On the diffusive–mean field limit of the kinetic interacting particle system.

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Collaborators

- Joint work with :
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  - **Gabriel Stoltz** (Ecole Nationale des Ponts et Chaussées, Inria Paris),
  - **Urbain Vaes** (Inria Paris, Ecole Nationale des Ponts et Chaussées).

$\rightarrow$ Prepint before next year (I hope so)

- **Physical motivation**
  - Provide realistic models for complex systems : plasmas, gases, vortex dynamics, biological swarms, etc.
  - Bridge microscopic (particle-based) and macroscopic (continuum or field-based) descriptions.
  - Help understand how collective behaviors emerge from simple local interactions.

Introduction

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  - Provide realistic models for complex systems : plasmas, gases, vortex dynamics, biological swarms, etc.
  - Bridge microscopic (particle-based) and macroscopic (continuum or field-based) descriptions.
  - Help understand how collective behaviors emerge from simple local interactions.
- **Mathematical motivation**
  - Offer a rigorous framework for the limit $N \to \infty$, connecting discrete dynamics to kinetic equations (Vlasov, Boltzmann, etc.).
  - Raise fundamental questions of stability, propagation of chaos, and convergence.
  - Serve as a bridge between probability theory, analysis, and mathematical physics.

The Interacting Particle System

The interacting particle system

• Let $\mathcal{E} = \mathbb{T}^d \times \mathbb{R}^d$. We consider N indistinguishable interacting particles $(q_t^i, p_t^i)_{i \in \{1 \dots N\}}$

• $(\mathrm{q}^N, \mathrm{p}^N) := \left(q^1, \dots, q^N, p^1, \dots, p^N\right) \in \mathcal{E}^N$.

Interacting Particle System (IPS)

$$\begin{cases} \mathrm{d}q_t^i = p_t^i \,\mathrm{d}t, \\ \mathrm{d}p_t^i = -\nabla V(q_t^i)\,\mathrm{d}t - \frac{1}{N}\sum_{j \neq i}^{N} \nabla W(q_t^i - q_t^j)\,\mathrm{d}t - \gamma p_t^i \,\mathrm{d}t + \sqrt{\frac{2\gamma}{\beta}}\,\mathrm{d}B_t^i, \end{cases}$$

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- (q_0^i, p_0^i) i.i.d with law μ_0
- V and W in $\mathcal{C}^2(\mathbb{T}^d)$ (periodic and W is even)
- $(B_t^i)_{i\in\{1\dots N\}}$ are N independent d dimensional standard Brownian motion
- $\gamma > 0$ friction, $\beta > 0$ inverse temperature

A simpler case :
The Langevin dynamics

Langevin dynamics (1)

- Positions $q \in \mathcal{D} = \mathbb{T}^d$, momenta $p \in \mathbb{R}^d$
$\rightarrow$ phase-space $\mathcal{E} = \mathcal{D} \times \mathbb{R}^d$
- Hamiltonian $H(q, p) = V(q) + \frac{1}{2}|p|^2$

Underdamped Langevin

$$\begin{cases} \mathrm{d}q_t = p_t \, \mathrm{d}t \\ \mathrm{d}p_t = -\nabla V(q_t) \, \mathrm{d}t - \gamma p_t \, \mathrm{d}t + \sqrt{\frac{2\gamma}{\beta}} \, \mathrm{d}W_t \end{cases}$$

- (q_0, p_0) sample from the initial distribution μ_0.

Langevin dynamics (2)

- Evolution semigroup $\left(\mathrm{e}^{t\mathcal{L}}\varphi\right)(q,p) = \mathbb{E}\left[\varphi(q_t,p_t)\,\middle|(q_0,p_0)=(q,p)\right]$

- Generator of the dynamics $\mathcal{L}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbb{E}\left[\varphi(q_t,p_t)\,\middle|(q_0,p_0)=(q,p)\right]\right) = \mathbb{E}\left[(\mathcal{L}\varphi)(q_t,p_t)\,\middle|(q_0,p_0)=(q,p)\right]$$

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Generator of the Langevin dynamics $\mathcal{L} = \mathcal{L}_{\mathrm{ham}} + \gamma\mathcal{L}_{\mathrm{FD}}$

$$\mathcal{L}_{\mathrm{ham}} = p\cdot\nabla_q - \nabla V\cdot\nabla_p, \qquad \mathcal{L}_{\mathrm{FD}} = -p\cdot\nabla_p + \frac{1}{\beta}\Delta_p$$

- Existence and uniqueness of the invariant measure characterized by

$$\forall\varphi\in C_{\mathrm{c}}^{\infty}(\mathcal{E}), \qquad \int_{\mathcal{E}}\mathcal{L}\varphi\,\mathrm{d}\mu = 0$$

- Here, canonical measure

$$\mu(dq\,dp) = Z^{-1}\mathrm{e}^{-\beta H(q,p)}\,\mathrm{d}q\,\mathrm{d}p = \nu(\mathrm{d}q)\,\kappa(\mathrm{d}p)$$

