Grushin problem method

Francis Nier, LAGA, Univ. Paris XIII

Cermics March 28th 2024

Warning:

- Semiclassical is more than semiclassical.
- A Grushin problem is not a Grushin problem.

A reference: J. Sjöstrand and M. Zworski "Elementary linear algebra for advanced spectral problems" (Ann. Inst. Fourier (2007)).

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  Hörmander FIO I (1971):*"The purpose of the present paper is not to extend the more or less formal methods used in geometrical optics but to extract from them a precise operator theory which can be applied to the theory of partial differential equations"*
- A Grushin problem is not a Grushin problem.
  The name and the notations were actually introduced in the early works of J. Sjöstrand (phD, 1973)

A reference: J. Sjöstrand and M. Zworski "Elementary linear algebra for advanced spectral problems" (Ann. Inst. Fourier (2007)).

Outline

- Schur complement and Grushin problem. First applications.
- Multiple wells, resonances
- Comparison of Langevin and overdamped Langevin.

Schur complement

The Grushin problem method can be viewed as a variation of the Schur complement formula of Linear Algebra (Numerical Analysis), like the Feshbach formula (Math. Phys.) or the Lyapunov-Schmidt method (Dynamical systems and non linear analysis).

Schur complement formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -A^{-1}B \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & (D - CA^{-1}B)^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -CA^{-1} & 1 \end{pmatrix},$$

whenever the block A is invertible as well as its Schur complement $(D - CA^{-1}B)$.

Grushin problem works in a different way

The block matrix is constructed in such a way that it is invertible and its relates the invertibility of A to the invertibility of a block of D-size (finite dimension for Fredholm theory).

Let $\theta \mapsto A_\theta$ be a continuous map from the metric space (X, d) to $\mathcal{L}(E; F)$.
Important example:
F Banach space; $E = D(A) \subset F$ the domain of a closed operator A;
(X, d) a Banach space of relatively bounded perturbations of A with bound less than 1 ($\Rightarrow A_\theta$ with $D(A_\theta) = D(A) = E$).

Definition

The operator $A \in \mathcal{L}(E, F)$ is Fredholm if $\dim \ker A = a_+ < \infty$, if its range $\mathrm{Ran} A$ is closed and if $\mathrm{codim}(\mathrm{Ran} A) = \dim \mathrm{coker} A = a_- < \infty$.
$\mathrm{Fred}(E; F)$ set of Fredholm operators.
$\mathrm{Ind}(A) = a_+ - a_-$.

Remember (checked below)

- $\mathrm{Fred}(E; F)$ is an open set of $\mathcal{L}(E; F)$;
- Ind is constant on connected components of $\mathrm{Fred}(E; F)$

Let us assume $A_{\theta_0} \in \mathrm{Fred}(E; F)$ (here $(X, d) = \mathcal{L}(E; F)$) with $\dim \ker A_{\theta_0} = a_+$ and $\dim \mathrm{coker} A_{\theta_0} = a_-$.

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$$R_- : \mathbb{C}^{a_-} \to F \quad \text{et} \quad R_+ : E \to \mathbb{C}^{a_+}, \quad R_+ \in \mathcal{L}(E;\mathbb{C}^{a_+}), \quad R_+\big|_{\ker A_{\theta_0}} \text{ bij.}$$

R_- (resp. R_+^{-1}) parametrizes $\mathrm{coker}(A_{\theta_0})$ (resp. $\ker(A_{\theta_0})$) and

$$\mathcal{A}_\theta = \begin{pmatrix} A_\theta & R_- \\ R_+ & 0 \end{pmatrix} : \begin{matrix} E \\ \oplus \\ \mathbb{C}^{a_-} \end{matrix} \to \begin{matrix} F \\ \oplus \\ \mathbb{C}^{a_+} \end{matrix} .$$

is invertible for $\theta \sim \theta_0$ and we set

$$\mathcal{A}(\theta)^{-1} = \begin{pmatrix} E(\theta) & E_+(\theta) \\ E_-(\theta) & E_{-+}(\theta) \end{pmatrix} : \begin{matrix} F \\ \oplus \\ \mathbb{C}^{a_+} \end{matrix} \to \begin{matrix} E \\ \oplus \\ \mathbb{C}^{a_-} \end{matrix} .$$

For $\theta = \theta_0$ solving $\mathcal{A}_{\theta_0}\begin{pmatrix} u \\ u_- \end{pmatrix} = \begin{pmatrix} v \\ v_+ \end{pmatrix}$ gives

$$\begin{pmatrix} E(\theta_0) & E_+(\theta_0) \\ E_-(\theta_0) & E_{-+}(\theta_0) \end{pmatrix} = \begin{pmatrix} (A_{\theta_0}\big|_{\ker R_+})^{-1}\pi_{\mathrm{Ran}A_{\theta_0}} & (R_+\big|_{\ker A_{\theta_0}})^{-1} \\ (R_-)^{-1}\pi_{\mathrm{Ran}R_-} & 0 \end{pmatrix} .$$

In particular $E_-(\theta_0)$ (resp. $E_+(\theta_0)$) has the maximal rank a_- (resp. a_+), which is an open condition ($\Rightarrow$ the same holds true for $E_-(\theta)$ and $E_+(\theta)$ when $\theta \sim \theta_0$).
Now by setting $R_+ u = v_+$, the equivalences

$$(A_\theta u = v) \Leftrightarrow \left\{ \begin{array}{lcl} A_\theta u + R_- 0 & = & v \\ R_+ u & = & v_+ \end{array} \right. \Leftrightarrow \left\{ \begin{array}{l} u = E(\theta)v + E_+(\theta)v_+ \\ 0 = E_-(\theta)v + E_{-+}(\theta)v_+ \end{array} \right. ,$$

lead to $\quad (v \in \mathrm{Ran}A_\theta) \Leftrightarrow \big(E_-(\theta)v \in \mathrm{Ran}E_{-+}(\theta)\big)$

and $(v = 0)$ $\quad (u \in \ker A_\theta) \Leftrightarrow \big(u \in E_+(\theta)\ker E_{-+}(\theta)\big)$.

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$$\mathrm{rank} E_-(\theta) = a_- \quad , \quad \mathrm{rank} E_+(\theta) = a_+ \quad \text{when } \theta \sim \theta_+ .$$

$\dim \mathrm{Ran} E_{-+}(\theta) < \infty$ and $E_-(\theta) \in \mathcal{L}(F; \mathbb{C}^{a_-}) \Rightarrow \mathrm{Ran} A_\theta$ closed.
Because $E_-(\theta)$ is onto and $E_+(\theta)$ is one to one, we deduce

- $\dim \mathrm{coker} A_\theta = \dim \mathrm{coker} E_{-+}(\theta)$;
- $\dim \ker A_\theta = \dim \ker E_{-+}(\theta)$.

