Numerical approximation of the Mortensen observer in high-dimension

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Outline

Problem setting

We consider linear dynamics with a perturbation term

$$\begin{cases} \dot{x}(t) = f(x(t)) + B\nu(t), \quad t \in (0,T), \\ x(0) = x_0 + \zeta \in \mathcal{X}, \end{cases} \tag{1}$$

where

- $\mathcal{X} = \mathbb{R}^d$ is the state space ;
- $f : \mathcal{X} \to \mathcal{X}$ is the dynamic ;
- $\mathcal{U} \subset \mathbb{R}^p$ is the control space ;
- $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ is a linear operator ;
- $\nu : (0,T) \to \mathcal{U}$ is the perturbation.

Problem setting

We assume that the solution to (1) is only accessible through a noisy observation

$$y(t) = h(x(t)) + \eta(t), \quad t \in (0,T), \tag{2}$$

with $h : \mathcal{X} \to \mathcal{Y}$ the observation function, and $\eta \in L^2((0,T);\mathcal{Y})$.

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Assumptions

We will assume that $\mathcal{U} = \mathcal{X}$ and $B = \mathrm{Id}$ is the identity and further:

- $f \in C^1(\mathcal{X};\mathcal{X})$ with ∇f bounded and f sub-linear

$$\|f(x)\|_{\mathcal{X}} \le C(1 + \|x\|_{\mathcal{X}}), \quad \forall x \in \mathcal{X};$$

- $h \in C^1(\mathcal{X};\mathcal{Y})$ with ∇h bounded and h sub-linear

$$\|h(x)\|_{\mathcal{Y}} \le C(1 + \|x\|_{\mathcal{X}}), \quad \forall x \in \mathcal{X}.$$

Well-posedness

We denote

$$\mathcal{V}_T := L^2((0,T);\mathcal{X}),$$

the set of admissible controls, then we have

Proposition

For all $(\zeta,\nu) \in \mathcal{X} \times \mathcal{V}_T$, there exists a unique solution $x|_{\zeta,\nu} \in H^1((0,T);\mathcal{X})$ to (1) with the property that there exists a constant $C \in \mathbb{R}_+$ such that for all $(\zeta',\nu') \in \mathcal{X} \times \mathcal{V}_T$,

$$\|x|_{\zeta,\nu} - x|_{\zeta',\nu'}\|_{H^1((0,T);\mathcal{X})} \leq C(\|\zeta - \zeta'\|_{\mathcal{X}} + \|\nu - \nu'\|_{\mathcal{V}_T})$$

proof : standard fixed point argument *à la* Cauchy-Lipschitz.

The Optimal Estimation problem

The problem is as follows:

Consider a real trajectory $\check{x} = x|_{\check{\zeta},\check{\nu}}$ that is *unknown* but accessible through observations $(y(t))_{0\leq t\leq T}$. We wish to compute the best possible *estimation* of the state as a function $(\hat{x}(t))_{0\leq t\leq T}$.

Which criterium for optimality ?

A quadratic criterium

For $0 \leq t \leq T$, consider the quadratic criterium

$$J_t(\zeta, x) = \frac{1}{2}\|\zeta\|^2 + \frac{1}{2}\int_0^t \left[\|\dot{x}(s) - f(x(s))\|_{\mathcal{X}}^2 + \|y(s) - h(x(s))\|_{\mathcal{Y}}^2\right] \mathrm{d}s$$

defined on $\mathcal{X} \times H^1((0,T);\mathcal{X})$. Now, we can replace $\dot{x} - f(x)$ by ν assuming that x satisfies (1).

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We define accordingly

Quadratic criterium

$$\mathscr{J}_t(\zeta, \nu) := J_t(\zeta, x|_{\zeta,\nu}) = \frac{1}{2}\|\zeta\|^2 + \frac{1}{2}\int_0^t \left[\|\nu(s)\|_{\mathcal{X}}^2 + \|y(s) - h(x|_{\zeta,\nu}(s))\|_{\mathcal{Y}}^2\right] \mathrm{d}s \quad (3)$$

for $(\zeta, \nu) \in \mathcal{X} \times \mathcal{V}_t$.