Fokker–Planck equations

- Evolution of the law $\psi_t(q,p)$ of the process at time $t \geqslant 0$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\mathcal{E}} \varphi\, \psi_t\right) = \int_{\mathcal{E}} (\mathcal{L}\varphi)\, \psi_t$$

- Fokker–Planck equation (with $\mathcal{L}^\dagger$ adjoint of $\mathcal{L}$ on $L^2(\mathcal{E})$)

$$\partial_t \psi_t = \mathcal{L}^\dagger \psi_t$$

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- It is convenient to work in $L^2(\mu)$ with $f_t = \psi_t/\mu$
  - denote the adjoint of $\mathcal{L}$ on $L^2(\mu)$ by $\mathcal{L}^*$

$$\mathcal{L}^* = -\mathcal{L}_{\mathrm{ham}} + \gamma\mathcal{L}_{\mathrm{FD}}, \quad \mathcal{L}_{\mathrm{FD}} = -\frac{1}{\beta}\sum_{i=1}^d \partial_{p_i}^* \partial_{p_i}, \quad \mathcal{L}_{\mathrm{ham}} = \frac{1}{\beta}\sum_{i=1}^d \partial_{p_i}^* \partial_{q_i} - \partial_{q_i}^* \partial_{p_i}$$

  - Fokker–Planck equation $\partial_t f_t = \mathcal{L}^* f_t$

Statistical error (1)

- **Ergodic theorem :** for an observable $\varphi \in L^1(\mu)$,

$$\widehat{\varphi}_t = \frac{1}{t}\int_0^t \varphi(q_s, p_s)\,\mathrm{d}t \xrightarrow[t\to+\infty]{\text{a.s.}} \mathbb{E}_\mu[\varphi].$$

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• **Asymptotic variance** $\sigma_\varphi^2 = \lim_{t\to+\infty} t\,\mathrm{Var}_\mu(\widehat{\varphi}_t)$: with $\Pi\varphi = \varphi - \int_{\mathcal{E}} \varphi\,\mathrm{d}\mu$,

$$\begin{aligned}\sigma_\varphi^2 &= \lim_{t\to+\infty} 2\int_0^t \left(1 - \frac{s}{t}\right)\mathbb{E}_\mu\left[\Pi\varphi(q_s,p_s)\Pi\varphi(q_0,p_0)\right]\mathrm{d}s \\ &= 2\int_0^{+\infty}\int_{\mathcal{E}} (\mathrm{e}^{s\mathcal{L}}\Pi\varphi)\Pi\varphi\,\mathrm{d}\mu\, ds = 2\int_{\mathcal{E}} (-\mathcal{L}^{-1}\Pi\varphi)\Pi\varphi\,\mathrm{d}\mu\end{aligned}$$

Well-defined if the **Poisson equation** :

$$-\mathcal{L}\Phi = \Pi\varphi$$

has a solution in $L_0^2(\mu) = \Pi L^2(\mu)$.

Statistical error (2)

- A Central Limit Theorem [1] holds in this case :

$$\sqrt{t}(\widehat{\varphi}_t - \mathbb{E}_\mu[\varphi]) \xrightarrow[t\to+\infty]{\text{Law}} \mathcal{N}(0, \sigma_\varphi^2), \qquad \sigma_\varphi^2 = \langle \Phi, \varphi - \mathbb{E}_\mu[\varphi] \rangle_{L^2(\mu)}.$$

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- Sufficient condition : integrability of the semigroup, *e.g.*

$$\left\| \mathrm{e}^{t\mathcal{L}} \right\|_{\mathcal{B}(L_0^2(\mu))} \leqslant C\mathrm{e}^{-\lambda t}, \qquad -\mathcal{L}^{-1} = \int_0^{+\infty} \mathrm{e}^{s\mathcal{L}}\, ds$$

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Nice, but, how to obtain such decay ?

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Statistical error (2)

Prove **exponential convergence** of the semigroup $\mathrm{e}^{t\mathcal{L}}$ on $E \subset L^2_0(\mu)$

- Lyapunov techniques [2] $L^\infty_K(\mathcal{E}) = \left\{ \varphi \,\text{measurable}, \sup \left| \frac{\varphi}{K} \right| < +\infty \right\}$
- "historic" hypocoercive [3] setup $H^1(\mu)$
- $L^2(\mu)$ after hypoelliptic regularization [4] from $H^1(\mu)$
- direct transfer from $H^1(\mu)$ to $L^2(\mu)$ by spectral argument [5]
- **directly** [6] $L^2(\mu)$ (recently [7] Poincaré using $\partial_t - \mathcal{L}_{\mathrm{ham}}$)
- coupling arguments [8]
- **direct estimates on the resolvent using Schur complements** [9]