Proposition

With the above notations and with $\theta \in \mathcal{V}_{\theta_0}$ small neighborhood of θ_0, $A_\theta \in \mathrm{Fred}(E; F)$ and

$$\mathrm{Ind} A_\theta = \mathrm{Ind} E_{-+}(\theta) = a_+ - a_-$$

which does not depend on $\theta \in \mathcal{V}_{\theta_0}$. Additionally when $\mathrm{Ind}(A_{\theta_0}) = 0$, $a_+ = a_-$, we have

$$(A_\theta \text{ invertible}) \Leftrightarrow (E_{-+}(\theta) \text{ invertible}) \Leftrightarrow (\det E_{-+}(\theta) \neq 0) ,$$

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Examples of non s.a. operators

When $A : D(A) \to F$ is closed and the imbedding $D(A) \to F$ is compact then either $\sigma(A) = \mathbb{C}$ or $\sigma(A) = \sigma_{disc}(A)$.

For $a = \frac{\partial_x + x}{\sqrt{2}}$ with $D(a) = D(\mathcal{O}^{1/2})$, $\mathcal{O} = \frac{-\Delta + x^2}{2}$, compactly included in $L^2(\mathbb{R}, dx)$, $\sigma(a) = \sigma(a^*) = \mathbb{C}$.

$\sigma(-\Delta_x + ix) = \emptyset$ for the complex Airy operator $-\Delta_x + ix$ with $D(A) = \{u \in L^2(\mathbb{R}, dx), \ -\Delta_x u \text{ and } xu \in L^2(\mathbb{R}, dx)\}$.
Subelliptic consequence:

$$\forall u \in \mathcal{C}_K^\infty(\mathbb{R}^2), \quad \|(-\Delta_{x_1} + x_1 \partial_{x_2})u\|_{H^s} \geq C_K \|u\|_{H^{s+2/3}}$$

Weyl's theorem when $B : D(A) \to F$ is compact the set $\{(t, z) \in \mathbb{C} \times (\mathbb{C} \setminus \sigma_{ess}(A)), z \in \sigma(A + tB)\}$ is an analytic set. (locally $\det(E_{-+}(z, t)) = 0$)
When $F = \ell^2(\mathbb{Z})$ consider the operators A et $C \in \mathcal{L}(F)$ given by

$$(A\varphi)_n = \varphi_{n+1} \quad \text{et} \quad (C\varphi)_n = \delta_{n,0}\varphi_1$$

Then $\sigma_{ess}(A) = \mathbb{S}^1$ and

- $\sigma_{ess}(A - tC) = \mathbb{S}^1$ for $t \neq 1$;
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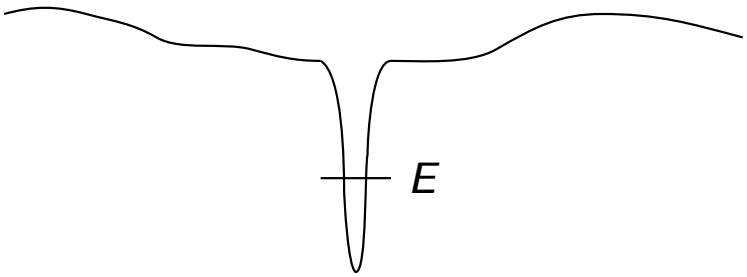
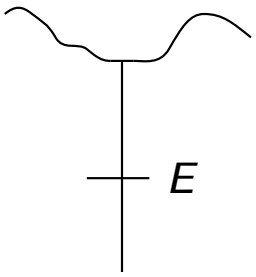
Quantum/semiclassical separation





Spectrum around E of $-\Delta + U(x - \frac{x_0}{h}) + \tilde{V}(hx)$ or $-h^2\Delta + U(\frac{x}{h}) + \tilde{V}(x_0 + x)$

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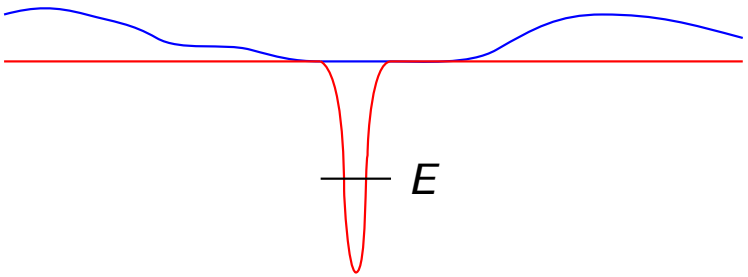

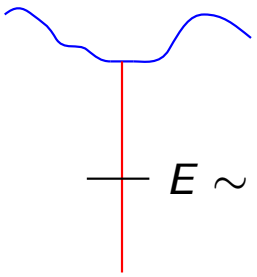

There is an eigenvalue $E^h = \tilde{V}(x_0) + \lambda_U + o(h^0)$ where $\lambda_U \in \sigma(-\Delta + U)$
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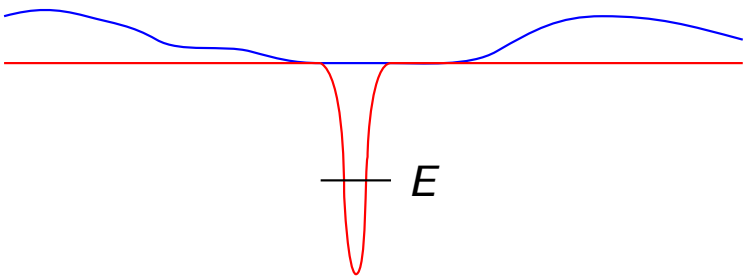
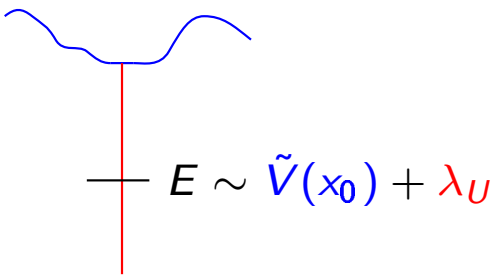
Quantum/semiclassical separation





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Exponential decay:

Quantum Hamiltonian

$$H_U = -\Delta + U(x)$$

$$H_U \psi_U = \lambda_U \psi_U \,,\ \lambda_U \leq E/2 < 0$$

$$\psi_U(x) = \tilde{\mathcal{O}}(e^{-\alpha_E |x|})$$

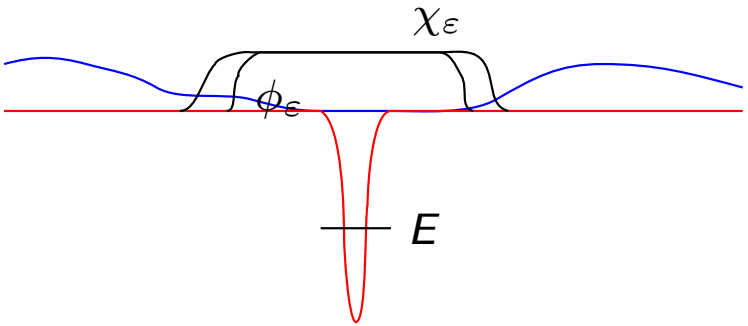
Filled well Hamiltonian (semiclassical)

