamiltonian (semiclassical)**

\[ \tilde{H}^h = -\Delta + \tilde{V}(hx) \]

\[ \tilde{V} \geq \frac{E}{4} \text{ and } \Re z \leq \frac{E}{2}, \]

\[ (z - \tilde{H}^h)^{-1}(x, y) = \mathcal{O}(e^{-\alpha_E|x-y|}) \]
Quantum/Semiclassical separation

**Grushin problem for** $H_U$:

\[
A^0(z) = \begin{pmatrix}
H_U - z & R^0_-
R^0_+ & 0
\end{pmatrix} : D(H_U) \oplus \mathbb{C} \to \oplus \mathbb{C}
\]

With $R^0_- = |\psi_U\rangle$ and $R^0_+ = \langle \psi_U|$, $A^0(\lambda_U)$ is invertible $\Rightarrow$ so $A^0(z)$ is invertible for $z \in \mathcal{V}(\lambda_U)$:

\[
G^0(z) = [A^0(z)]^{-1} = \begin{pmatrix}
E^0_0(z) & E^0_-
E^0_+ & E^0_{-+}(z)
\end{pmatrix}.
\]

We consider now for $z$ close to $\tilde{V}(x_0) + \lambda_U$

\[
A(z) = \begin{pmatrix}
H^h - z & R_-
R_+ = R^0_+ & 0
\end{pmatrix}
\]

and take

\[
\mathcal{F}(z) = \begin{pmatrix}
\chi_\varepsilon E^0_0(z - \tilde{V}(x_0))\phi_\varepsilon + (\tilde{H}^h - z)^{-1}(1 - \phi_\varepsilon)
E^0_0\phi_\varepsilon
\end{pmatrix}
\]

A direct computation shows (with $h \leq \varepsilon^2$)

\[
A(z)\mathcal{F}(z) = \text{Id} + O(\varepsilon) + \tilde{O}(e^{-\frac{\varepsilon}{h}}) \quad \text{and} \quad E_{-+}(z) = E^0_{-+}(z - \tilde{V}(x_0)) + O(\varepsilon) + \tilde{O}(e^{-\frac{\varepsilon}{h}})
\]

Combined with $(H^h - z)^{-1} = E(z) - E_+(z)(E_{-+}(z))^{-1}E_-(z)$ this allows to compare spectral quantities (eigenvalues, projector) around $E$. 

Changing $H_U$, changing $\tilde{H}^h$

In the construction of the approximate inverse $\mathcal{F}(z)$, $H_U$ can be replaced by any quantum local problem (think of $\phi_\varepsilon$ and $\chi_\varepsilon$-truncations)

The long range exponential decay of eigenvector with energy $\sim E$ comes from the exponential decay estimate of $\tilde{H}^h$, expressed here in terms of

$$d_{Ag}(x',y'; E) = \inf_{\gamma(0)=x', \gamma(1)=y'} \int_0^1 \sqrt{\tilde{V}(\gamma(t)) - E} |\dot{\gamma}(t)| \, dt, \quad x' = hx, \, y' = hy.$$ 

A perturbative analysis of $E_{-+}(z)^{-1}$ for two different choices $\tilde{V}_1$ and $\tilde{V}_2$, $\tilde{V}_1 = \tilde{V}_2$ in $|h(x - x_0)| \leq R$, leads to an error of size $\tilde{O}(e^{-2 SR \frac{h}{h}})$ where $S_R = \inf_{|x'-x_0|=R} d_{Ag}(x', x_0)$ as long as $\inf_{x \in \mathbb{R}} \tilde{V}_k(x) > E$. 

\begin{align*}
\int_0^1 \sqrt{\tilde{V}(\gamma(t)) - E} |\dot{\gamma}(t)| \, dt, \\
\inf_{\gamma(0)=x', \gamma(1)=y'}
\end{align*}
Changing $H_U$, changing $\tilde{H}^h$

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Changing $H_U$, changing $\tilde{H}^h$

In the construction of the approximate inverse $F(z)$, $H_U$ can be replaced by any quantum local problem (think of $\phi_\varepsilon$ and $\chi_\varepsilon$-truncations)