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2. Wu ('01) ; Mattingly/Stuart/Higham ('02) ; Rey-Bellet ('06) ; Hairer/Mattingly ('11)
3. Villani (2009) and before Talay (2002), Eckmann/Hairer (2003), Hérau/Nier (2004),...
4. Hérau, *J. Funct. Anal.* (2007)
5. Deligiannidis/Paulin/Doucet, *Ann. Appl. Probab.* (2020)
6. Hérau (2006), Dolbeaut/Mouhot/Schmeiser (2009, 2015)
7. Albritton/Armstrong/Mourrat/Novack (2019), Cao/Lu/Wang (2019), Brigatti (2021), Dietert/Hérau/Hutridurga/Mouhot (2022), Brigati/Stoltz (2023)
8. Eberle/Guillin/Zimmer, *Ann. Probab.* (2019)
9. Bernard/Fathi/Levitt/Stoltz, *Annales Henri Lebesgue* (2022)

An important application : The diffusive limit

• In the particular case where $\varphi = p$, the CLT (and homogenization theory) gives

$$\varepsilon Q_{t/\varepsilon^2} \xrightarrow[\varepsilon \to 0]{} \mathcal{N}(0, 2\mathcal{D}t), \qquad Q_t - Q_0 = \int_0^t p_s \, \mathrm{d}s.$$

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• Gives the **longtime behavior** of the underdamped dynamics ! Behave like a Brownian motion with

$$\mathcal{D} = \frac{1}{\gamma} \int_{\mathcal{E}} \nabla_p \Phi^\top \nabla_p \Phi \, \mathrm{d}\mu = \int_{\mathcal{E}} \Phi \otimes p \, \mathrm{d}\mu.$$

• Consequence : Einstein's formula :

$$\lim_{t \to +\infty} \mathbb{E}_\mu \left[\frac{|Q_t - Q_0|^2}{t} \right] = \mathcal{D}.$$

Key ideas of the proof

• Applying Itô's formula :

$$\Phi(q_t, p_t) - \Phi(q_0, p_0) = \int_0^t \mathcal{L}\Phi(q_s, p_s)\,\mathrm{d}t + \sqrt{\frac{2\gamma}{\beta}} \int_0^t \nabla_p \Phi(q_s, p_s) \cdot \mathrm{d}B_t.$$

• Rescaling by ε :

$$\begin{aligned} \varepsilon Q_{t/\varepsilon^2} - \varepsilon Q_0 &= \varepsilon \int_0^{t/\varepsilon^2} p_s \mathrm{d}t \\ &= \varepsilon(\Phi(q_0, p_0) - \Phi(q_{t/\varepsilon^2}, p_{t/\varepsilon^2})) + \varepsilon \sqrt{\frac{2\gamma}{\beta}} \int_0^{t/\varepsilon^2} \nabla_p \Phi(q_s, p_s) \cdot \mathrm{d}B_t \end{aligned}$$

• Φ bounded polynomially by p [10] and by admits bounded moments. Conclusion by FCLT.

10. Talay (2002), Kopec (2015)

Convergence of Langevin dynamics

Direct $L^2(\mu)$ approach : lack of coercivity

- The generator, considered on $L^2(\mu)$, is the sum of :
  - a **degenerate** symmetric part $\mathcal{L}_{\mathrm{FD}} = -p \cdot \nabla_p + \frac{1}{\beta}\Delta_p$
  - an **antisymmetric** part $\mathcal{L}_{\mathrm{ham}} = p \cdot \nabla_q - \nabla V \cdot \nabla_p$

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- Standard strategy for coercive generators :

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\mathrm{e}^{t\mathcal{L}}\varphi\right\|^2_{L^2(\mu)}\right) = 2\left\langle \mathrm{e}^{t\mathcal{L}}\varphi, \mathcal{L}\mathrm{e}^{t\mathcal{L}}\right\rangle_{L^2(\mu)} = 2\left\langle \mathrm{e}^{t\mathcal{L}}\varphi, \mathcal{L}_{\mathrm{FD}}\mathrm{e}^{t\mathcal{L}}\right\rangle_{L^2(\mu)}$$
$$= -\frac{2}{\beta}\left\|\nabla_p \mathrm{e}^{t\mathcal{L}}\varphi\right\|^2_{L^2(\mu)} \leqslant 0,$$

but **no control** of $\|\phi\|_{L^2(\mu)}$ by $\|\nabla_q \phi\|_{L^2(\mu)}$ for a Gronwall estimate...

