Lattice structures of Gog and Magog triangles

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Alternating sign matrices (ASM)

- ASMs are square matrices having coefficients equal to $0, 1$ or -1, so that, in each row and in each column, nonzero coefficients alternate between 1 and -1 and their sum is 1.

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

- (Robbins, Rumsey 1986) For $n \times n$ matrix M, we have

$$\det_\lambda(M) = \sum_{A \in ASM(n)} (1+\lambda)^{s(A)} \lambda^{i(A)} \prod_{i,j} M_{i,j}^{A_{i,j}}.$$

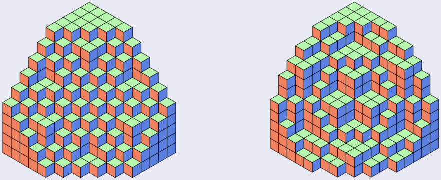
- (Zeilberger 1994) The number of ASMs of size n is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

Other objects are enumerated by the same sequence:

Totally symmetric and self-complementary plane partitions (PPTSAC)

- A TSSCPP of size n is a plane partitions inside a cube of side $2n$, having all the symmetries of the triangle, and the same shape as its complement in the cube.



- (ANDREWS 1994) The number of PPTSACs of size n is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

Descending plane partitions (DPPs)

- A DPP of order n is an array of positive integers less than or equal to n, with nonincreasing rows and decreasing columns, such that the left-hand edges are successively indented, and the number of entries in each row is strictly less than the largest entry in that row and greater than or equal to the largest entry in the next row.

$$\begin{array}{ccccc} 6 & 6 & 6 & 6 & 5 \\ & 5 & 4 & 4 & 4 \\ & & 3 & 2 & \end{array} \qquad \begin{array}{cccccc} 7 & 7 & 6 & 4 & \color{red}{3} & \color{red}{2} \\ & 6 & 4 & 4 & \color{red}{1} & \\ & & 2 & & & \end{array}$$

- (Mills, Robbins et Rumsey 1982) The number of DPPs of order n is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

Recently, a new family of objects equienumerated with ASMs has been discovered:

Alternating sign triangles (ASTs)

- An AST of size n is a triangular array with n rows of length $2n-1, \dots, 3, 1$ with coefficients equal to $0, 1$ or -1, so that, in each row, nonzero coefficients alternate between 1 and -1 and their sum is 1, and in each column they alternate between 1 and -1 starting from the top.

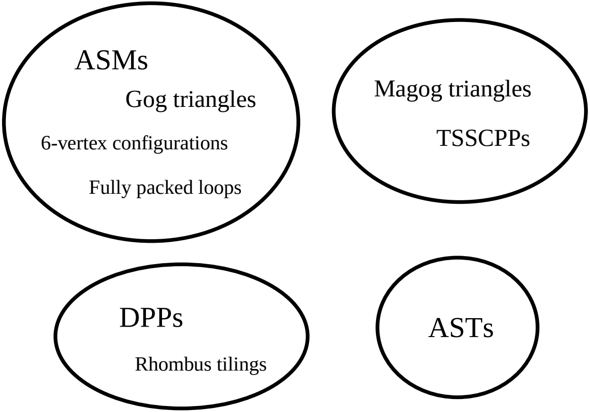
$$\begin{array}{ccccccc} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ & 1 & -1 & 0 & 1 & 0 & \\ & & 1 & 0 & 0 & & \\ & & & 1 & & & \end{array} \qquad \begin{array}{ccccccc} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & 1 & 0 & 0 & -1 & 1 & \\ & & 1 & 0 & 0 & & \\ & & & 1 & & & \end{array}$$

- (Ayyer, Behrend and Fischer 2020) The number of ASTs of order n is

$$\prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!}.$$

Bijections

We have four families of objects that counted by the same sequence. Yet we do not know any bijection between them:



Gelfand-Tsetlin triangles $\Longleftrightarrow$ Semistandard Young tableaux

Definition

A Gelfand-Tsetlin triangle of size n is a triangular array of strictly positive integers $X = (X_{i,j})_{n \geq i \geq j \geq 1}$ such that $\forall i, j\ X_{i+1,j} \leq X_{i,j} \leq X_{i+1,j+1}$.