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Multiple wells

Initiate the Grushin problem with

\[ A^0(z) = \begin{pmatrix} H_{U_1} \oplus H_{U_2} - z & R^0_+ \\ R^0_- & 0 \end{pmatrix} : D(H_{U_1}) \oplus D(H_{U_2}) \rightarrow \mathbb{C}^2 \]
Multiple wells

There are three different cases

Non resonant wells: $|E^h_1 - E^h_2| \geq c > 0$, $E^h_k = V(x_k) + \lambda(U_k) + o(h^0)$
Multiple wells

There are three different cases

**Resonant wells:** \(|E_1^h - E_2^h| = \tilde{O}(e^{-\frac{S_{12}}{h}})\), \(E_k^h \sim V(x_k) + \lambda(U_k) + o(h^0)\),
\(S_{12} = d_{Ag}(x_1, x_2)\)
There are three different cases

Weakly resonant wells: \( |E_1^h - E_2^h| \geq e^{-\frac{S_{12} + c}{h}} \), \( \lim_{h \to 0} E_1^h = E = \lim_{h \to 0} E_2^h \).
Multiple wells

Actually for self-adjoint spectral problems (Schrödinger type), Grushin method can be replaced by variational principles, min-max, \( \|(z - A)^{-1}\| = d(z, \sigma(H))^{-1} \), and Agmon type estimates. In the series of articles Puits-multiples I-VI, Helffer-Sjöstrand do not really use this method.

Things are different for non self-adjoint problems.
Quantum resonances=quantum metastable states

An example: $\tilde{H} = -h^2 \Delta + \frac{1}{1+x^2}$

$\tilde{H}_\theta - E = e^{-2i\theta} [-h^2 \Delta + \frac{1}{e^{-2i\theta} + x^2} - e^{2i\theta} E] \xrightarrow{\text{unitary}} e^{-2i\theta} \left[-h^2 \Delta + \frac{1}{x^2} - E + i \tan(2\theta) \left( \frac{1}{1+x^2} - E \right) + O(\theta^2) \right]$.

For $E \in \mathbb{R} \setminus \{0,1\}$ the principal semiclassical symbol

$$
(|\xi|^2 + \frac{1}{1+x^2} - E) + i \tan(2\theta) \left( \frac{1}{1+x^2} - E \right) + O(\theta^2)
$$

is elliptic for $\theta = \theta_E$ small enough.

For $h \in ]0, h_0]$, $\tilde{H}_\theta - E$ is invertible, $E \not\in \sigma(\tilde{H}_\theta)$ + stability via perturbations.

Perturbations: 1) $(\tilde{H}_\theta - z)$ for $|z - E| \leq \varepsilon$.
2) $e^{-\frac{\varphi}{h}} (\tilde{H}_\theta - z) e^{-\frac{\varphi}{h}}$ with $|\partial^\alpha_x \varphi(x)| \leq C_{\alpha, \varepsilon}$ for $|\alpha| \geq 1$ has the semiclassical symbol

$$
((\xi + i\partial_x \varphi)^2 + \frac{1}{1+x^2} - z) + i \tan(2\theta) \left( \frac{1}{1+x^2} - z \right) + O(\theta^2)
$$

This allows to prove $(\tilde{H}_\theta - z)^{-1} (x, y) = \tilde{O}(e^{-\frac{\varepsilon |x-y|}{h}})$.
Quantum resonances=quantum metastable states

$e^{\lambda(iA)} u = e^{\frac{d\lambda}{2}} u(e^{\lambda} x) \quad iA = x.\partial_x + d/2$

$e^{\lambda(iA)} p(x, D_x)e^{-\lambda(iA)} = p(e^{\lambda} x, e^{-\lambda} D_x) \quad \lambda = i\theta.$

$H_\theta = e^{-\theta A}(-\Delta + W(x))e^{\theta A} = -e^{-2i\theta} \Delta + W(e^{i\theta} x)$