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Two options :

- change of scalar product (use antisymmetric part)
- direct linear algebra approach

Almost direct $L^2(\mu)$ approach : convergence result

- Assume that the potential V is smooth and [11, 12]
  - the marginal measure ν satisfies a Poincaré inequality

$$\|\varphi - \nu(\varphi)\|_{L^2(\nu)} \leqslant \frac{1}{R_\nu} \|\nabla_q \varphi\|_{L^2(\nu)}$$

  - there exist $c_1 > 0$, $c_2 \in [0, 1)$ and $c_3 > 0$ such that V satisfies

$$\Delta V \leqslant c_1 + \frac{c_2}{2} |\nabla V|^2, \qquad \left|\nabla^2 V\right| \leqslant c_3 \left(1 + |\nabla V|\right)$$

There exist $C > 0$ and $\lambda > 0$ such that, for any $\varphi \in L_0^2(\mu)$,

$$\forall t \geqslant 0, \qquad \left\|\mathrm{e}^{t\mathcal{L}} \varphi\right\|_{L^2(\mu)} \leqslant C \mathrm{e}^{-\lambda t} \|\varphi\|_{L^2(\mu)}$$

11. Dolbeault, Mouhot and Schmeiser, *C. R. Math. Acad. Sci. Paris* (2009)
12. Dolbeault, Mouhot and Schmeiser, *Trans. AMS*, **367**, 3807–3828 (2015)

Sketch of proof (1)

- Change of scalar product to use the antisymmetric part $\mathcal{L}_{\mathrm{ham}}$:
  - bilinear form $\mathcal{H}[\varphi] = \frac{1}{2}\|\varphi\|^2_{L^2(\mu)} - \varepsilon \langle A\varphi, \varphi\rangle$ with [13]

$$A = \Big(1 + (\mathcal{L}_{\mathrm{ham}}\Pi_\kappa)^*(\mathcal{L}_{\mathrm{ham}}\Pi_\kappa)\Big)^{-1}(\mathcal{L}_{\mathrm{ham}}\Pi_\kappa)^*, \quad \Pi_\kappa\varphi = \int_{p\in\mathbb{R}^d} \varphi \, d\kappa$$

13. Hérau (2006), Dolbeault/Mouhot/Schmeiser (2009, 2015), ...

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  - $A = \Pi_\kappa A(1-\Pi_\kappa)$ and $\mathcal{L}_{\mathrm{ham}}A$ are bounded
  - modified square norm $\mathcal{H} \sim \|\cdot\|^2_{L^2(\mu)}$ for $\varepsilon \in (-1, 1)$
  - Approach not fully quantitative (optimize scalar product, here ε)

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- **Interest** : $(\mathcal{L}_{\mathrm{ham}}\Pi_\kappa)^*(\mathcal{L}_{\mathrm{ham}}\Pi_\kappa) = \beta^{-1}\nabla_q^*\nabla_q$ coercive in q, and

$$A\mathcal{L}_{\mathrm{ham}}\Pi_\kappa = \frac{(\mathcal{L}_{\mathrm{ham}}\Pi_\kappa)^*(\mathcal{L}_{\mathrm{ham}}\Pi_\kappa)}{1 + (\mathcal{L}_{\mathrm{ham}}\Pi_\kappa)^*(\mathcal{L}_{\mathrm{ham}}\Pi_\kappa)}$$

13. Hérau (2006), Dolbeault/Mouhot/Schmeiser (2009, 2015), ...

Sketch of proof (2)

- Recall Poincaré inequalities : $\nabla_p^*\nabla_p \geqslant K_\kappa^2(1-\Pi_\kappa)$ and $\nabla_q^*\nabla_q \geqslant R_\nu^2\Pi_\kappa$

Coercivity in the scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ induced by $\mathcal{H}$

$$D[\varphi] := \langle\langle -\mathcal{L}\varphi, \varphi\rangle\rangle \geqslant \lambda\|\varphi\|^2$$

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Coercivity in the scalar product $\langle\langle \cdot, \cdot \rangle\rangle$ induced by $\mathcal{H}$

$$D[\varphi] := \langle\langle -\mathcal{L}\varphi, \varphi \rangle\rangle \geqslant \lambda \|\varphi\|^2$$

- Upon controlling the remainder terms (some elliptic estimates)

$$\begin{aligned} D[\varphi] &= \gamma \left\langle -\mathcal{L}_{\mathrm{FD}}\varphi, \varphi \right\rangle + \varepsilon \left\langle R\mathcal{L}_{\mathrm{ham}}\Pi_\kappa \varphi, \varphi \right\rangle + \mathrm{O}(\gamma\varepsilon) \\ &= \frac{\gamma}{\beta} \|\nabla_p \varphi\|_{L^2(\mu)}^2 + \varepsilon \left\langle \frac{\nabla_q^* \nabla_q}{\beta + \nabla_q^* \nabla_q} \Pi_\kappa \varphi, \Pi_\kappa \varphi \right\rangle + \mathrm{O}(\gamma\varepsilon) \\ &\geqslant \frac{\gamma K_\kappa^2}{\beta} \|(1 - \Pi_\kappa)\varphi\|_{L^2(\mu)}^2 + \frac{\varepsilon R_\nu^2}{\beta + R_\nu^2} \|\Pi_\kappa \varphi\|_{L^2(\mu)}^2 + \mathrm{O}(\gamma\varepsilon) \end{aligned}$$

- Gronwall inequality $\dfrac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{H}\left[\mathrm{e}^{t\mathcal{L}}\varphi\right]\right) = -D\left[\mathrm{e}^{t\mathcal{L}}\varphi\right] \leqslant -\dfrac{2\lambda}{1+\varepsilon}\mathcal{H}\left[\mathrm{e}^{t\mathcal{L}}\varphi\right]$