Gog and Magog triangles

A Gog triangle of size n is a Gelfand-Tsetlin triangle $X = (X_{i,j})_{n \geq i \geq j \geq 1}$ such that $\forall i,\ X_{n,i} = i$ et $\forall j < j'\ X_{i,j} < X_{i,j'}$.
A Magog triangle of size n is a Gelfand-Tsetlin triangle $X = (X_{i,j})_{n \geq i \geq j \geq 1}$ such that $\forall i, j,\ X_{i,j} \leq j$.

$$\begin{array}{ccccccccccc}
1 & & 2 & & 3 & & 4 & & 5 & & 6 \\
& 1 & & 2 & & 4 & & 5 & & 6 & \\
& & 2 & & 3 & & 5 & & 6 & & \\
& & & 2 & & 4 & & 6 & & & \\
& & & & 2 & & 6 & & & & \\
& & & & & 4 & & & & &
\end{array}$$

A Gog triangle.

$$\begin{array}{ccccccccccc}
1 & & 1 & & 1 & & 3 & & 5 & & 5 \\
& 1 & & 1 & & 2 & & 4 & & 5 & \\
& & 1 & & 2 & & 2 & & 4 & & \\
& & & 1 & & 2 & & 3 & & & \\
& & & & 1 & & 2 & & & & \\
& & & & & 1 & & & & &
\end{array}$$

A Magog triangle.

Bijections

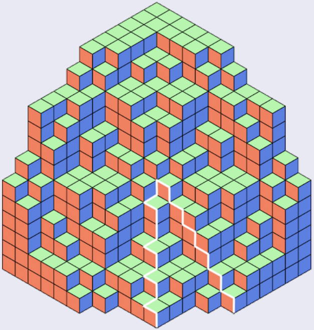
Gelfand-Tsetlin triangles are in bijection with semistandard Young tableaux with all distinct values appearing in the first column:

$$
\begin{array}{ccccccccccc}
1 & & 1 & & 3 & & 4 & & 5 & & 7 \\
& 1 & & 3 & & 3 & & 5 & & 6 & \\
& & 1 & & 3 & & 4 & & 6 & & \\
& & & 2 & & 3 & & 5 & & & \\
& & & & 2 & & 4 & & & & \\
& & & & & 4 & & & & &
\end{array}
\iff
\begin{array}{|c|c|c|c|c|c|c|}
\cline{1-1}
6 \\
\cline{1-1}
5 \\
\cline{1-3}
4 & 5 & 5 \\
\cline{1-4}
3 & 3 & 4 & 6 \\
\cline{1-5}
2 & 2 & 3 & 4 & 5 \\
\hline
1 & 1 & 1 & 1 & 3 & 4 & 6 \\
\hline
\end{array}
$$

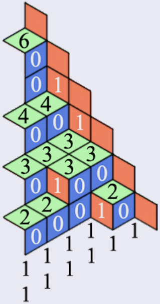
Gog triangles are in bijection with ASMs:

$$
\begin{array}{ccccccccccc}
1 & & 2 & & 3 & & 4 & & 5 & & 6 \\
& 1 & & 3 & & 4 & & 5 & & 6 & \\
& & 1 & & 3 & & 4 & & 6 & & \\
& & & 1 & & 3 & & 5 & & & \\
& & & & 2 & & 4 & & & & \\
& & & & & 4 & & & & &
\end{array}
\longleftrightarrow
\begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & -1 & 1 \\
1 & -1 & 1 & -1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
$$

Magog triangles are in bijection with TSSCPPs:



$\Longleftrightarrow$

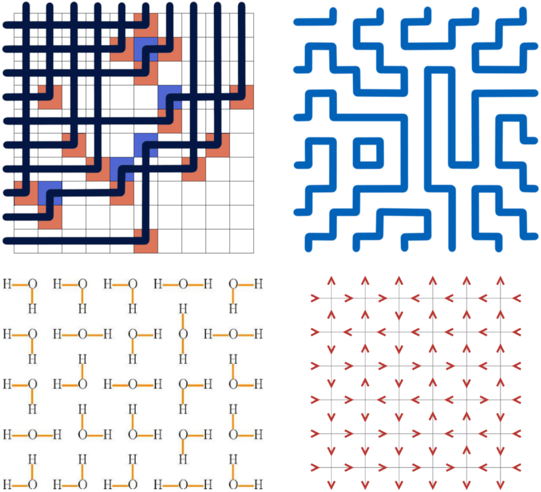


$\Longleftrightarrow$

$$\begin{array}{ccccccccccc}
1 & & 1 & & 2 & & 3 & & 4 & & 6 \\
& 1 & & 1 & & 2 & & 3 & & 4 & \\
& & 1 & & 1 & & 3 & & 3 & & \\
& & & 1 & & 1 & & 3 & & & \\
& & & & 1 & & 2 & & & & \\
& & & & & 1 & & & & &
\end{array}$$

Bijections

ASMs are also in bijection with 6-vertex configurations, fully packed loops, the square ice model etc...