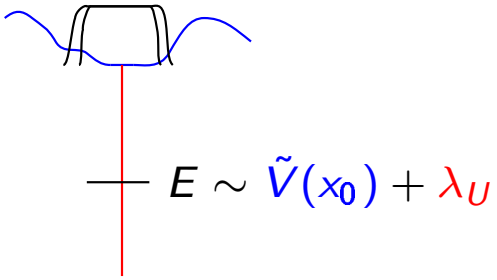
$$\tilde{H}^h = -\Delta + \tilde{V}(hx)$$

$$\tilde{V} \geq \tfrac{E}{4} \text{ and } \mathbb{R}e\ z \leq \tfrac{E}{2}\,,$$

$$(z - \tilde{H}^h)^{-1}(x,y) = \tilde{\mathcal{O}}(e^{-\alpha_E |x-y|})$$

Quantum/semiclassical separation





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Cut-off: $\phi, \chi \in \mathcal{C}_0^\infty$, $\phi \preceq \chi$,

$$1 = \underbrace{\phi(\frac{hx - x_0}{\varepsilon})}_{\phi_\varepsilon(hx)} + (1-\phi)(\frac{hx - x_0}{\varepsilon}) = \underbrace{\chi(\frac{hx - x_0}{\varepsilon})}_{\chi_\varepsilon(hx)} + (1-\chi)(\frac{hx - x_0}{\varepsilon})$$

Quantum/Semiclassical separation

Grushin problem for H_U:

$$\mathcal{A}^0(z) = \begin{pmatrix} H_U - z & R^0_- \\ R^0_+ & 0 \end{pmatrix} : \begin{matrix} D(H_U) \\ \oplus \\ \mathbb{C} \end{matrix} \to \begin{matrix} L^2 \\ \oplus \\ \mathbb{C} \end{matrix}$$

With $R^0_- = |\psi_U\rangle$ and $R^0_+ = \langle\psi_U|$, $\mathcal{A}^0(\lambda_U)$ is invertible $\Rightarrow$ so $\mathcal{A}^0(z)$ is invertible for $z \in \mathcal{V}(\lambda_U)$:

$$\mathcal{G}^0(z) = [\mathcal{A}^0(z)]^{-1} = \begin{pmatrix} E^0_0(z) & E^0_+ \\ E^0_- & E^0_{-+}(z) \end{pmatrix}.$$

We consider now for z close to $\tilde{V}(x_0) + \lambda_U$

$$\mathcal{A}(z) = \begin{pmatrix} H^h - z & R_- = \chi_\varepsilon R^0_- \\ R_+ = R^0_+ & 0 \end{pmatrix}$$

and take

$$\mathcal{F}(z) = \begin{pmatrix} \chi_\varepsilon E^0_0(z - \tilde{V}(x_0))\phi_\varepsilon + (\tilde{H}^h - z)^{-1}(1 - \phi_\varepsilon) & \chi_\varepsilon E^0_+ \\ E^0_- \phi_\varepsilon & E^0_{-+}(z - \tilde{V}(x_0)) \end{pmatrix}$$

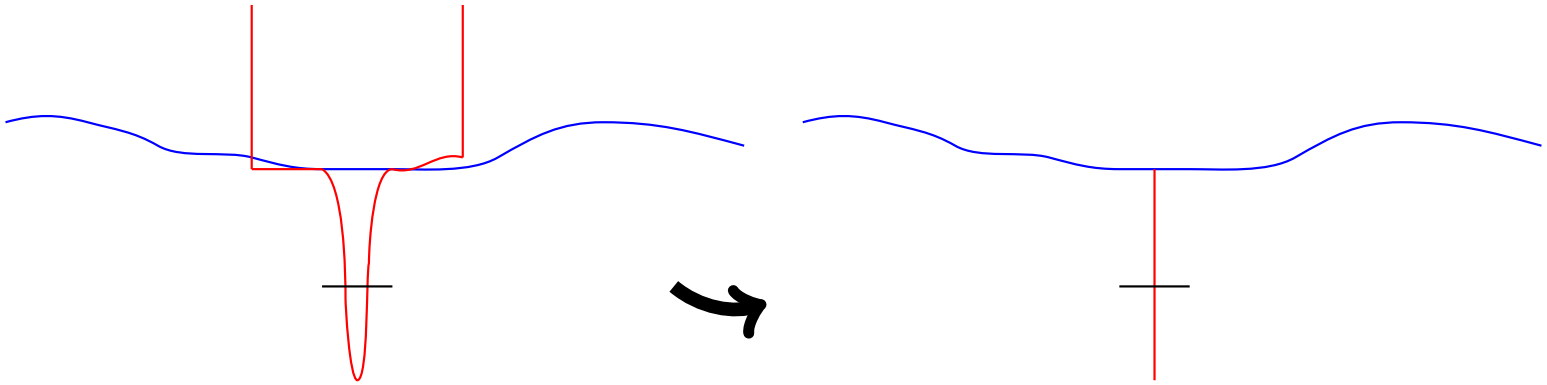
A direct computation shows (with $h \leq \varepsilon^2$)

$$\mathcal{A}(z)\mathcal{F}(z) = \mathrm{Id} + \mathcal{O}(\varepsilon) + \tilde{O}(e^{-\frac{\varepsilon}{h}}) \quad \text{and} \quad E_{-+}(z) = E^0_{-+}(z - \tilde{V}(x_0)) + \mathcal{O}(\varepsilon) + \tilde{O}(e^{-\frac{\varepsilon}{h}}))$$

Combined with $(H^h - z)^{-1} = E(z) - E_+(z)(E_{-+}(z))^{-1}E_-(z)$ this allows to compare spectral quantities (eigenvalues, projector) around E.

Changing H_U, changing $\tilde{H}^h$

In the construction of the approximate inverse $\mathcal{F}(z)$, H_U can be replaced by any quantum local problem (think of ϕ_ε and χ_ε-truncations)



The long range exponential decay of eigenvector with energy $\sim E$ comes from the exponential decay estimate of $\tilde{H}^h$, expressed here in terms of

$$d_{Ag}(x', y'; E) = \inf_{\substack{\gamma \\ \gamma(0)=x' \\ \gamma(1)=y'}} \int_0^1 \sqrt{\tilde{V}(\gamma(t)) - E}|\dot{\gamma}(t)|\, dt\,,\ x' = hx\,,\ y' = hy\,.$$

A perturbative analysis of $E_{-+}(z)^{-1}$ for two different choices $\tilde{V}_1$ and $\tilde{V}_2$, $\tilde{V}_1 = \tilde{V}_2$ in $|h(x - x_0)| \leq R$, leads to an error of size $\tilde{O}(e^{-2\frac{S_R}{h}})$ where $S_R = \inf_{|x'-x_0|=R} d_{Ag}(x', x_0)$ as long as $\inf_{x\in\mathbb{R}} \tilde{V}_k(x) > E$.