Obtaining directly bounds on the resolvent (1)

"Saddle-point like" structure [14] for typical hypocoercive operators on $L^2_0(\mu)$

$$\mathcal{L} = \begin{pmatrix} 0 & \mathcal{A}_{0+} \\ \mathcal{A}_{+0} & \mathcal{L}_{++} \end{pmatrix}, \qquad \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_+, \qquad \mathcal{H}_0 = \Pi_\kappa \mathcal{H}, \qquad \mathcal{A} = \mathcal{L}_{\mathrm{ham}}$$

Formal inverse with Schur complement $\mathfrak{S}_0 = \mathcal{A}_{+0}^* \mathcal{L}_{++}^{-1} \mathcal{A}_{+0}$

$$\mathcal{L}^{-1} = \begin{pmatrix} \mathfrak{S}_0^{-1} & -\mathfrak{S}_0^{-1} \mathcal{A}_{0+} \mathcal{L}_{++}^{-1} \\ -\mathcal{L}_{++}^{-1} \mathcal{A}_{+0} \mathfrak{S}_0^{-1} & \mathcal{L}_{++}^{-1} + \mathcal{L}_{++}^{-1} \mathcal{A}_{+0} \mathfrak{S}_0^{-1} \mathcal{A}_{0+} \mathcal{L}_{++}^{-1} \end{pmatrix}$$

14. E. Bernard, M. Fathi, A. Levitt and G. Stoltz, *Annales Henri Lebesgue* (2022)

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Invertibility of $\mathfrak{S}_0$ is the crucial element : two ingredients

- $-\dfrac{1}{2}(\mathcal{L} + \mathcal{L}^*) \geqslant s\Pi_+ = s(1 - \Pi_\kappa)$ (Poincaré on $\kappa(dp)$ for Langevin)
- "macroscopic coercivity" $\|\mathcal{A}_{+0}\varphi\|_{L^2(\mu)} \geqslant a\|\Pi_\kappa \varphi\|_{L^2(\mu)}$
  Amounts to $\mathcal{A}_{+0}^* \mathcal{A}_{+0} \geqslant a^2 \Pi_\kappa$
  Guaranteed here by a Poincaré inequality for $\nu(dq)$, with $a^2 = R_\nu^2/\beta$

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Scaling with the friction and the dimension

- Final estimate for Fokker–Planck operators : scaling $\max(\gamma, \gamma^{-1})$

$$\left\|\mathcal{L}^{-1}\right\|_{\mathcal{B}(L_0^2(\mu))} \leqslant \frac{2\beta\gamma}{R_\nu^2} + \frac{4}{\gamma}\left(\frac{3}{4} + \left\|\Pi_+ \mathcal{L}_{\mathrm{ham}}^2 \Pi_\kappa \left(\mathcal{A}_{+0}^* \mathcal{A}_{+0}\right)^{-1}\right\|^2\right)$$

Scaling with the friction and the dimension

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- $C = 1$ and $C' = 0$ when V is convex ;
- $C = 1$ **and** $C' = K$ **when** $\nabla_q^2 V \geqslant -K\mathrm{Id}$ **for some** $K \geqslant 0$;
- $C = 2$ and $C' = \mathrm{O}(\sqrt{d})$ when $\Delta V \leqslant c_1 d + \frac{c_2\beta}{2}|\nabla V|^2$ (with $c_2 \leqslant 1$) and $\left|\nabla^2 V\right|^2 \leqslant c_3^2\left(d + |\nabla V|^2\right)$

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• In our case (V smooth on the torus) :

$$\left\|\mathcal{L}^{-1}\right\|_{\mathcal{B}(L^2_0(\mu))} \leqslant \frac{11}{\gamma} + \frac{1}{R_\nu^2}\left(\frac{4(\left\|\nabla^2 V\right\|_{L^\infty(\mathbb{T}^d)} + \left\|\nabla^2 W\right\|_{L^\infty(\mathbb{T}^d)})}{\gamma} + 2\beta\gamma\right)$$

Let's go back to our particle system !

The limit when N goes to infinity

- Recall the N **interacting particle system** :

$$\begin{cases} \mathrm{d}q_t^i = p_t^i \,\mathrm{d}t, \\ \mathrm{d}p_t^i = -\nabla V(q_t^i)\,\mathrm{d}t - \dfrac{1}{N}\displaystyle\sum_{j\neq i}^{N} \nabla W(q_t^i - q_t^j)\,\mathrm{d}t - \gamma p_t^i \,\mathrm{d}t + \sqrt{\dfrac{2\gamma}{\beta}}\,\mathrm{d}B_t^i. \end{cases}$$

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- The limit of each particle, when N goes to infinity, is

Nonlinear Mckean–Vlasov dynamics

$$\begin{cases} \mathrm{d}q_t = p_t \,\mathrm{d}t \\ \mathrm{d}p_t = -\nabla V(q_t)\,\mathrm{d}t - \nabla W \star \Pi\mu_t(q_t)\,\mathrm{d}t - \gamma p_t \,\mathrm{d}t + \sqrt{\dfrac{2\gamma}{\beta}}\,\mathrm{d}B_t, \end{cases}$$

$\star$ stands for the convolution and Π is the integration according to p.