Order relation on Gelfand-Tsetlin triangles

Gelfand-Tsetlin triangles can be ordered by coordinatewise comparison, giving them a poset structure:

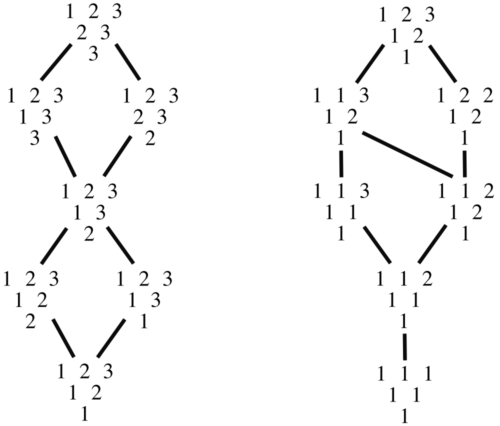


Figure: The lattices of Gog and Magog triangles of size 3.

Distributive lattices

Gog and Magog triangles, but also descending plane partitions and alternating sign triangles form distributive lattices. This means:

- they are endowed with a partial order relation $<$;
- for all pairs x, y there exists a smallest element $x \vee y$ such that $x < x \vee y$ and $y < x \vee y$, and there exists a biggest element $x \wedge y$ such that $x \wedge y < x$ and $x \wedge y < y$. In particular they have a minimal and a maximal element;
- the operations $\vee$ and $\wedge$ distributive over each other:

$$x \wedge (y \vee y) = (x \wedge y) \vee (x \wedge z) \qquad x \vee (y \wedge y) = (x \vee y) \wedge (x \vee z)$$

On Gog and Magog triangles, the operations $\vee$ and $\wedge$ are respectively the coordinatewise maximum and minimum.

$$\begin{array}{ccccccc} 1 & & 2 & & 3 & & 4 \\ & 1 & & 2 & & 4 & \\ & & 2 & & 4 & & \\ & & & 2 & & & \end{array} \wedge \begin{array}{ccccccc} 1 & & 2 & & 3 & & 4 \\ & 2 & & 3 & & 4 & \\ & & 2 & & 3 & & \\ & & & 3 & & & \end{array} = \begin{array}{ccccccc} 1 & & 2 & & 3 & & 4 \\ & 1 & & 2 & & 4 & \\ & & 2 & & 3 & & \\ & & & 2 & & & \end{array}$$

Distributive lattices can be encoded nicely by smaller posets:

Theorem (Birkhoff 1937)

Elements of a finite distributive lattice $\mathbb{L}$ are in bijection with lower sets of $\mathrm{Irr}(\mathbb{L})$, *i.e. the poset of join-irreducible elements $\mathbb{L}$.*

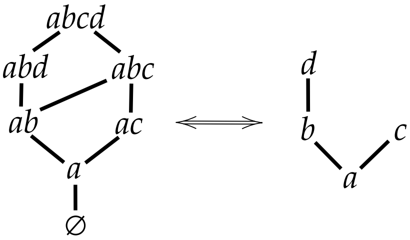
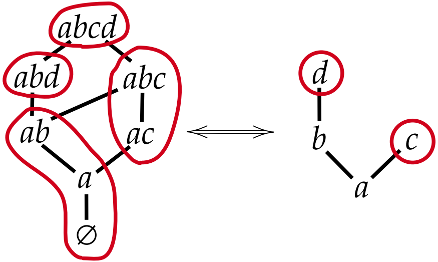


Figure: A distributive lattice $\mathbb{L}$ and its join-irreducibles poset $\mathrm{Irr}(\mathbb{L})$.

A lattice congruence is an equivalence relation $\equiv$ such that, if $x_1 \equiv y_1$ and $x_2 \equiv y_2$, then we must have $x_1 \wedge x_2 \equiv y_1 \wedge y_2$ and $x_1 \vee x_2 \equiv y_1 \vee y_2$.