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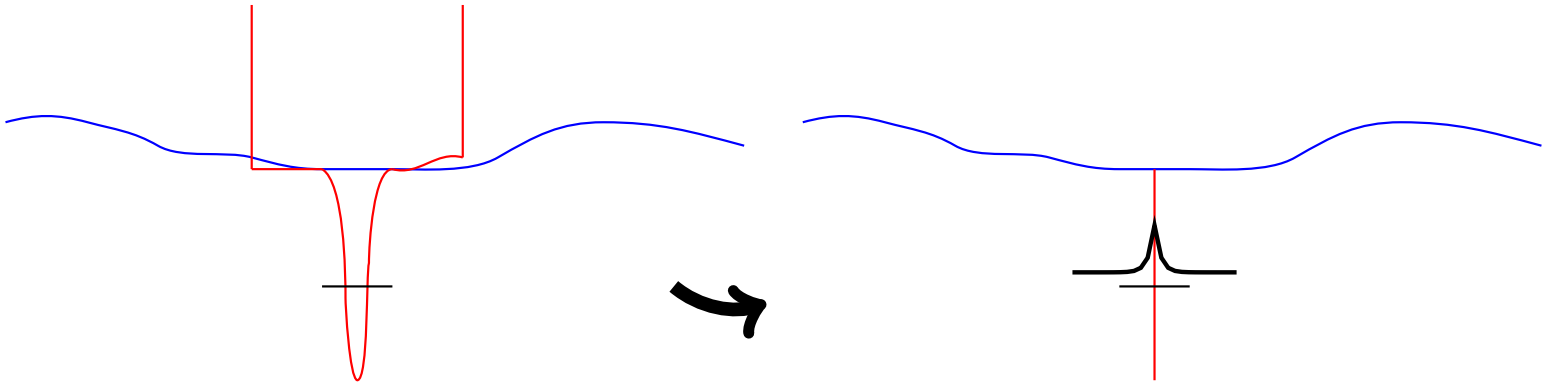

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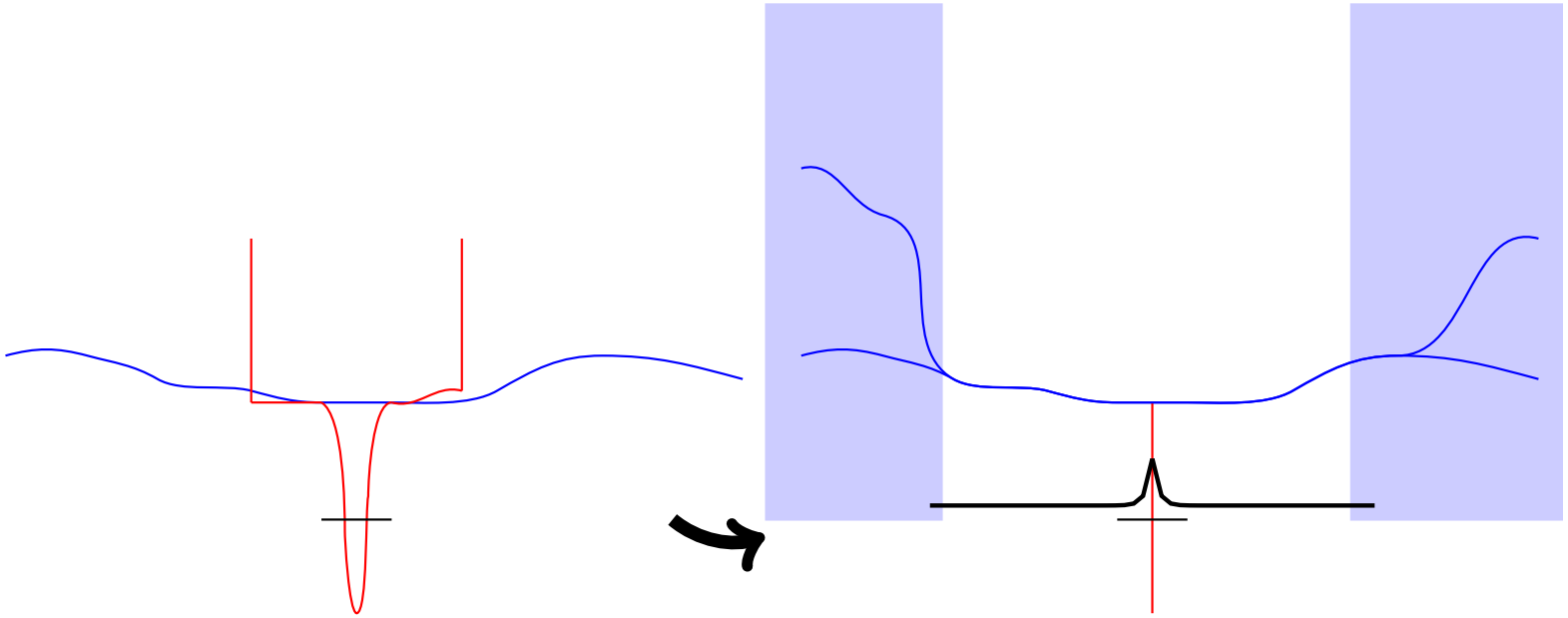

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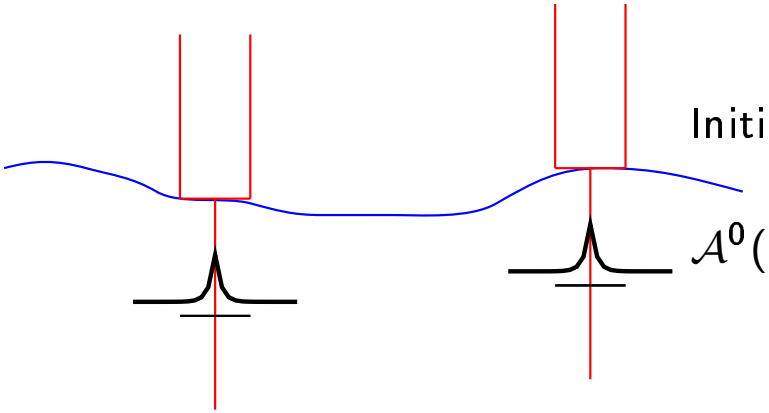
Multiple wells