Behavior of the nonlinear Mckean–Vlasov dynamics

- Problem : **nonlinear dynamics = nonlinear Fokker–Planck equation** (called Vlasov–Foker–Planck)

- Multiple stationary solutions. Let μ_∞ be one, it is solution of

$$\mu(\mathrm{d}q\,\mathrm{d}p) = \left(\frac{\beta}{2\pi}\right)^{d/2} \mathrm{e}^{-\beta\frac{|p|^2}{2}}\nu(q)\,\mathrm{d}q\,\mathrm{d}p, \qquad \nu(q) = \frac{1}{Z_\infty}\mathrm{e}^{-\beta(V(q)+W\star\nu(q))}.$$

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- Let $\mu_t = f_t\mu_\infty$, then :

$$\partial_t f_t = \mathsf{L}_t^* f_t - \beta p\cdot\nabla W \star \Pi_0(\mu_t - \mu_\infty) f_t,$$

where we informally define

$$\mathsf{L}_t = \mathsf{L}_t^{\mathrm{ham}} + \gamma\mathsf{L}^{\mathrm{FD}}, \qquad \mathsf{L}_t^* = -\mathsf{L}_t^{\mathrm{ham}} + \gamma\mathsf{L}^{\mathrm{FD}},$$

with

$$\mathsf{L}_t^{\mathrm{ham}} = p\cdot\nabla_q - \nabla V\cdot\nabla_p - \nabla W \star \Pi\mu_t\cdot\nabla_p, \qquad \mathsf{L}^{\mathrm{FD}} = -\frac{1}{\beta}\nabla_p^*\nabla_p.$$

The classical trick : linearize the equation

Linearized McKean–Vlasov dynamics

$$\begin{cases} \mathrm{d}\overline{q}_t = \overline{p}_t \mathrm{d}t \\ \mathrm{d}\overline{p}_t = -\nabla V(\overline{q}_t)\mathrm{d}t - \nabla W \star \Pi_0 \mu_\infty(\overline{q}_t)\mathrm{d}t - \gamma \overline{p}_t \mathrm{d}t + \sqrt{\frac{2\gamma}{\beta}}\mathrm{d}B_t, \end{cases}$$

It is now a **"classical" Langevin equation** !

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It is now a **"classical" Langevin equation** !

- We have a diffusive limit with covariance matrix :

$$\mathcal{D}_\infty = \frac{\gamma}{\beta} \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla_p \Phi^\infty(q,p)(\nabla_p \Phi^\infty(q,p))^\top \mu_\infty(q,p),$$

with

$$-\mathsf{L}_\infty \Phi^\infty = p.$$

Application to the nonlinear dynamics 1/2

- By Itô's formula :

$$Q_t - Q_0 = \Phi^\infty(q_0, p_0) - \Phi^\infty(q_t, p_t) + \sqrt{\frac{2\gamma}{\beta}} \int_0^t \nabla_p \Phi^\infty(q_s, p_s) \cdot \mathrm{d}B_s$$
$$- \int_0^t (\mathsf{L}_\infty - \mathsf{L}_s)\Phi^\infty(q_s, p_s) \, \mathrm{d}s.$$

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- We need to control the remainder :

$$\begin{aligned}\left|\int_0^t (\mathsf{L}_\infty - \mathsf{L}_s)\Phi^\infty(q_s,p_s)\,\mathrm{d}s\right| &= \left|\int_0^t \nabla W \star \Pi(\mu_s - \mu_\infty)(q_s)\cdot \nabla_p\Phi^\infty(q_s,p_s)\,\mathrm{d}s\right| \\ &\leqslant \int_0^t |\nabla W \star \Pi(\mu_s - \mu_\infty)(q_s)|\,|\nabla_p\Phi^\infty(q_s,p_s)|\,\mathrm{d}s.\end{aligned}$$

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$$\leqslant \int_0^t |\nabla W \star \Pi(\mu_s - \mu_\infty)(q_s)|\, |\nabla_p \Phi^\infty(q_s, p_s)|\,\mathrm{d}s.$$

- For every $q \in \mathbb{T}^d$,

$$|\nabla W \star \Pi(\mu_s - \mu_\infty)(q)| = \left| \int_{\mathbb{T}^d \times \mathbb{R}^d} \nabla W(q-u)(\mu_s(u,p) - \mu_\infty(u,p))\,\mathrm{d}u\,\mathrm{d}p \right|$$
$$\leqslant \|\nabla W\|_{L^\infty(\mathbb{T}^d)} \|\mu_s - \mu_\infty\|_{L^1(\mathbb{T}^d \times \mathbb{R}^d)}.$$