The optimization problem

Proposition

There exists a unique minimizer for the functional $\mathscr{J}_t(\cdot,\cdot)$ on $\mathcal{X} \times \mathcal{V}_t$.

proof : weak compactness, injection $H^1((0,t);\mathcal{X}) \hookrightarrow C^0([0,t];\mathcal{X})$ and strict convexity.

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For a given fixed $x \in \mathcal{X}$ consider the constrained optimization problem

$$\begin{cases} \min\limits_{(\zeta,\nu)\in\mathcal{X}\times\mathcal{V}_t} \mathscr{J}_t(\zeta,\nu), \\ \text{s.t.} \quad x|_{\zeta,\nu}(t) = x. \end{cases} \tag{4}$$

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$$V(t,x) = \begin{cases} \min_{(\zeta,\nu)\in\mathcal{X}\times\mathcal{V}_t} \mathscr{J}_t(\zeta,\nu), \\ \text{s.t.} \quad x|_{\zeta,\nu}(t) = x. \end{cases} \tag{4}$$

We define the value function as the solution.

The HJB equation

One shows with a dynamic programming principle that the value function is (a viscosity) solution to the Hamilton-Jacobi-Bellman equation set on $(0,T) \times \mathcal{X}$

$$\begin{cases} \partial_t V(t,x) + \frac{1}{2}\langle f(t,x), \nabla V(t,x)\rangle_{\mathcal{X}} + \frac{1}{2}\|\nabla V(t,x)\|_{\mathcal{X}}^2 - \|y(t) - h(t,x)\|_{\mathcal{Y}}^2 = 0, \\ V(0,x) = \frac{1}{2}\|x - x_0\|_{\mathcal{X}}^2. \end{cases} \tag{HJB}$$

Outline

Definition of the Mortensen observer

Definition (Luenberger 1964)

Assume that $T = +\infty$. We say that a time function $z : (0,T) \to \mathcal{X}$ is a (non-linear) *observer* of the dynamics if

$$\|z(t) - \check{x}(t)\|_{\mathcal{X}} \to 0, \quad \text{as } t \to +\infty.$$

Under our assumptions, the value function is solution to (HJB) and it remains *strictly convex* for all time $t \in (0,T)$, hence we define

$$\hat{x}(t) := \underset{x \in \mathcal{X}}{\arg\min}\, V(t,x). \tag{5}$$

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Theorem (Mortensen 1968)

$(0, T) \ni t \mapsto \hat{x}(t)$ is an observer for the non-linear system.

The Kalman filter

Assume

- $f : x \mapsto Ax$ (linear dynamics);
- $h : x \mapsto Hx$ (linear observations).

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Then we have

Theorem

There exists a unique solution to the HJB equation, given by

$$V(t,x) = \langle x - \hat{x}(t), \Pi(t)^{-1}(x - \hat{x}(t)) \rangle_{\mathcal{X}} + \frac{1}{2}\int_0^t \|y(s) - H\hat{x}(s)\|_{\mathcal{Y}}^2 \, \mathrm{d}s, \quad (6)$$

where $(\hat{x}, \Pi)$ are given by the continuous-time Kalman-Bucy filter equations:

$$\begin{cases} \dot{\hat{x}} = A\hat{x} + \Pi H^\top (y - H\hat{x}), \\ \dot{\Pi} = A\Pi + \Pi A^\top - \Pi H^\top H \Pi + \mathrm{Id}\,. \end{cases} \quad (7)$$

proof : this ansatz is differentiable and satisfies (HJB). For unicity it requires the strong result that there is a unique viscosity solution (Crandall and Lions 1983).

Remark : V is strictly convex, with $\nabla^2 V = \Pi^{-1}$.