Initiate the Grushin problem with

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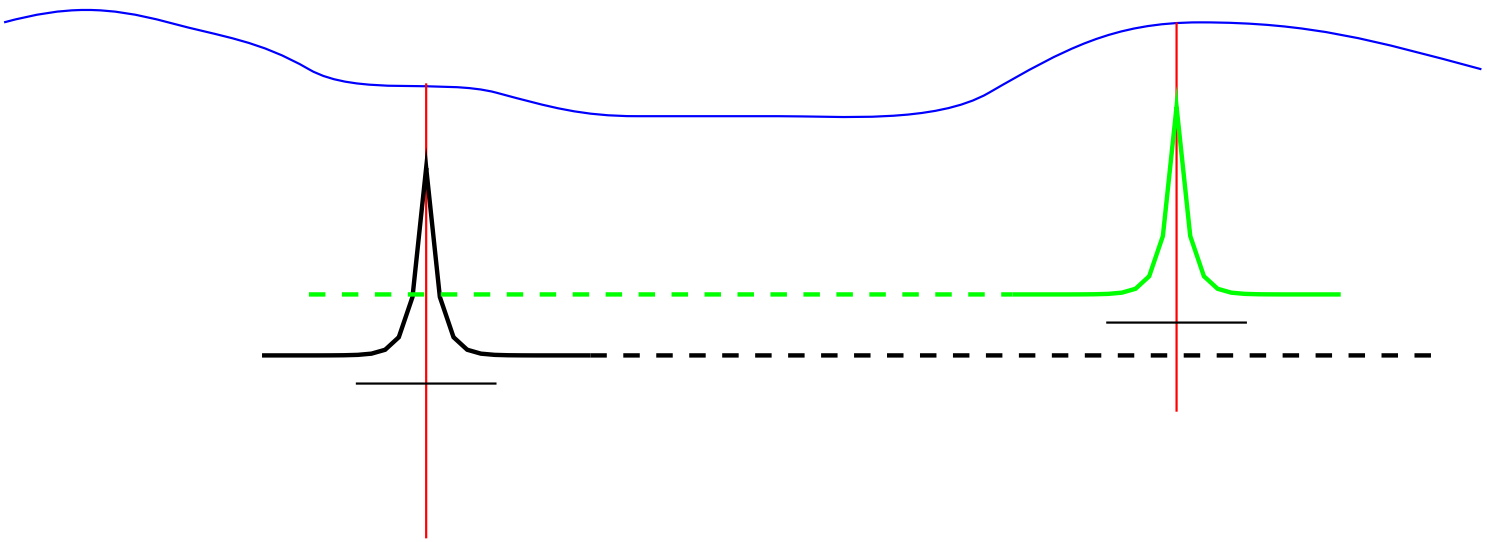
Multiple wells

There are three different cases

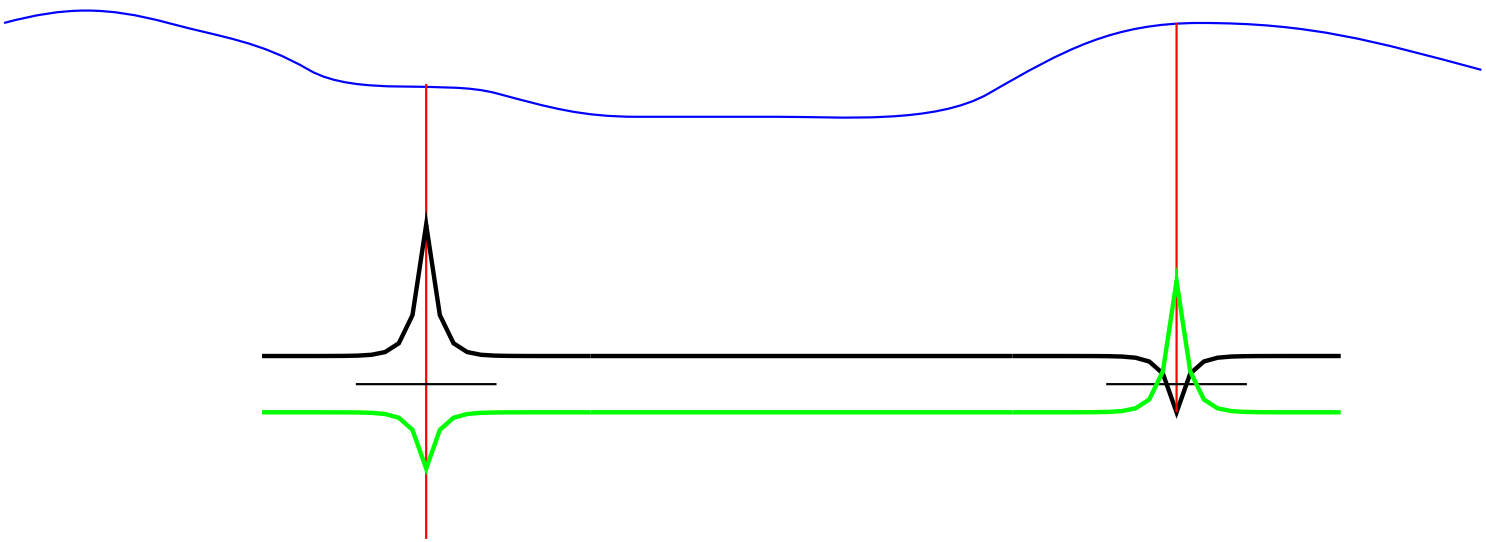
Non resonant wells: $|E_1^h - E_2^h| \geq c > 0$, $E_k^h = V(x_k) + \lambda(U_k) + o(h^0)$



Multiple wells

There are three different cases

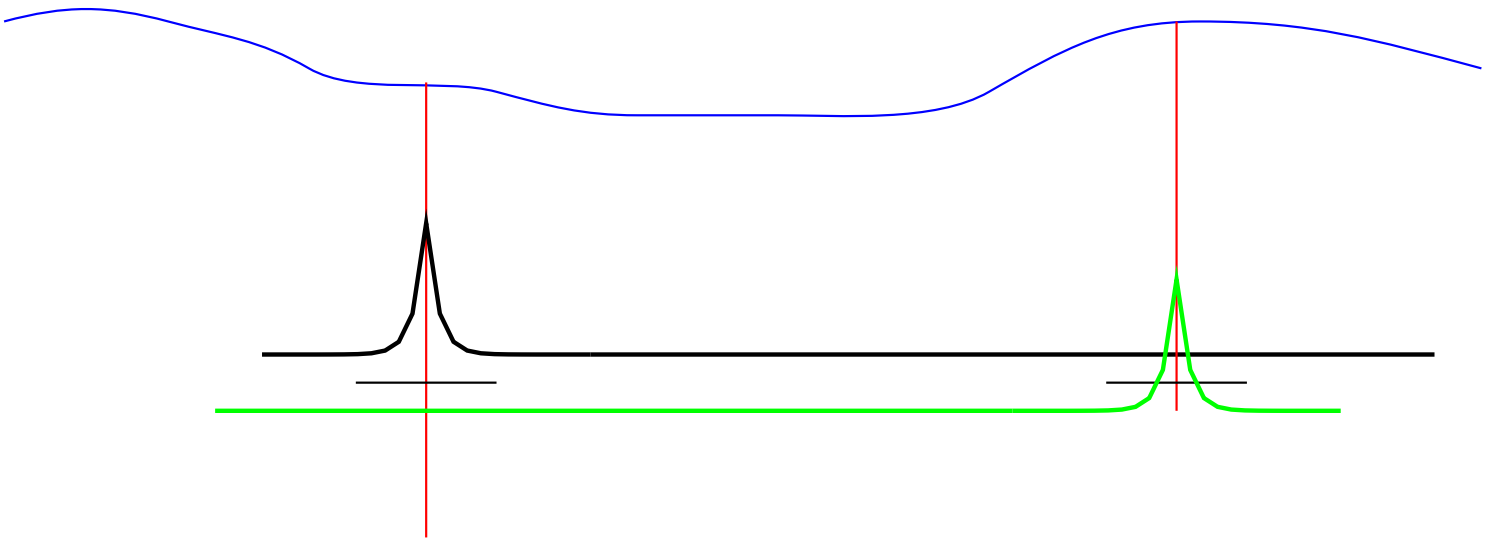
Resonant wells: $|E_1^h - E_2^h| = \tilde{O}(e^{-\frac{S_{12}}{h}})$, $E_k^h \sim V(x_k) + \lambda(U_k) + o(h^0)$, $S_{12} = d_{Ag}(x_1, x_2)$