$\sigma_{\text{ess}}(H_\theta) = e^{-2i\theta}[0, +\infty[.$

An example: $\tilde{H} = -h^2 \Delta + \frac{1}{1+x^2}$

$\tilde{H}_\theta - E = e^{-2i\theta}[-h^2 \Delta + \frac{1}{e^{-2i\theta} + x^2} - e^{2i\theta} E] \quad \text{unitary} \quad e^{-2i\theta}[-h^2 \Delta + \frac{1}{1+x^2} - E + i \tan(2\theta)(\frac{1}{(1+x^2)^2} - E) + \mathcal{O}(\theta^2)]$

For $E \in \mathbb{R} \setminus \{0, 1\}$ the principal semiclassical symbol

$(|\xi|^2 + \frac{1}{1+x^2} - E) + i \tan(2\theta)(\frac{1}{(1+x^2)^2} - E) + \mathcal{O}(\theta^2)$

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$((\xi + i\partial_x \varphi)^2 + \frac{1}{1+x^2} - z) + i \tan(2\theta)(\frac{1}{(1+x^2)^2} - z) + \mathcal{O}(\theta^2)$

This allows to prove $(\tilde{H}_\theta - z)^{-1}(x, y) = \tilde{O}(e^{-\frac{e|x-y|}{h}})$.
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\[ e^{\lambda (iA)} u = e^{\frac{d\lambda}{2}} u (e^{\lambda} x) \quad iA = x.\partial_x + d/2 \]

\[ e^{\lambda (iA)} p(x, D_x) e^{-\lambda (iA)} = p(e^{\lambda} x, e^{-\lambda} D_x) \quad \lambda = i\theta. \]

\[ H_\theta = e^{-\theta A} (-\Delta + W(x)) e^{\theta A} = -e^{-2i\theta} \Delta + W(e^{i\theta} x) \]

\[ \sigma_{\text{ess}} (H_\theta) = e^{-2i\theta} [0, +\infty[. \]

An example: \( \tilde{H} = -h^2 \Delta + \frac{1}{1+x^2} \)

\[ \tilde{H}_\theta - E = e^{-2i\theta} [-h^2 \Delta + \frac{1}{e^{-2i\theta} + x^2} - e^{2i\theta} E] \quad \text{unitary} \quad \frac{e^{-2i\theta}}{\cos(2\theta)} [-h^2 \Delta + \frac{1}{1+x^2} - E + i \tan(2\theta) (\frac{1}{1+x^2} - E) + O(\theta^2)] \]

For \( E \in \mathbb{R} \setminus \{0,1\} \) the principal semiclassical symbol

\[ (|\xi|^2 + \frac{1}{1+x^2} - E) + i \tan(2\theta) (\frac{1}{(1+x^2)^2} - E) + O(\theta^2) \]

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2) \( e^{\frac{\phi}{h}} (\tilde{H}_\theta - z)e^{-\frac{\phi}{h}} \) with \( |\partial_x^\alpha \phi(x)| \leq C_\alpha \varepsilon \) for \( |\alpha| \geq 1 \) has the semiclassical symbol

\[ ((\xi + i \partial_x \phi)^2 + \frac{1}{1+x^2} - z) + i \tan(2\theta) (\frac{1}{(1+x^2)^2} - z) + O(\theta^2) \]

This allows to prove \((\tilde{H}_\theta - z)^{-1} (x, y) = O(e^{-\frac{e|x-y|}{h}})\)
Quantum resonances=quantum metastable states

Replace $iA = x \cdot \partial_x + \frac{d}{2}$ by $iA_x = (1 - \chi(x))x \cdot \partial_x + \frac{1}{2} \text{div} (1 - \chi(x)x)$.

By the same argument as before, $\tilde{H}_\theta - z$ is invertible for $|z - E| \leq \varepsilon$ and

$$(\tilde{H}_\theta - E)^{-1}(x, y) = \tilde{O}(e^{-\frac{\varepsilon|x-y|}{h}})$$

$E$
Quantum resonances = quantum metastable states

Replace $iA = x \cdot \partial_x + \frac{d}{2}$ by $iA\chi = (1 - \chi(x))x \cdot \partial_x + \frac{1}{2} \text{div} (1 - \chi(x)x)$.