Proposition (Thurston 1950)

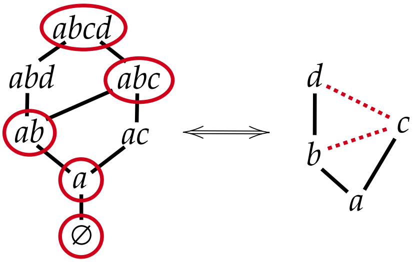
Congruence relations of a finite distributive lattice $\mathbb{L}$ are in bijection with subposets of $\mathrm{Irr}(\mathbb{L})$. A quotient lattice is isomorphic to the lattice of ideals of the corresponding subposet.



A sublattice of a lattice $\mathbb{L}$ is a subset of $\mathbb{L}$ which is closed under the operations $\vee$ and $\wedge$.

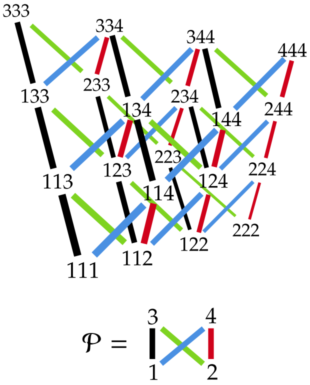
Proposition

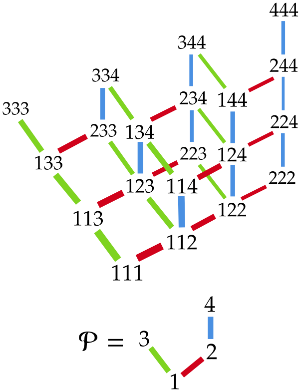
Sublattices of a finite distributive lattice $\mathbb{L}$ are in bijection with refinements of the poset $\mathrm{Irr}(\mathbb{L})$.
Maximal chains of $\mathbb{L}$ are in bijection with linear extensions of $\mathrm{Irr}(\mathbb{L})$.



Proposition

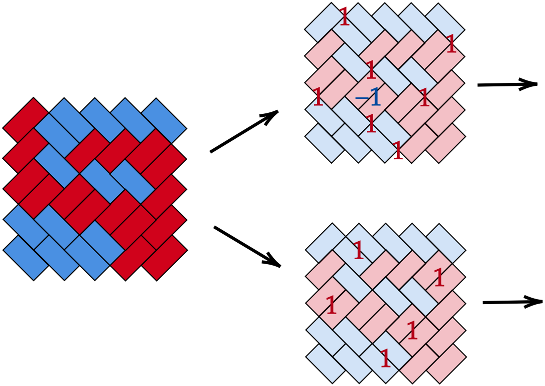
The poset of join-irreducible Gog (or Magog) triangles of size $n+2$ is isomorphic to $\mathcal{P}^n/\mathfrak{S}_n$ for some poset $\mathcal{P}$ of size 4.





ASMs and aztec diamond tillings

To each aztec diamond tilling of size n corresponds a certain pair of ASMs of size n and $n+1$:

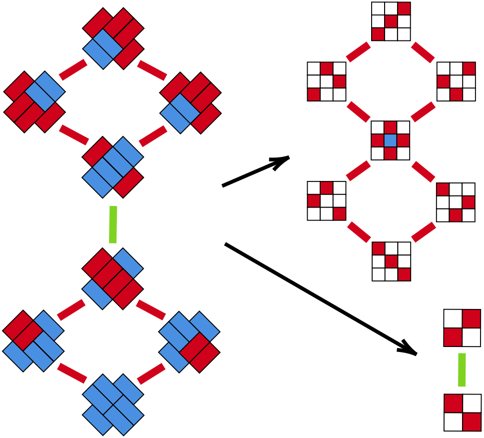


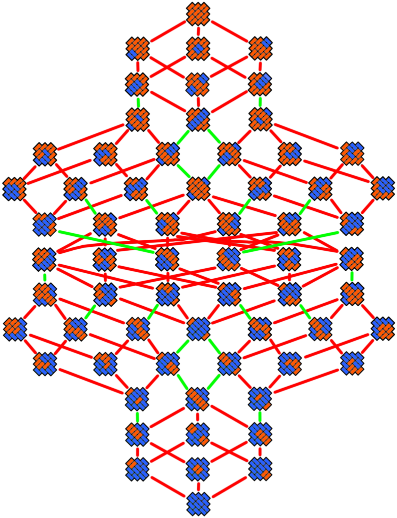
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

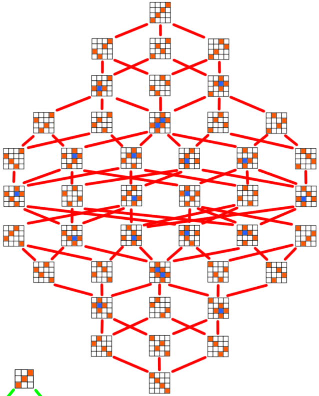
$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

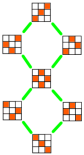
Figure: An aztec diamond tilling of size 5, and the corresponding ASMs.