Discrete-time dynamics

We introduce discrete-time dynamics

$$\begin{cases} x_{n+1} = \varphi_\tau(x_n) + \nu_{n+1}, \quad n \geq 0, \\ x_0 = \hat{x}_0 + \zeta \in \mathcal{X}, \end{cases} \tag{8}$$

for $\varphi_\tau : \mathcal{X} \to \mathcal{X}$ and observations

$$y_n = h(x_n).$$

One can apply the same procedure to define a discrete optimal estimation problem. We set

$$J_{n^-}(\zeta, x_{0:n}) = \frac{1}{2}\|\zeta\|^2_{\mathcal{X}} + \frac{1}{2\tau}\sum_{k=0}^{n-1}\|x_{k+1} - \varphi_\tau(x_k)\|^2_{\mathcal{X}} + \frac{\tau}{2}\sum_{k=0}^{n-1}\|y_k - h(x_k)\|^2_{\mathcal{Y}}.$$

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One can alternatively define

$$J_{n^+}(\zeta, x_{0:n}) = J_{n^-}(\zeta, x_{0:n}) + \|y_n - h(x_n)\|_{\mathcal{Y}}^2$$

The splitting scheme

Discrete-time optimization problem

$$V_{n^\pm}(x) := \begin{cases} \min\limits_{(\zeta,\nu)\in\mathcal{X}\times\mathcal{V}_n} \mathscr{J}_{n^\pm}(\zeta,\nu) := J_{n^\pm}(\zeta, x_{0:n|\zeta,\nu}) \\ \text{s.t.} \quad x_{n|\zeta,\nu} = x. \end{cases} \tag{9}$$

Using Bellman, we see that a time splitting scheme emerges naturally with

1. *initialization*:
$$V_{0^-}(x) = V_0(x) = \frac{1}{2}\|x - \hat{x}_0\|^2_{\mathcal{X}};$$
2. *correction*:
$$V_{n^+}(x) = V_{n^-}(x) + \frac{\tau}{2}\|y_n - h(x)\|^2_{\mathcal{Y}}, \quad \forall n \geq 0;$$
3. *prediction*:
$$V_{n+1^-}(x) = \inf_{z\in\mathcal{X}} \left\{ V_{n^+}(z) + \frac{1}{2\tau}\|x - \varphi_\tau(z)\|^2_{\mathcal{X}} \right\}, \quad \forall n \geq 0.$$

The splitting scheme

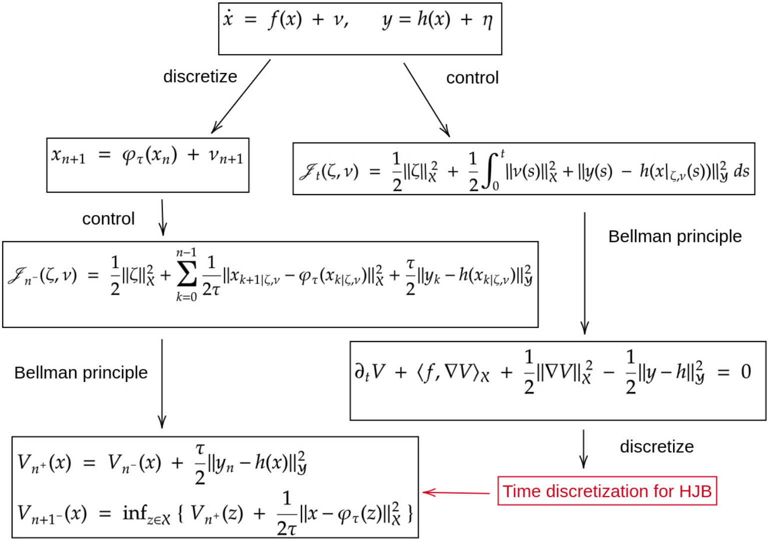
Theorem (Krener 2015; Moireau 2018)

Assume that V is smooth (at least C^1) and that φ_τ provides a consistent and unconditionally stable numerical scheme of the dynamics (1) as $\tau \to 0$. Then, the time splitting scheme is consistent and unconditionally stable and we have convergence

$$V_{n^\pm} \underset{\tau \to 0}{\longrightarrow} V(t_n, \cdot), \quad \text{in } C^1(\mathcal{X}),$$

where V is the value function.