With the Grushin problem $H = -\Delta + V(hx) + U(x)$ has one resonance, an eigenvalue of $e^{-\theta A\chi} He^{\theta A\chi}$ near $E = V(x_0) - \lambda_U$, $E^h - i\Gamma^h$ with $\Gamma^h = \mathcal{O}(e^{-\frac{c}{h}})$.
Quantum resonances=quantum metastable states

With a better treatment of the transition between the classically forbidden and free admissible region (set $F_E = \{x, V(x) = E\}$) one can prove actually
\[
\lim h \log \Gamma^h = -2S_0 \quad S_0 = d_{Ag}(x_0, F_E, E).
\]

It must be done with $\theta = Ch, \ C \gg 1$, and the complex Airy model plays a role in the neighborhood of $F_E$ with the subelliptic semiclassical estimate
\[
\|(-\partial_x^2 + i(hx))u\| \geq \frac{h^{2/3}}{C} \|u\|.
\]

and
\[
h^{2/3} \gg h \quad \text{or} \quad (h^{2/3} \gg h^2 \quad \text{with subprincipal computations}).
\]
Remember that the exponential decay estimates for $p(x, hD_x)$ requires the study of the symbol $p(x, \xi + i\partial_x \phi)$.

This can be microlocalized by more general deformations into the complex phase-space: Fourier-Bros-Iagolnitzer (FBI)-transform and Sjöstrand spaces $H_\phi$.

A spectral (quantum) accident occurring at a point $(x_0, \xi_0) \in \mathbb{R}^{2d}$ can be treated by Grushin techniques combined with (sub)-elliptic and exponential decay estimates for the “filled well” problem.

**Figure**: Microlocal tunnel effect between $(x_1, \xi_1)$ and $(x_2, \xi_2)$ via complex deformation.
Langevin process

\[ dq = pdt \quad , \quad dp = -\frac{1}{h} \partial_{q} Vdt - \frac{1}{b} pdt + \frac{1}{\sqrt{b}} dW \]

Semigroup generator on \( C^0 \)

\[ K = -p.\partial_{q} + \frac{1}{h} \partial_{q} V.\partial_{p} + \frac{1}{2b} (-\partial_{p} + 2p).\partial_{p} \]

Invariant measure \( K'\mu = 0 \) gives \( \mu = e^{-\left( |p|^2 + \frac{2}{h} V(q) \right)} dqdp = M^2(q,p)dqdp \) and \( L^2(\mathbb{R}^d, \mu) \xrightarrow{M} L^2(\mathbb{R}^d, dqdp) \).

\[
\begin{align*}
MKM^{-1} &= -(p.\partial_{q} - \frac{1}{h} \partial_{q} V(q)\partial_{p}) + \frac{1}{b} \frac{-\Delta p + |p|^2 - d}{2} = -\gamma + \frac{1}{b} \mathcal{O}, \\
\text{unitary} \quad &\xrightarrow{\text{unitary}} \quad \frac{1}{bh} \left[ -(b\sqrt{h}p.\partial_{q} - b\sqrt{h}\partial_{q} V(q)\partial_{p}) + \frac{-h^2 \Delta p + |p|^2 - d}{2} \right].
\end{align*}
\]

Subellipticity \( p.\partial_{q} - \Delta_{p} = ip.\xi - \Delta_{p} \) (complex Airy operator) \( \rightarrow \) regularity gain of \( 2/3 \) for \( 1 + K^{-1} \).
Bismut-Lebeau (2008) consider $h = 1$ and $b \to 0$ for the hypoelliptic Laplacian ($p$-forms). They prove that $\sigma(B_{\pm,b,V})$ converges to $\sigma(\Delta_{V,1})$ in a fixed disc around $z = 0$. (Schur complement + very tricky analysis)

Hérau-Hitrik-Sjöstrand (2011): $V$ is a Morse function $b = \sqrt{h}$ (pure semiclassical pde problem). They consider the scalar case in $\mathbb{R}^{2d}$. They prove that the number of $o(h^{1/b})$-eigenvalues are given by the number of local minima with a full asymptotic description of eigenvalues. Method: Grushin problem, FBI-transforms, quadratic local model, supersymmetric trick and information from the semiclassical Witten Laplacian (degree 0 and 1 max).

Geometry in $\mathbb{C}^2 \times 2 \times d \sim \mathbb{R}^{8d}$ for the Brownian motion in $\mathbb{R}^{d}$.