These operations are morphisms from the lattice of aztec diamond tillings to the lattice of ASMs. Their preimages are boolean intervals, whose dimensions are the number of 1 or the number of -1 in the corresponding ASM.









All these properties can be explained by the structure of their irreducible posets:

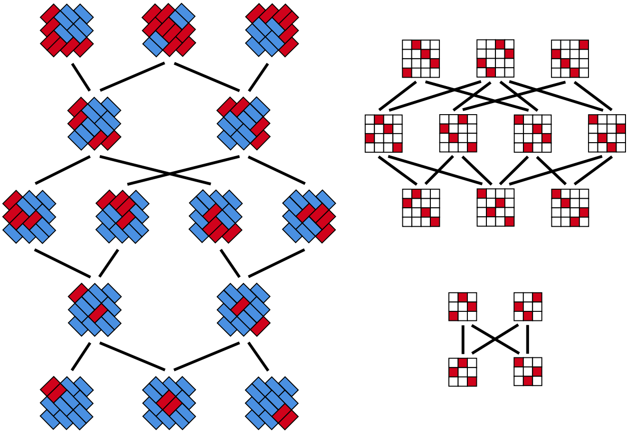


Figure: Join-irreducible aztec diamond tilings of size 3, and the join-irreducible ASMs of size 3 and 4.

Other lattices exhibit this property, such as Dyck and Motzkin paths:

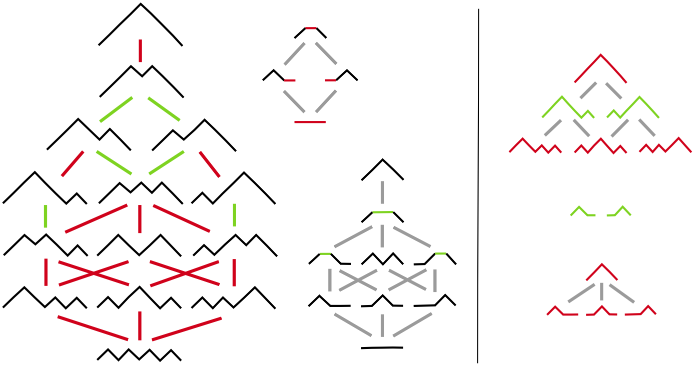


Figure: Lattices of Dyck paths and Motzkin paths, and their irreducibles posets.

Schröder paths can also be seen as pairs of Dyck paths:

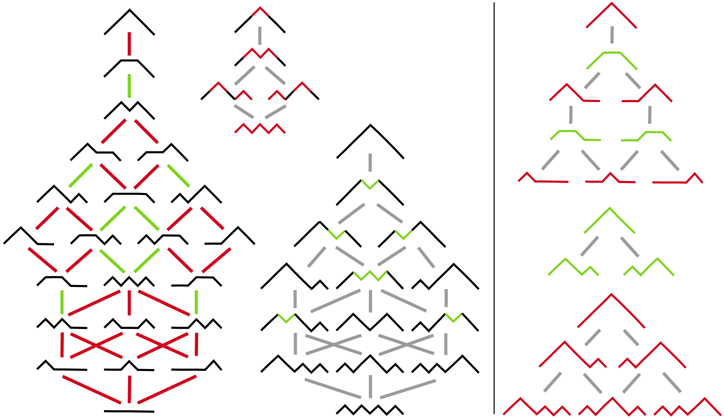


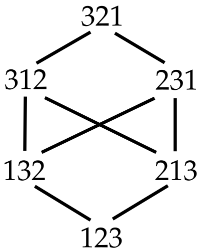
Figure: Lattices of Schröder paths and Dyck paths, and their irreducibles posets.