The discretization strategy



Space discretization

We are left with space discretization. High-dimensional setting $\mathcal{X} = \mathbb{R}^d$ with $10 < d < 100$ typically $\rightarrow$ we need a *scalable* method which means **no**:

1. FD;
2. FEM;
3. FV;
4. methods that require a mesh...

Many possibilities : meshfree methods, tensor methods, adaptive grids, machine learning approaches (regression, PINNs, Deep Ritz,...)

A stepback on SciML

Scientific Machine Learning (SciML) is the field of research that is devoted to combining scientific insight with data-driven approaches. The definition is *very broad* as it is still an emerging field. Some literature:

- (E and Yu 2018) The Deep Ritz method ;
- (Raissi, Perdikaris, and Karniadakis 2019) Physics Informed Neural Networks (PINNs);
- (Y. Lu, J. Lu, and Wang 2021) extension of the Deep Ritz method in high-dim. for elliptic problems;
- (Müller and Zeinhofer 2023) Energy Natural Gradient Descent (ENGD).

Neural Network approximation

Approximation with NNs goes back (at least) to Cybenko 1989 $\rightarrow$ **Universal Approximation Theorems**. More recent developments:

- (Barron 1993) gives convergence rates for functions in the Barron space using a two-layer NN;
- (Y. Lu, J. Lu, and Wang 2021) show that for the Poisson problem if the source term is Barron, the solution is Barron;
- (Abdeljawad and Dittrich 2023) introduce a space-time analog of Barron for general Sobolev/Bochner spaces;
- (Dus and Ehrlacher 2025) two-layer NN approximation for the Poisson problem with convergence analysis using optimal transport techniques.

Back to our problem: a first idea

We recall the time-splitting scheme starting from a given (convex) V_0:

$$\begin{cases} \text{for } n \geq 0, \\ \textit{correction}: \\ \quad V_{n^+}(x) = V_{n^-}(x) + \frac{\tau}{2} \|y_n - h(x)\|^2, \\ \textit{prediction}: \\ \quad V_{n+1^-}(x) = \inf_{z \in \mathcal{X}} \left\{ V_{n^+}(z) + \frac{1}{2\tau} \|x - \varphi_\tau(z)\|^2 \right\}, \end{cases} \tag{10}$$

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Proposition (Hopf-Lax formula)

Assume that the flow φ_τ is invertible. Then, for any $n \in \mathbb{N}$, the predicted value function satisfies $V_{n+1^-}(x) = v_n(\tau, x)$ where v_n is the unique solution of the Hamilton-Jacobi equation

$$\begin{cases} \partial_t v + \dfrac{1}{2}\|\nabla v\|^2 = 0, \quad \text{in } (0,\tau) \times \mathcal{X}, \\ v(0,\cdot) = V_{n^+} \circ \varphi_\tau^{-1}, \quad \text{in } \mathcal{X}. \end{cases} \tag{hj$_n$}$$

A first idea

We introduce a vanishing viscosity problem for $\varepsilon > 0$

$$\begin{cases} \partial_t v + \frac{1}{2}\|\nabla v\|^2 - \varepsilon \Delta v = 0, & \text{in } (0,\tau)\times\mathcal{X}, \\ v(0,\cdot) = V_{n^+} \circ \varphi_\tau^{-1}, & \text{in } \mathcal{X}, \end{cases} \tag{$\mathrm{hj}_n^\varepsilon$}$$

and we solve ($\mathrm{hj}_n^\varepsilon$) using a space-time PINN for each $n \in \mathbb{N}$.

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Why a PINN ?

- No variational formulation $\rightarrow$ Deep Ritz is forbidden;
- rich literature on PINNs for Hamilton-Jacobi equations in high-dimension;

Example in 1D

We consider two simple 1D problems ($\mathcal{X} = \mathcal{Y} = \mathbb{R}$) with complete observations:

1. *steady target*: $\dot{x} = 0, \quad y = x$;
2. *drunk man*: $\dot{x} = \nu, \quad y = x$, where ν is a white noise,

and run our PINN-time-splitting scheme on these examples. We compare our result with the discrete Kalman Filter in 1D.