Application to the nonlinear dynamics 2/2

- Considering the rescaling, we have

$$\varepsilon^2 \left| \int_0^{t/\varepsilon^2} (\mathsf{L}_\infty - \mathsf{L}_s)\Phi^\infty(q_s, p_s)\,\mathrm{d}s \right|^2$$
$$\leqslant \|\nabla W\|_{L^\infty(\mathbb{T}^d)}\,\varepsilon^2 \left(\int_0^{t/\varepsilon^2} \|\mu_s - \mu_\infty\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} \left|\nabla_p \Phi^\infty(q_s, p_s)\right| \mathrm{d}s \right)^2$$

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• From Langevin, $\nabla_p \Phi^\infty$ has uniform moments in t. Then, this term vanishes if $t \mapsto \|\mu_t - \mu_\infty\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)}$ is integrable on $(0, +\infty)$!

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When does such a decay holds ?

Answer : can be obtained by hypocoercivity !

Obtaining exponential decay 1/2

- First

$$\|\mu_t - \mu_\infty\|_{L^1(\mathbb{T}^d\times\mathbb{R}^d)} = \left\|\frac{\mu_t}{\mu_\infty} - \mathbf{1}\right\|_{L^1(\mu_\infty)} = \|f_t\|_{L^1(\mu_\infty)} \leqslant \|f_t\|_{L^2(\mu_\infty)}\,.$$

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- By differentiating according to t

$$\frac{\mathrm{d}}{\mathrm{d}t} \|f_t\|^2_{L^2(\mu_\infty)} = 2 \langle \mathrm{L}_t^* f_t, f_t \rangle_{L^2(\mu_\infty)} - 2\beta \langle p \cdot \nabla W \star \Pi_0(\mu_t - \mu_\infty) f_t, f_t \rangle_{L^2(\mu_\infty)}.$$

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- We have the expression

$$\mathsf{L}_t^* = \mathsf{L}_\infty^* + \nabla W \star \Pi(\mu_t - \mu_\infty)\cdot\nabla_p.$$

And

$$\frac{\mathrm{d}}{\mathrm{d}t}\|f_t\|^2_{L^2(\mu_\infty)} = 2\left\langle \mathsf{L}_\infty^* f_t, f_t\right\rangle_{L^2(\mu_\infty)} - 2\left\langle \nabla W \star \Pi_0(\mu_t-\mu_\infty)\cdot\nabla_p^* f_t, f_t\right\rangle_{L^2(\mu_\infty)}.$$

Obtaining exponential decay 2/2

Exponential decay

There exists $\beta_\star > 0$ such that, for any initial condition μ_0 satisfying

$$\frac{\mu_0}{\mu_\infty} \in L^2(\mu_\infty),$$

the following holds : for any $\beta < \beta_\star$, there exist constants $L, C_\beta, \lambda_\beta > 0$ such that

$$\|\mu_t - \mu_\infty\|_{L^1(\mathbb{T}^d \times \mathbb{R}^d)} \leqslant L\mathrm{e}^{-\lambda_\beta t}.$$

In other words, for any μ_∞, there exists β small enough such that the decay holds.

Here, $N \to \infty$ then $\varepsilon \to 0$. What about the other way around ?

- Interacting particle system : **Langevin dynamics in big dimension**
- We can write the Poisson equation

$$-\mathcal{L}_N \Phi^N = \mathrm{p}^N$$

where each coordinate is defined by $-\mathcal{L}_N \Phi_i^N = \mathrm{p}_i^N$.

- The diffusion matrix is

$$\mathcal{D}^N = \frac{1}{\gamma} \int_{\mathcal{E}} (\nabla_p \Phi^N)^\top \nabla_p \Phi^N \, \mathrm{d}\mu_N = \int_{\mathcal{E}} \Phi^N \otimes \mathrm{p}^N \, \mathrm{d}\mu_N.$$

Question : What's happening when $N \to \infty$?

$\varepsilon \to 0$ then $N \to +\infty$

• Propagation of chaos theory : $\mathcal{D}^N$ converges to a **diagonal by block matrix** (the particles become i.i.d).

• Let $K \in \mathbb{N}$ and $(\mathcal{D}^N)_K$ be the extracted matrix of size $Kd \times Kd$, then

$$\lim_{N\to+\infty} (\mathcal{D}^N)_K = \begin{pmatrix} \mathcal{D}_\infty & 0 & \cdots & 0 \\ 0 & \mathcal{D}_\infty & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{D}_\infty \end{pmatrix} \in \mathbb{R}^{Kd\times Kd}.$$

But which $\mathcal{D}_\infty$? There could be multiple stationary states !