Multiple wells

There are three different cases

Weakly resonant wells: $|E_1^h - E_2^h| \geq e^{\frac{-S_{\mathbf{12}}+c}{h}}$, $\lim_{h\to 0} E_1^h = E = \lim_{h\to 0} = E_2^h$.

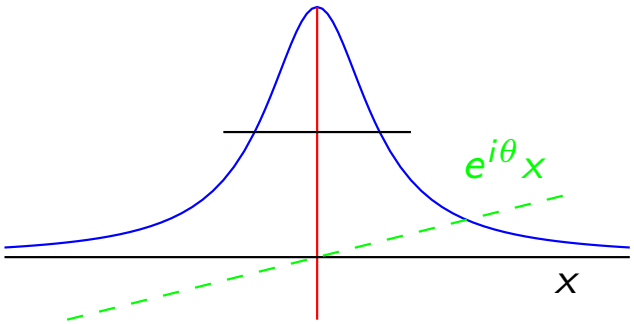


Multiple wells

Actually for self-adjoint spectral problems (Schrödinger type), Grushin method can be replaced by variational principles, min-max, $\|(z-A)^{-1}\| = d(z,\sigma(H))^{-1}$, and Agmon type estimates. In the series of articles Puits-multiples I-VI, Helffer-Sjöstrand do not really use this method.

Things are different for non self-adjoint problems.

Quantum resonances=quantum metastable states



$$e^{\lambda(iA)}u = e^{\frac{d\lambda}{2}}u(e^{\lambda}x) \quad iA = x.\partial_x + d/2$$

$$e^{\lambda(iA)}p(x,D_x)e^{-\lambda(iA)} = p(e^{\lambda}x, e^{-\lambda}D_x) \quad \lambda = i\theta.$$

$$H_\theta = e^{-\theta A}(-\Delta + W(x))e^{\theta A} = -e^{-2i\theta}\Delta + W(e^{i\theta}x)$$

$$\sigma_{\text{ess}}(H_\theta) = e^{-2i\theta}[0,+\infty[.$$

An example: $\tilde{H} = -h^2\Delta + \frac{1}{1+x^2}$

$$\tilde{H}_\theta - E = e^{-2i\theta}[-h^2\Delta + \frac{1}{e^{-2i\theta}+x^2} - e^{2i\theta}E] \overset{\text{unitary}}{\to} \frac{e^{-2i\theta}}{\cos(2\theta)}[-h^2\Delta + \frac{1}{1+x^2} - E + i\tan(2\theta)(\frac{1}{(1+x^2)^2} - E) + \mathcal{O}(\theta^2)]$$

For $E \in \mathbb{R}\setminus\{0,1\}$ the principal semiclassical symbol

$$(|\xi|^2 + \frac{1}{1+x^2} - E) + i\tan(2\theta)(\frac{1}{(1+x^2)^2} - E) + \mathcal{O}(\theta^2)$$

is elliptic for $\theta = \theta_E$ small enough.

For $h \in]0,h_0]$, $\tilde{H}_\theta - E$ is invertible, $E \notin \sigma(\tilde{H}_\theta)$ + stability via perturbations.

Perturbations: 1) $(\tilde{H}_\theta - z)$ for $|z-E| \le \varepsilon$.

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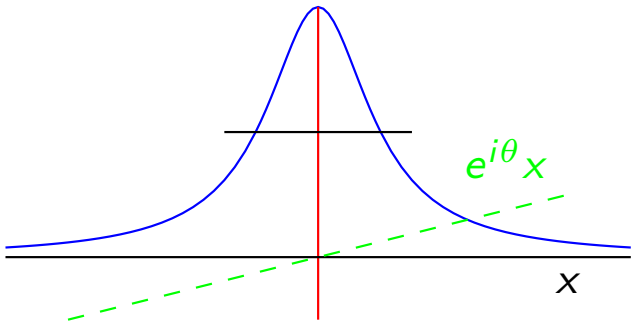

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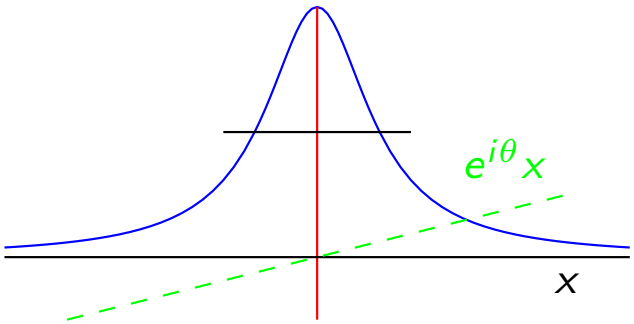