Shu Shen (2016) consider $b \asymp \sqrt{h}$, $V$ is a Morse function, simplifies the riemannian metric around critical points. Eigenvalues of $B_{\pm,b,V_h}$ counted like the ones of $\frac{1}{2h^2} \Delta_{V,h}$ lead to Morse theory and Reidemeister torsion. Method: Schur complement technique + subelliptic estimate + intuition coming from Feynman-Kac formula.


Ren and Z. Tao in 2023 studied a simplified scalar model via a Grushin problem technique combined with subelliptic estimates.

N., Sang, White: $V$ is a $C^\infty$ potential $b \leq h \times \varrho_h$ where $\varrho_h$ is the spectral gap for the Witten Laplacian (when $V$ is Morse function, the method must be adapted to recover $b \leq \sqrt{h}$ instead of $b \leq h^2$). Then the eigenvalues of $B_{\pm,b,V_h}$ with modulus $\leq \varrho_h/2$ are real, non negative, exponentially small. Their number is given by the bar code, in persistent homology, of the function $V$. Method: 1) accurate control of the constant w.r.t $b$ in subelliptic estimates; 2) adapted Grushin problem approach of Ren-Tao.
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Shu Shen (2016) consider $b \approx \sqrt{h}$, $V$ is a Morse function, simplifies the riemannian metric around critical points. Eigenvalues of $B_{\pm}, b, \frac{V}{h}$ counted like the ones of $\frac{1}{2h^2} \Delta V, h \Rightarrow$ Morse theory and Reidemeister torsion. Method: Schur complement technique + subelliptic estimate + intuition coming from Feynman-Kac formula.


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Langevin process: accurate spectral results

Bismut-Lebeau (2008) consider $h = 1$ and $b \to 0$ for the hypoelliptic Laplacian ($p$-forms). They prove that $\sigma(B_{\pm,b,V})$ converges to $\sigma(\Delta_{V,1})$ in a fixed disc around $z = 0$. (Schur complement + very tricky analysis)

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N., Sang, White: $V$ is a $C^{\infty}$ potential $b \leq h \times \varrho_{h}$ where $\varrho_{h}$ is the spectral gap for the Witten Laplacian (when $V$ is Morse function, the method must be adapted to recover $b \leq \sqrt{h}$ instead of $b \leq h^2$). Then the eigenvalues of $B_{\pm,b,V,h}$ with modulus $\leq \varrho_{h}/2$ are real, non negative, exponentially small. Their number is given the bar code, in persistent homology, of the function $V$. Method: 1) accurate control of the constant w.r.t $b$ in subelliptic estimates; 2) adapted Grushin problem approach of Ren-Tao.
Bismut-Lebeau (2008) consider $h = 1$ and $b \to 0$ for the hypoelliptic Laplacian ($p$-forms). They prove that $\sigma(B_{\pm,b,V})$ converges to $\sigma(\Delta V,1)$ in a fixed disc around $z = 0$. (Schur complement + very tricky analysis)

Hérau-Hitrik-Sjöstrand (2011): $V$ is a Morse function $b = \sqrt{h}$ (pure semiclassical pde problem). They consider the scalar case in $\mathbb{R}^{2d}$. They prove that the number of $o(h^{1/h})$-eigenvalues are given by the number of local minima with a full asymptotic description of eigenvalues. Method: Grushin problem, FBI-transforms, quadratic local model, supersymmetric trick and information from the semiclassical Witten Laplacian (degree 0 and 1 max).

Geometry in $\mathbb{C}^2 \times \mathbb{R} \times \mathbb{R} \sim \mathbb{R}^{8d}$ for the Brownian motion in $\mathbb{R}^d$.

Shu Shen (2016) consider $b \approx \sqrt{h}$, $V$ is a Morse function, simplifies the riemannian metric around critical points. Eigenvalues of $B_{\pm,b,V,h}$ counted like the ones of $\frac{1}{2h^2}\Delta V,h \Rightarrow$ Morse theory and Reidemeister torsion. Method: Schur complement technique + subelliptic estimate + intuition coming from Feynman-Kac formula.


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