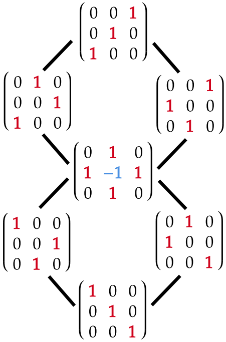
ASMs and completion of Bruhat order

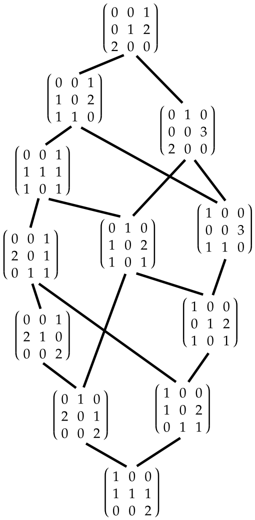
The completion of a poset is the smallest lattice containing it.

Theorem (Lascoux, Schützenberger 1996)

The lattice of ASMs of size n is isomorphic to the completion of Bruhat order on permutations of length n.







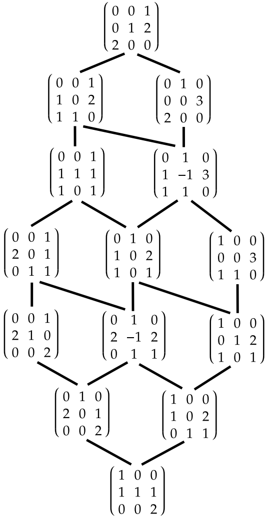


Figure: The Bruhat order on $\mathfrak{S}_{132}\backslash\mathfrak{S}_6/\mathfrak{S}_{213}$ and its completion.

ASMs and completion of Bruhat order

Theorem (Lascoux, Schützenberger 1996)

The completion of the Bruhat order on Coxeter groups of type B is a distributive lattice.

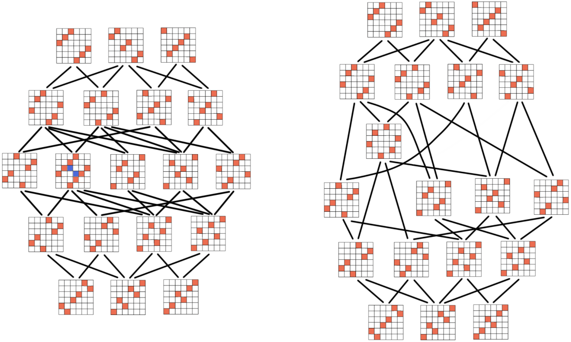


Figure: Join-irreducible centrally symmetric ASMs of size 6, and join-irreducibles of the completion of the Bruhat order on B_3.

Proposition

The lattice of ASMs of size n is a sublattice of the lattice of congruences of the weak order on the permutations of size $n+1$.

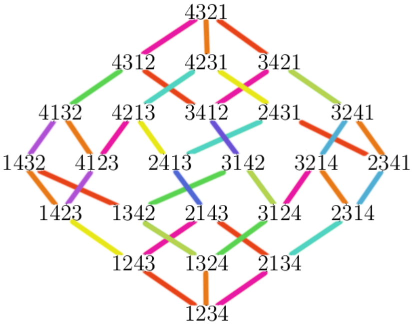
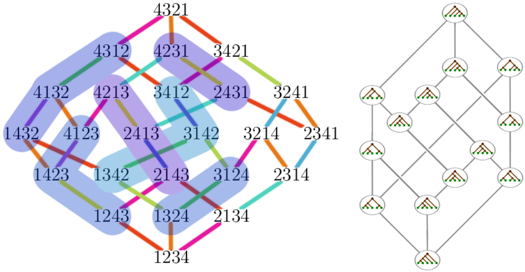


Figure: The weak order on $\mathfrak{S}_4$, and the poset of its join-irreducible congruences.

ASMs and congruences of the weak order

The Tamari lattice can be obtained as a quotient of the weak order by a congruence corresponding to an ASM:

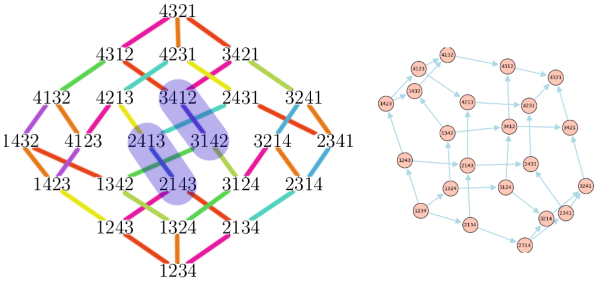


$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure: The Tamari congruence, the Tamari lattice and the corresponding ASM.

ASMs and congruences of the weak order

The lattice of Baxter permutations is another example of such a quotient:



$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure: The lattice of Baxter permutations and the corresponding ASM.

Thank you for your attention !