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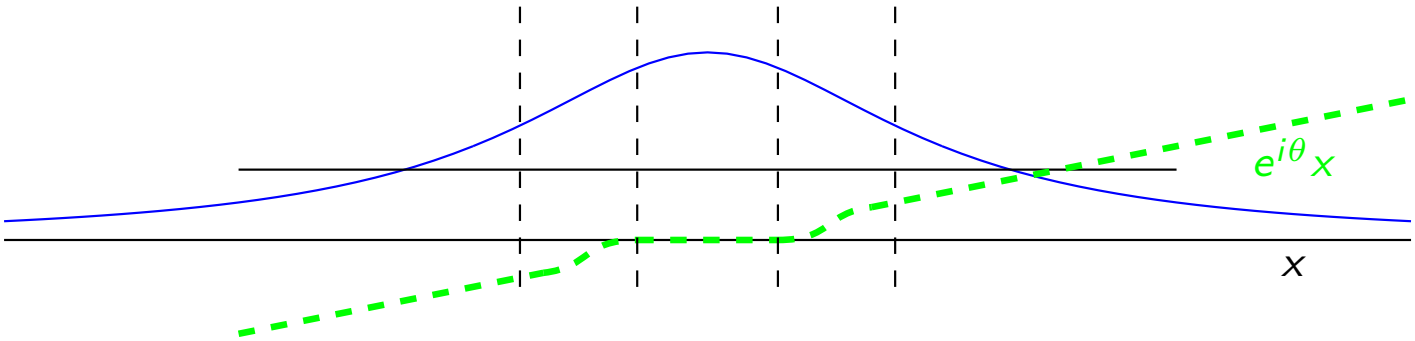
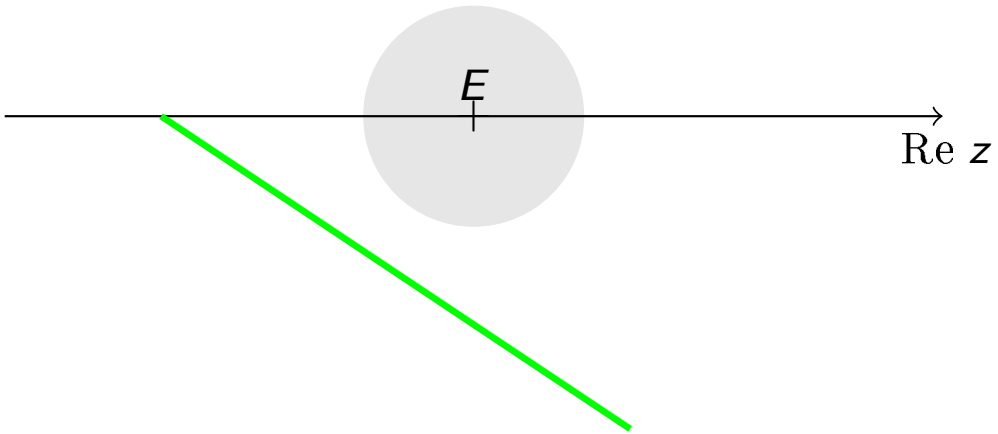
Quantum resonances=quantum metastable states

Replace $iA = x.\partial_x + \frac{d}{2}$ by $iA_\chi = (1-\chi(x))x.\partial_x + \frac{1}{2}\mathrm{div}\,(1-\chi(x)x)$.



By the same argument as before, $\tilde{H}_\theta - z$ is invertible for $|z-E| \leq \varepsilon$ and $(\tilde{H}_\theta - E)^{-1}(x,y) = \tilde{O}(e^{-\frac{\varepsilon|x-y|}{h}})$



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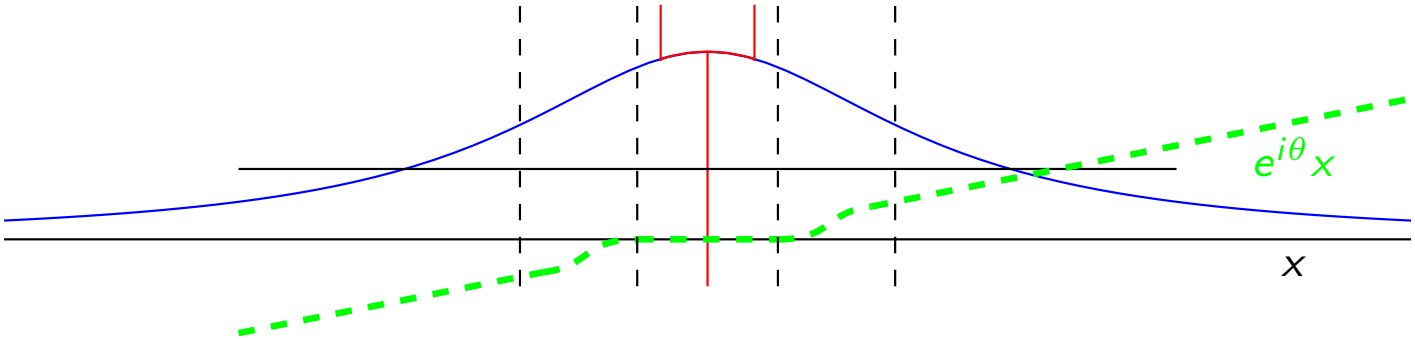
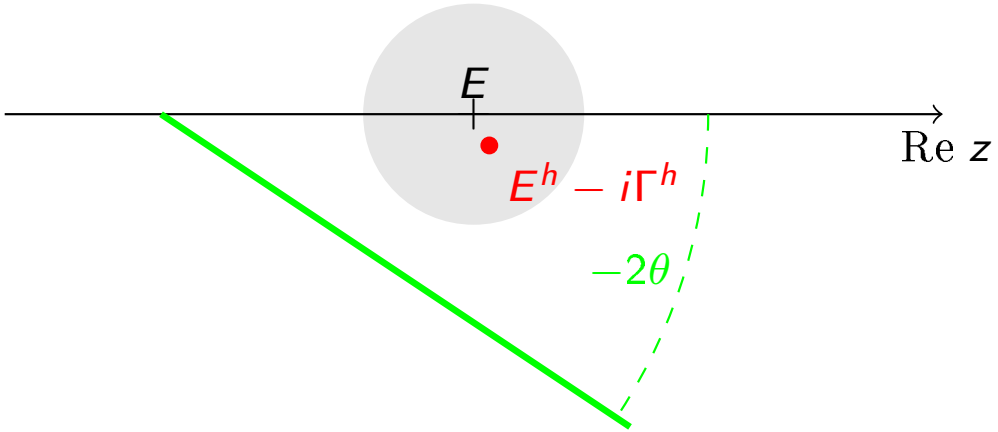


With the Grushin problem $H = -\Delta + V(hx) + U(x)$ has one resonance, an eigenvalue of $e^{-\theta A_\chi} H e^{\theta A_\chi}$ near $E = V(x_0) - \lambda_U$, $E^h - i\Gamma^h$ with $\Gamma^h = \mathcal{O}(e^{-\frac{c}{h}})$