Numerical results

Steady test

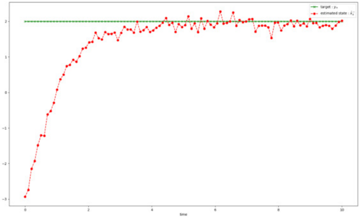


Figure: PINN-time-splitting, $[64, 64]$ hidden layers, 2000 samples, 1000 epochs, $\tau = 0.1$, $\varepsilon = 0.1$

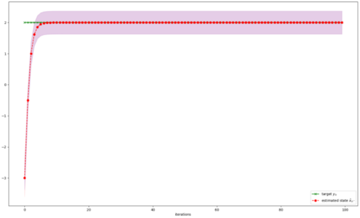
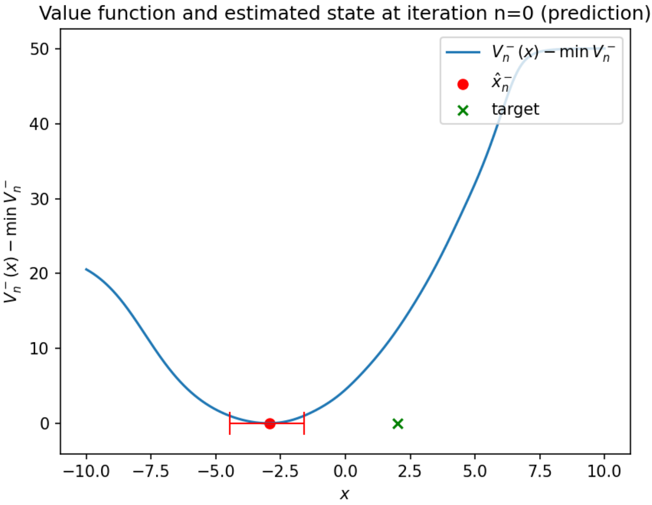


Figure: Kalman filter

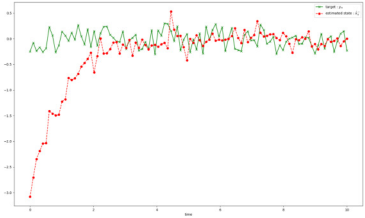
Numerical results

steady test



Numerical results

Drunk man



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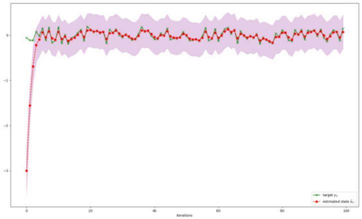

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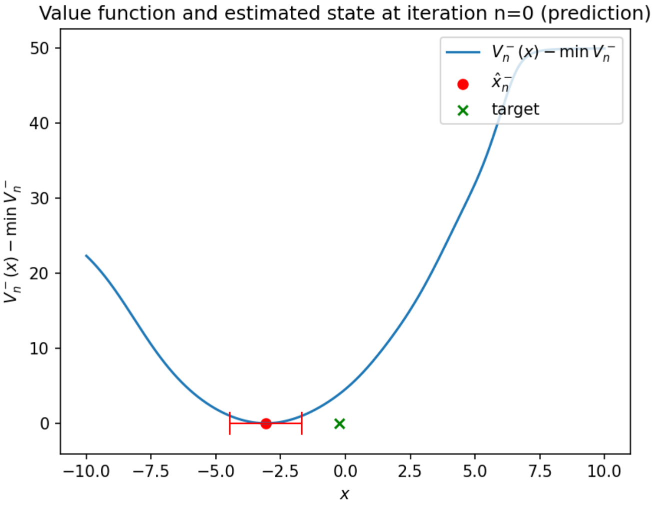
Numerical results

drunk man



Perspectives

- Scale up and try the method in more dimensions, next steps : harmonic oscillator, pendulum, and then Lorenz attractor;
- test new methods:
  1. currently working on a quadratic representation (ICNN) based on the linear (Kalman) setting with regression instead of PINN, links with optimal transport;
  2. solving the heat equation using deep Ritz techniques and obtain convergence results ? (*à la* Dus and Ehrlacher);
- first year so many new ideas to come...


To Be Continued

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