Limit of the matrix

Two conditions to pass to the limit :

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- Uniform estimate on the resolvant of $-\mathcal{L}_N \Phi_i^N = \mathrm{p}_i^N$

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$$(\mu_N)_1 \to \mu_\infty,$$

where μ_∞ is the **unique global minimiser** of the free energy functional

$$E_{\mathrm{MF}}[\rho] = \int_{\mathcal{E}} \left(\frac{1}{\beta}\log\rho(q,p) + \frac{p^2}{2} + V(q) + \frac{1}{2} W \star \Pi\rho(q)\right)\rho(q,p)\,\mathrm{d}q\mathrm{d}p,$$

Everything holds if β small enough !

Commutativity of the limits

- Let gather all the results :

  - If β small enough : **uniform Poincaré**[15] constant and **unique minimiser**[16] = limit for $\mathcal{D}^N$!

  - Again, if β small enough : recall the result for the nonlinear process. Convergence to $\mathcal{D}_\infty$ with μ_∞ the unique minimiser of the free energy !

The limit does commute !

15. Otto/Reznikoff (2007), Guillin/Liu/Wu/Zhang (2021)
16. Carillo/McCann/Villani (2003)

Numerical illustrations

The Kuramoto model

- The Kuramoto model captures how many interacting oscillators gradually synchronize their phases despite having different natural frequencies.

$$\begin{cases} \mathrm{d}q_t^i = p_t^i \,\mathrm{d}t, \\ \mathrm{d}p_t^i = -V'(q_t^i)\,\mathrm{d}t - \dfrac{1}{N}\displaystyle\sum_{j\neq i}^{N} W'(q_t^i - q_t^j)\,\mathrm{d}t - \gamma p_t^i \,\mathrm{d}t + \sqrt{\dfrac{2\gamma}{\beta}}\,\mathrm{d}B_t^i, \end{cases} \quad (2)$$

where $V(q) = \eta W(q) = -\eta \cos(2\pi q)$ and with its mean-field limit

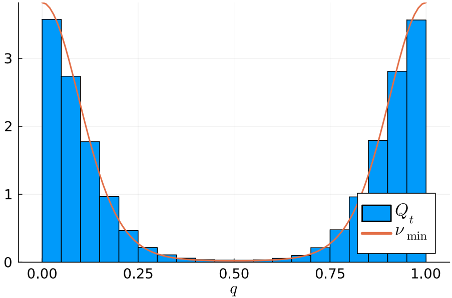
$$\begin{cases} \mathrm{d}q_t = p_t \,\mathrm{d}t, \\ \mathrm{d}p_t = -V'(q_t)\,\mathrm{d}t - W' \star \Pi\mu_t \,\mathrm{d}t - \gamma p_t^i \,\mathrm{d}t + \sqrt{\dfrac{2\gamma}{\beta}}\,\mathrm{d}B_t, \end{cases} \quad (3)$$

- **Stationnary states** of the form :

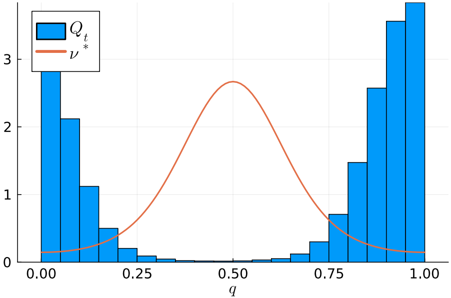
$$\mu(q,p) = Z^{-1}\mathrm{e}^{a_\beta \cos(2\pi q)}\kappa(p), \qquad Z = \int_{\mathbb{T}} \mathrm{e}^{a_\beta \cos(2\pi q)}\,\mathrm{d}q,$$

- Multiple value of a_β possible !

Numerical illustration 1/2

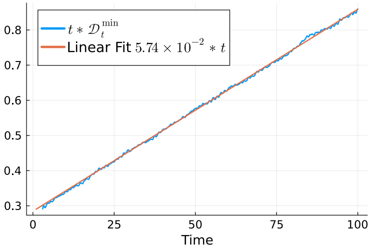


(a) Distribution of the particle system starting from ν_{min}

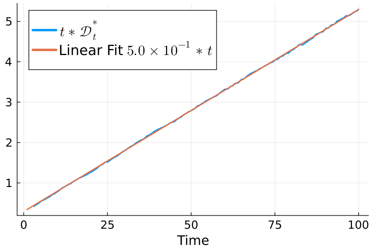


(b) Distribution of the particle system starting from ν^*

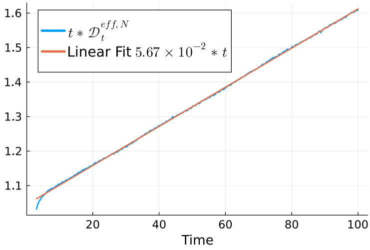
Numerical illustration 2/2



(a) $\mathcal{D}^{\min}$



(b) $\mathcal{D}^*$



(c) $\mathcal{D}^N$

Conclusion and perspectives

Synthesis of the work :

- The limit does commute when β is small enough.

Perspectives :

- Establish a general non-commutativity result
- What happens when the kernel in not the convolution ?
- In any case, scaling in N. What are the minimal hypothesis to have a uniform estimate on N on Φ^N ?
- Is a numerical illustration