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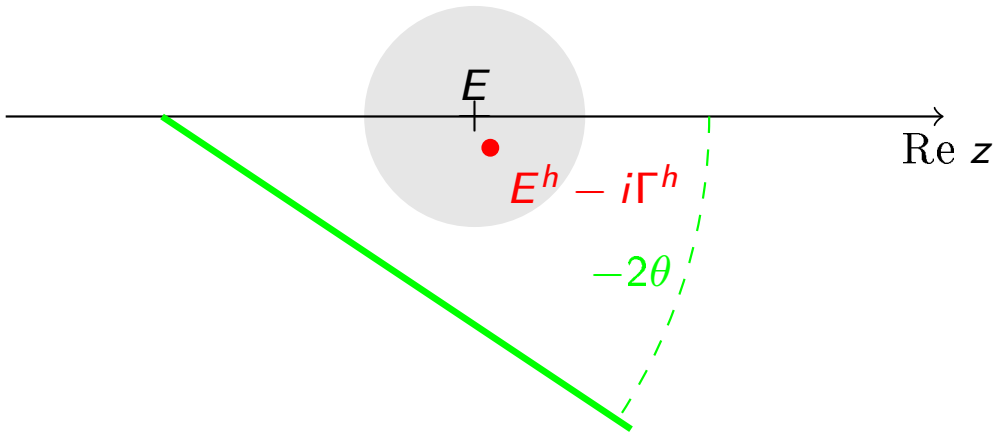

With a better treatment of the transition between the classically forbidden and free admissible region (set $F_E = \{x,\ V(x) = E\}$) one can prove actually

$$\lim h \log \Gamma^h = -2S_0 \quad S_0 = d_{Ag}(x_0, F_E, E).$$

It must be done with $\theta = Ch$, $C \gg 1$, and the complex Airy model plays a role in the neighborhood of F_E with the subelliptic semiclassical estimate

$$\|(-\partial_x^2 + i(hx))u\| \geq \frac{h^{2/3}}{C}\|u\|.$$

and

$$h^{2/3} \gg h \quad \text{or} \quad (h^{2/3} \gg h^2 \quad \text{with subprincipal computations}).$$

Microlocal tunnel effect

Remember that the exponential decay estimates for $p(x, hD_x)$ requires the study of the symbol $p(x, \xi + i\partial_x\varphi)$.
This can be microlocalized by more general deformations into the complex phase-space: Fourier-Bros-Iagolnitzer (FBI)-transform and Sjöstrand spaces H_ϕ.
A spectral (quantum) accident occuring at a point $(x_0, \xi_0) \in \mathbb{R}^{2d}$ can be treated by Grushin techniques combined with (sub)-elliptic and exponential decay estimates for the "filled well" problem.

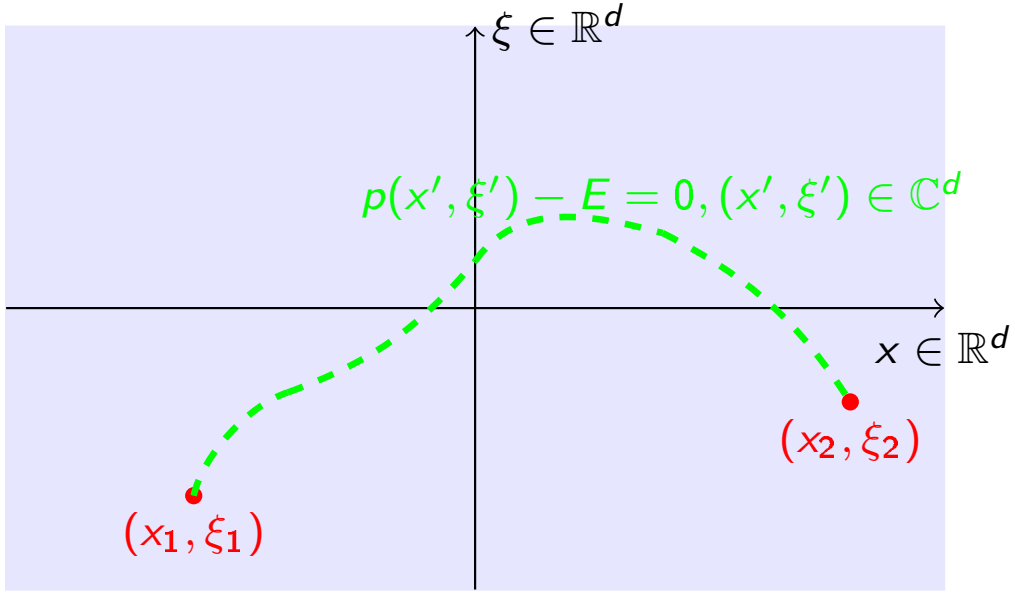

Figure: Microlocal tunnel effect between (x_1, ξ_1) and (x_2, ξ_2) via complex deformation

Langevin process

$$dq = pdt \quad , \quad dp = -\frac{1}{h}\partial_q V dt - \frac{1}{b} p dt + \frac{1}{\sqrt{b}} dW$$

Semigroup generator on $\mathcal{C}^0$

$$K = -p.\partial_q + \frac{1}{h}\partial_q V.\partial_p + \frac{1}{2b}(-\partial_p + 2p).\partial_p$$

Invariant measure $K'\mu = 0$ gives $\mu = e^{-(|p|^2 + \frac{2}{h}V(q))} dqdp = M^2(q,p)dqdp$ and $L^2(\mathbb{R}^{2d}, \mu) \overset{M\times}{\mapsto} L^2(\mathbb{R}^{2d}, dqdp)$.

$$\begin{aligned} MKM^{-1} &= -(p.\partial_q - \frac{1}{h}\partial_q V(q)\partial_p) + \frac{1}{b}\frac{-\Delta_p + |p|^2 - d}{2} = -\mathcal{Y} + \frac{1}{b}\mathcal{O}\,, \\ &\overset{\text{unitary}}{\to} \frac{1}{bh}\left[-(b\sqrt{h}p.\partial_q - b\sqrt{h}\partial_q V(q)\partial_p) + \frac{-h^2\Delta_p + |p|^2 - d}{2}\right]. \end{aligned}$$

Subellipticity $p.\partial_q - \Delta_p = ip.\xi - \Delta_p$ (complex Airy operator) $\to$ regularity gain of $2/3$ for $1 + K^{-1}$.

Langevin process: accurate spectral results

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Hérau-Hitrik-Sjöstrand (2011): V is a Morse function $b = \sqrt{h}$ (pure semiclassical pde problem). They consider the scalar case in $\mathbb{R}^{2d}$. They prove that the number of $o(h\frac{1}{bh})$-eigenvalues are given by the number of local minima with a full asymptotic description of eigenvalues. Method : Grushin problem, FBI-transforms, quadratic local model, supersymmetric trick and information from the semiclassical Witten Laplacian (degree 0 and 1 max).
Geometry in $\mathbb{C}^{2\times 2\times d} \sim \mathbb{R}^{8d}$ for the Brownian motion in $\mathbb{R}^d$.

Shu Shen (2016) consider $b \asymp \sqrt{h}$, V is a Morse function, simplifies the riemannian metric around critical points. Eigenvalues of $B_{\pm,b,\frac{V}{h}}$ counted like the ones of $\frac{1}{2h^2}\Delta_{V,h}$ $\Rightarrow$ Morse theory and Reidemeister torsion. Method: Schur complement technique + subelliptic estimate + intuition coming from Feynman-Kac formula.

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